INFORMATIQUE THÉORIQUE ET APPLICATIONS

DOMINIQUE BARTH
FRANÇOIS PELLEGRINI
ANDRÉ RASPAUD
JEAN ROMAN

On bandwidth, cutwidth, and quotient graphs

Informatique théorique et applications, tome 29, n° 6 (1995), p. 487-508

<http://www.numdam.org/item?id=ITA_1995__29_6_487_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol. 29, n° 6, 1995, p. 487 à 508)

ON BANDWIDTH, CUTWIDTH, AND QUOTIENT GRAPHS (*)

by Dominique BARTH ([1]), François PELLEGRINI ([2]),
André RASPAUD ([2]) and Jean ROMAN ([2])

Communicated by R. CORI

Abstract. – *The bandwidth and the cutwidth are fundamental parameters related to many problems modeled in terms of graphs. In this paper, we present a general method for finding upper bounds of the bandwidth of a graph from the ones the quotient graph and induced subgraphs issued from any of its partitions, as well as an equivalent result for cutwidth. Moreover, general lower bounds are obtained by using vertex- and edge-bisection notions.*

These results are applied, in a second time, to several interconnection networks. By choosing convenient vertex partitions and judicious internal numberings of the vertices of the partition blocks, we prove in this paper original bounds for the binary de Bruijn and Butterfly graphs, and summarize results for the Shuffle-Exchange, FTT, and CCC graphs.

Résumé. – *Les largeurs de bande et de coupe sont deux paramètres fondamentaux qui interviennent dans la formulation de nombreux problèmes modélisés en termes de graphes.*

Dans cet article, nous présentons une méthode permettant de majorer la largeur de bande d'un graphe en fonction de celle du graphe quotient et des sous-graphes induits par n'importe laquelle de ses partitions, ainsi qu'un résultat équivalent pour la largeur de coupe. De plus, nous proposons des minorations générales de ces paramètres à partir des bissections sommet et arête du graphe.

Ces résultats sont appliqués à plusieurs réseaux d'interconnexion. En choisissant des partitions adaptées et des numérotations judicieuses des sommets des parties de ces partitions, nous prouvons des majorations originales pour les graphes de de Bruijn binaire et Butterfly, et récapitulons nos résultats pour les graphes Shuffle-Exchange, FFT, et CCC.

1. INTRODUCTION

In all the paper, G is a graph with n vertices, the vertex and edge sets of which are denoted $V(G)$ and $E(G)$ respectively. An ordering φ of G is a one-to-one mapping of $V(G)$ in the set $\{0, 1, ..., (n-1)\}$, and $\Phi(G)$ is the set of all orderings φ of G.

This work was supported by the French GDR PRS.

(*) Received April 1994; revised and accepted June 1995.

([1]) L.R.I., Université Paris-XI, Centre d'Orsay, Bât. 490, 91405 Orsay, France.

([2]) LaBRI, URA CNRS 1304, Université Bordeaux-I, 351, cours de la Libération, 33405 Talence, France.

Informatique théorique et Applications/Theoretical Informatics and Applications
0988-3754/95/06/$ 4.00/

The ordering of graphs is equivalent to their embedding into the path graph of same order, which allows us to define the dilation $\mathrm{dil}(\varphi)$ and congestion $\mathrm{cg}(\varphi)$ of an ordering φ as the ones of the corresponding embedding:

$$\mathrm{dil}(\varphi) \stackrel{\text{def}}{=} \max_{\{v', v''\} \in E(G)} (|\varphi(v') - \varphi(v'')|),$$

$$\mathrm{cg}(\varphi) \stackrel{\text{def}}{=} \max_{v \in V(G)} (|\{\{v', v''\} \in E(G) / \varphi(v') \leq \varphi(v) < \varphi(v'')\}|).$$

The bandwidth of G is then defined as

$$\mathrm{Bd}(G) \stackrel{\text{def}}{=} \min_{\varphi \in \Phi(G)} (\mathrm{dil}(\varphi)),$$

and the cutwidth of G is

$$\mathrm{Cw}(G) \stackrel{\text{def}}{=} \min_{\varphi \in \Phi(G)} (\mathrm{cg}(\varphi)).$$

The bandwidth and the cutwidth are two formal parameters of graphs which appear in the formulation of many problems modeled in terms of graphs. For instance, bandwidth has been used in code correction [14], gaussian elimination for sparse matrices [6, 12], and V.L.S.I. layout [18, 30], and cutwidth in V.L.S.I. design [9, 20, 25, 29].

General bounds have been proven for bandwidth in [6, 7], but since it has been shown in [10, 11, 12] that the problem of finding the bandwith and the cutwidth of any given graph is NP-complete, the main way to compute them is by the means of heuristics, as in [12, 13, 28].

