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UET FLOW SHOP SCHEDULING WITH DELAYS (*)

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Abstract. – *$F|UET$, delays $|C_{\max}$ is introduced and shown to be NP-complete.*

Résumé. – *Le problème du flow shop avec des temps de transport est introduit. Il est montré qu'il est NP-difficile même si les temps opératoires sont unitaires.*

1. INTRODUCTION

The usual flow shop problem can be described as follows.

Given are a set of n jobs and a set of m machines. Each machine can handle at most one job at a time and each job can be processed by at most one machine at a time. Each job consists of m tasks indexed by $1, \ldots, m$ and the i-th task of a job precedes its $(i+1)$-th task for $i = 1, \ldots, m-1$. Further, the i-th task of the j-th job has to be carried out on the i-th machine, during an uninterrupted period of a given length of time, l_{ij}. The purpose is to find a schedule of all the jobs which minimises the overall completion time.

Flow shop scheduling is shown to be NP-complete in the strong sense [1], even for the case $m = 3$. However, for the special case $m = 2$, there exists a polynomial time algorithm [2].

In this paper, we introduce the concept of an interprocessor time delay. This models the situation where there is a time delay when a job is transferred from one machine to another.

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Let d_{ij}, $1 \leq i \leq m-1$, $1 \leq j \leq n$, denote the time delay encountered in transferring job j from machine i to machine $i+1$ and c_{ij} and $s_{i+1\,j}$, respectively, denote the completion time of job j on machine i and the starting time of job j on machine $i+1$. Thus, the length of the i-th task of job j which must be scheduled on machine i is given by

$$l_{ij} = c_{ij} - s_{ij} + 1.$$

If in a given schedule,

$$s_{i+1\,j} \geq c_{ij} + d_{ij}, \quad \text{for } 1 \leq i \leq m-1,\ 1 \leq j \leq n,$$

then the schedule is called valid.

The delays are uniform if $d_{ij} = d$ for $1 \leq i \leq m-1$ and $1 \leq j \leq n$; otherwise the delays are non-uniform and, in general, this is the case we will be considering.

By setting $d_{ij} = 0$, for all $1 \leq i \leq m-1$, $1 \leq j \leq n$, it follows immediately from the NP-completeness of flow shop that flow shop with delays is NP-complete in the strong sense for any fixed $m \geq 3$. In [5] it is shown that the problem of flow shop with delays is NP-complete in the strong sense even for $m = 2$. However, if this problem is for a permutation flow shop, *i.e.* the order of the jobs is the same on all machines, then it can be solved in polynomial time [4].

When all the processing times are unit execution times (UET), the optimal flow shop schedule might not be achieved by a permutation flow shop nor by greedily ordering the jobs by nonincreasing delays even for $m = 2$. For example, consider four UET jobs on two machines with delays 5, 3, 3 and 1. The optimal schedule is of length 8 but the optimal permutation schedule is of length 10 and the greedy schedule is of length 9.

In section 2 of this paper, we prove that the problem of scheduling UET jobs with arbitrary delays in a (non-permutation) flow shop becomes NP-complete if we allow an arbitrary number of processors. The complexity of UET flow shop scheduling with delays for fixed $m \geq 2$ is open.

We summise that the two machine case is in *P* but we have been unable to prove this. A useful observation is that, for the two machine case only, there exists a valid schedule of optimal length such that the n jobs are processed continuously on machine 1 and continuously on machine 2. Moreover, for the two machine case, it is relatively straightforward to establish the bound

$$\omega_{\text{opt}} \geq \max\left\{\left\lceil \sum_{j=1}^{k} d_j/k \right\rceil + k + 1 : 1 \leq k \leq n\right\}.$$

However, this bound is not tight [3]; consider six jobs with delays 4, 4, 4, 0, 0, 0. The optimal schedule has length 10 which is greater than the bound of 9 given by the formula.

2. THE NP-COMPLETENESS RESULT

In this section, we prove the following decision problem is NP-complete.

UET FLOW SHOP WITH DELAYS (FUD)

Instance: number $p \in Z^+$ of processors, set J of jobs, for each job $j \in J$ and $1 \leq k < p$, a delay $d(j, k) \in Z^+$, and an overall deadline $D \in Z^+$.

Question: Is there a valid flow shop schedule of the jobs in J meeting the deadline where each job $j \in J$ has an associated UET task on each processor and such that for each $j \in J$ and each $1 \leq k < p$, if j is processed at time t on processor k then it is processed at time $\geq t + d(j, k) + 1$ on processor $k + 1$?

We will show that FUD is NP-complete. Our first step is to show that the following well-known NP-complete problem can be polynomially transformed into FUD.

VERTEX COVER (VC)

Instance: Graph $G = (V, E)$ and a positive integer, $k < |V|$.

Question: Is there a vertex cover of size $\leq k$ for G, *i.e.* a subset $V' \subset V$ with $|V'| \leq k$ where each edge in E is adjacent to some element in V'?

LEMMA 1: $VC \propto FUD$.