In several occasions [7, 12], the method used to compute bandwidth involves a two-stage process, where vertices are clustered before being numbered within these clusters. Formalizing this approach by the means of quotient graphs, we present in this paper a method for finding upper bounds of the bandwidth and cutwidth of a graph, based on the bandwidth and cutwidth of its quotient graph and induced subgraphs with respect to the blocks of partitions of $V(G)$. All the necessary definitions and notations are given in Section 2.

This method leads to general upper bounds, stated in Theorems 1 and 5. When the structures of the graphs are convenient, the vertex orderings induced by the partitions can be finely tuned to improve the upper bounds; in that sense, our results can be seen as a framework for finding good

orderings rather than plain theorems. Lower bounds are also proven in Propositions 3 and 7, using the vertex- and edge-bisections of G. Section 3 contains all these results.

In Section 4, we use these results to prove original bounds for the binary de Bruijn and Butterfly graphs. Equivalent results for the Shuffle-Exchange, FFT, and CCC graphs are also summarized.

2. DEFINITIONS AND NOTATIONS

It is assumed that the reader is familiar with standard graph theoretic notations; *see* [5] for reference. In particular, $\mathrm{diam}(G)$ is the diameter of graph G, $\delta(v)$ denotes the degree of a vertex v, and $\delta(G) \stackrel{\text{def}}{=} \min_{v \in V(G)} (\delta(v))$ and $\Delta(G) \stackrel{\text{def}}{=} \max_{v \in V(G)} (\delta(v))$ are the minimum and maximum degrees of graph G, respectively.

In the following, a mod b denotes the remainder of the euclidian division of a by b, and $\oplus$ the exclusive-or operator on binary representations of integer numbers.

2.1. Partitions

Given a non-empty n-vertex set $V(G)$ of a graph G, a partition Π of $V(G)$ is a family of N non-empty mutually disjoint subsets called blocks, the union of which is equal to $V(G)$. We denote $\mathcal{P}(G)$ the set of all the partitions of $V(G)$.

For all vertices v in $V(G)$, $\pi(v)$ denotes the block of Π containing v. The size of the biggest block of a partition is $\max_\pi \stackrel{\text{def}}{=} \max_{\pi \in \Pi} (|\pi|)$.

For all blocks π in Π, $\omega(\pi)$ is the cocycle of π in G, *i.e.* the edge set $\{\{v', v''\} \in E(G) / v' \in \pi, v'' \notin \pi\}$. The size of the biggest cocycle in a partition is $\max_\omega \stackrel{\text{def}}{=} \max_{\pi \in \Pi} (|\omega(\pi)|)$. $E_I(\pi)$ and $E_E(\pi)$ respectively denote the sets of edges which have both ends in π (*i.e. internal* edges) and none of their ends in π (*i.e. external* edges). These notations are extended to the whole partition, with $E_I(\Pi) \stackrel{\text{def}}{=} \bigcup_{\pi \in \Pi} (E_I(\pi))$ and $E_E(\Pi) \stackrel{\text{def}}{=} E(G) - E_I(\Pi)$.

2.2. Quotient graphs

For all blocks π in Π, $G[\pi]$ denotes the subgraph of G induced by π. The quotient graph $Q = G_{/\Pi}$ of a graph G with respect to a partition Π of $V(G)$

is the graph such that $V(Q) = \Pi$, and $\{\pi', \pi''\}$ belongs to $E(Q)$ if and only if there exist v' in π' and v'' in π'' such that $\{v', v''\}$ belongs to $E(G)$.

2.3. Orderings

An *ordering* φ of a graph G is a one-to-one mapping of $V(G)$ in $\{0, 1, ..., (n-1)\}$. The *restriction* of φ to a block π of G, denoted $\varphi_{|\pi}$, is the ordering of $G[\pi]$ such that for all vertices v' and v'' in π, $(\varphi(v') < \varphi(v'')) \Leftrightarrow (\varphi_{|\pi}(v') < \varphi_{|\pi}(v''))$.