Proof: Let an instance of VC comprise a graph, $G = (V, E)$ with $|V| = n$, $|E| = m < n^2$ and a positive integer $k < n$. Assume $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$.

We construct an instance of FUD as follows.

$$\begin{aligned}
p &= 2m+3,\\
J &= \{x, b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_{k+1}\} \cup V,\\
d(x, 1) &= k+1,\\
d(c_i, 1) &= 3k+1,\ 1 \leq i \leq k+1,\\
d(b_i, 1) &= 0,\ 1 \leq i \leq k,\\
d(v_i, 1) &= 0,\ 1 \leq i \leq n,
\end{aligned}$$

and then, for $1 \leq r \leq m$,

$$\begin{aligned}
d(x, 2r) &= 2k+1,\\
d(c_i, 2r) &= k,\ 1 \leq i \leq k+1,\\
d(b_i, 2r) &= k-1,\ 1 \leq i \leq k,\\
d(x, 2r+1) &= 0,\\
d(c_i, 2r+1) &= k+1,\ 1 \leq i \leq k+1,\\
d(b_i, 2\ +1) &= k+2,\ 1 \leq i \leq k,
\end{aligned}$$

and, for $1 \leq i \leq n$, we define

$$d(v_i, 2r) = \begin{cases} 0, & \text{if } v_i \text{ is adjacent to } e_r,\\ k, & \text{otherwise} \end{cases}$$

and

$$d(v_i, 2r+1) = \begin{cases} 2k, & \text{if } v_i \text{ is adjacent to } e_r,\\ k, & \text{otherwise.} \end{cases}$$

Finally,

$$\begin{aligned}
d(x, 2m+2) &= 4k+n+3,\\
d(c_i, 2m+2) &= 2k+n+3-2i,\ 1 \leq i \leq k+1,\\
d(b_i, 2m+2) &= 3k+n+2-2i,\ 1 \leq i \leq k,\\
d(v_i, 2m+2) &= 0,\ 1 \leq i \leq n.
\end{aligned}$$

The deadline, D, is set by

$$D = 5k + 2mk + 3m + n + 7.$$

Since k is $O(n)$, this is clearly a polynomial transformation.

Our first observation concerns x. The total delays for x are

$$(k+1)+m\,(2\,k+1)+4\,k+n+3.$$

The total computation time for job x, as for all jobs, is $p = 2\,m+3$. The sum of these two values is exactly D. Thus, the deadline is met iff x is the first job processed on machine 1 and the last job processed on machine p. Moreover, the processing time on every intervening machine is also determined uniquely by the delays.

Now, we consider the job c_i. This has a total delay of

$$\begin{aligned}&3\,k+1+m\,(2\,k+1)+2\,k+n+3-2\,i\\&\quad=5\,k+2\,mk+m+n+4-2\,i\end{aligned}$$

and a processing time of $p = 2\,m+3$. The sum of these is

$$5\,k+2\,mk+3\,m+n+7-2\,i = D-2\,i.$$

A simple induction argument can then be used to determine that the deadline is met iff c_i is processed at time $1+i$ on machine 1 and at time $D-i$ on machine p. Moreover, the processing times of each c_i on every intervening machine are also determined uniquely by the delays.

Next, consider the job, b_i. The earliest it can be processed on machine 1 is $k+3$ and the latest it can be processed on machine p is

$$D-k-2 = 4\,k+2\,mk+3\,m+n+5.$$

The total of the delays for b_i is

$$m\,(2\,k+1)+3\,k+n+2-2\,i.$$

Adding the processing time of $2\,m+3$ gives a total of

$$3\,k+2\,mk+3\,m+n+5-2\,i.$$

Again, a simple induction argument shows that b_i must be processed at time $k+2+i$ on machine 1 and at time $D-k-i-1$ on machine p with all the intervening times uniquely determined by the delays.

The necessary scheduling of the job x and those of types b and c is described in figure 1. This shows that a valid schedule of all these jobs is achievable within the deadline.

We note that on machine $2r$, $(1 \leq r \leq m+1)$, x is processed at time

$$\begin{aligned} t_{2r} &= 2r + \sum_{j=1}^{2r-1} d(x, j) \\ &= 2r + k + 1 + (r-1)(2k+1) \\ &= 3r + 2rk - k, \end{aligned}$$

b_i is processed at time

$$\begin{aligned} &k + 1 + i + 2r + \sum_{j=1}^{2r-1} d(b_i, j) \\ &= k + 1 + i + 2r + (r-1)(2k+1) \\ &= t_{2r} + i \end{aligned}$$

and c_i is processed at time

$$\begin{aligned} &i + 2r + \sum_{j=1}^{2r-1} d(c_i, j) \\ &= i + 2r + 3k + 1 + (r-1)(2k+1) \\ &= t_{2r} + 2k + i. \end{aligned}$$

On machine $2r+1$,

b_i is proceesed at $t_{2r} + k + i$,
x is processed at $t_{2r} + 2k + 2$, and
c_i is processed at $t_{2r} + 3k + i + 1$.

On machine $2r+1$, spare processing time is thus available in a one unit slot at time $t_{2r} + 2k + 1$ and in a $(k-1)$ continuous block, $t_{2r} + 2k + 3, \ldots, t_{2r} + 3k + 1$. On machine $2m+3$, all the available times $\geq D - 2k - 1$ are allocated jobs. The lastest a job in V could be processed on machine $2m+3$ is

$$D - 2k - 2 = 3k + 2mk + 3m + n + 5$$

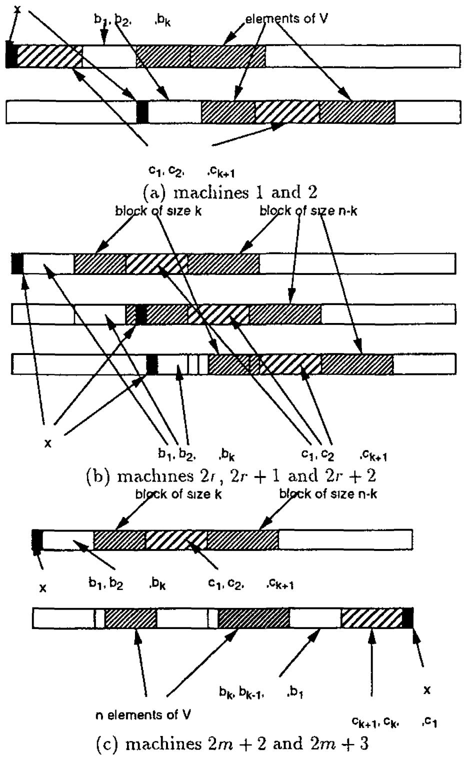

Figure 1. – The schedule structure

and hence on machine $2m + 2$ is

$$\begin{aligned} & 3k + 2mk + 3m + n + 4 \\ & = 3(m+1) + 2(m+1)k - k + 2k + 2 + n - 1 \\ & = t_{2m+2} + |J| - 1. \end{aligned}$$

Thus on machine $2m + 2$, the block of size k between the b-jobs and the c-jobs must be allocated to k jobs in V. The remaining $n - k$ jobs in V must be processed immediately after c_{k+1}. This partitions the jobs in V into two sets V_1 and V_2. The k jobs in V_1 are processed on machine $2m + 2$ in the early block of size k.

If an element $v \in V$ is processed at time $< t_{2r}$ on machine $2r$ for any $1 < r \leq m$ then, since the b-block on machine $2r - 1$ is fixed, this v must be processed before that block on machine $2r - 1$ and hence at time $< t_{2r-2}$ on machine $2r - 2$. Hence, we can deduce that such a v would be processed at time $< t_2$ on machine 2 which is impossible. We thus know every $v \in V$ is processed at time $> t_{2r}$ on machine $2r$ for all $1 \leq r \leq m+1$.

Having established that the block of size k on machine $2m+2$ must be allocated to V_1, an induction argument can then be used to show that V_1 always occupies the k locations between the b-block and the c-block on machines $2m$, $2m-2, \ldots, 2$.

Now, consider machine $2r+1$ $(1 \leq r \leq m)$. The single, isolated location between the location allocated to b_k and that allocated to x can be used iff V_1 contains a vertex adjacent to e_r. A schedule is valid iff this location is used. Hence, we have a valid schedule within the deadline iff V_1 is a vertex cover. Thus, the lemma is established.

It is now easy to establish

THEOREM 1: *FUD is NP-complete.*

Proof: Having established that an NP-complete problem polynomially transforms to FUD, all we need to establish is that FUD $\in$ NP.

Given the $|J|$ jobs and p machines, we simply guess an integer, $1 \leq t_{j,r} \leq D + r - p$, for each $j \in J$, $1 \leq t \leq p$. Then, in polynomial time, we check that

1. $t_{j,r} = t_{j',r} \Rightarrow j = j'$, for all $1 \leq r \leq p$, and
2. $t_{j,r+1} \geq t_{j,r} + d(j, r)$ for each $j \in J$ and $1 \leq r < p$.

An instance of FUD is a yes-instance iff there is some guess which passes all these checks.

REFERENCES

1. M. R. GAREY, D. S. JOHNSON and R. SETHI, The complexity of flow shop and job shop scheduling, *Math. Oper. Res.*, 1976, *1*, pp. 117-129.
2. S. JOHNSON, Optimal two and three stage production schedules with set-up times included, *Nav. Res. Log. Quart.*, 1954, *1*, pp. 61-68.
3. J. K. LENSTRA, *Private communication*, 1992.
4. P. L. MAGGU and G. DAS, On $2 \times n$ sequencing problem with transportation times of jobs, *Pure and Applied Math. Sci.*, 1980, *12*, No. 1-2, pp. 1-6.
5. R. J. M. VAESSENS and M. DELL'AMICO, Flow and open shop on two machines with transportation times and machine independent processing times is NP-hard, *unpublished manuscript*, 1995.