φ defines an implicit orientation of the edges of G, which allows us to define the lower and upper half-degrees of any vertex:

$$\delta_{\varphi}^{-}(v) \stackrel{\text{def}}{=} |\{\{v', v\} \in E(G)/\varphi(v') < \varphi(v)\}|,$$

and

$$\delta_{\varphi}^{+}(v) \stackrel{\text{def}}{=} |\{\{v, v''\} \in E(G)/\varphi(v) < \varphi(v'')\}|.$$

Let $\Phi_Q(G, \Pi)$ be the set of all orderings φ_Q of $Q = G_{/\Pi}$. For all blocks π of Π ordered by φ_Q, we define the numbers

$$\delta_{\Pi, \varphi_Q}^{-}(\pi) \stackrel{\text{def}}{=} \min_{v \in \pi} (|\{\{v', v\} \in \omega(\pi(v))/\varphi_Q(\pi(v')) < \varphi_Q(\pi(v))\}|),$$

and

$$\delta_{\Pi, \varphi_Q}^{+}(\pi) \stackrel{\text{def}}{=} \min_{v \in \pi} (|\{\{v, v''\} \in \omega(\pi(v))/\varphi_Q(\pi(v'')) > \varphi_Q(\pi(v))\}|).$$

They are the minimum number of edges which link any vertex of block π to vertices belonging to blocks of smaller and bigger numbers, respectively.

Let G be a graph, Π a partition of $V(G)$ into N blocks, and φ_Q an ordering in $\Phi_Q(G, \Pi)$. An ordering φ_G of $V(G)$ is *compatible* with Π and φ_Q if and only if vertices in blocks of increasing numbers have increasing numbers, *i.e.*:

$$(\varphi_Q(\pi') < \varphi_Q(\pi'')) \quad \Leftrightarrow \quad \begin{array}{l} \forall \pi', \pi'' \in \Pi, \\ (\forall v' \in \pi', \forall v'' \in \pi'', \varphi_G(v') < \varphi_G(v'')), \end{array}$$

which amounts to the following set of properties:

$$\begin{cases} \bigcup_{v \in \pi} \{\varphi_G(v)\} = (p_{\varphi_Q(\pi)}, \ldots, P_{\varphi_Q(\pi)}) & \text{the numbers in each block form an interval;} \\ \bigcup_{i=0}^{N-1} \{p_i, \ldots, P_i\} = \{0, \ldots, n-1\} & \text{the intervals are disjoint and cover } \{0, \ldots, n-1\}; \\ \forall i \in \{1, \ldots, N-1\},\ p_i = P_{i-1} + 1 & \text{partitions of increasing numbers give intervals of increasing extrema.} \end{cases}$$

In particular, $p_0 = 0$, $P_0 = |\varphi_Q^{-1}(0)| - 1$, $p_1 = |\varphi_Q^{-1}(0)|$, $P_1 = |\varphi_Q^{-1}(0)| + |\varphi_Q^{-1}(1)| - 1, \ldots, P_{N-1} = n - 1$. We denote $\Phi_G(G, \Pi, \varphi_Q)$ the set of all these orderings. We call *configuration* of G any triplet $(\Pi, \varphi_Q, \varphi_G)$ such that Π belongs to $\mathcal{P}(G)$, φ_Q belongs to $\Phi_Q(G, \Pi)$, and φ_G belongs to $\Phi_G(G, \Pi, \varphi_Q)$.

Since the partition $\Pi \cong V(G)$ with n blocks always allows orderings $\varphi_G \cong \varphi_Q$ which achieve the bandwidth and the cutwidth of G, we have

$$\mathrm{Bd}(G) = \min_{\substack{(\Pi, \varphi_Q, \varphi_G) \\ \text{configuration of } G}} (\mathrm{dil}(\varphi_G)),$$

$$\mathrm{Cw}(G) = \min_{\substack{(\Pi, \varphi_Q, \varphi_G) \\ \text{configuration of } G}} (\mathrm{cg}(\varphi_G)).$$

2.4. Bisections

The edge-bisection $\mathrm{bis}_e(G)$ of a graph G is the size of the edge-set of minimum cardinality whose removal splits G into two subgraphs the vertex-cardinalities of which differ by at most one.

The vertex-bisection $\mathrm{bis}_v(G)$ of a graph G is the size of the vertex-set of minimum cardinality whose removal splits G into two subgraphs of same vertex-cardinality. In some parts of our computations, we will be more

interested in the minimality of the size of a disconnecting vertex-set rather than in the strict equality between the sizes of the two resulting subgraphs. Therefore, we define the *almost-vertex-bisection* $\mathrm{bis}'_v(G)$ of a graph G to be the size of the vertex-set of minimum cardinality whose removal splits G into two subgraphs the vertex-cardinalities of which differ by at most one. Therefore,

$$\mathrm{bis}_v(G) = \begin{cases} \mathrm{bis}'_v(G) & \text{if} \quad n - \mathrm{bis}'_v(G) \text{ is even} \\ \mathrm{bis}'_v(G) + 1 & \text{if} \quad n - \mathrm{bis}'_v(G) \text{ is odd.} \end{cases}$$

3. GENERAL RESULTS

In this section, we prove upper and lower bound theorems, both for bandwidth and cutwidth. The proofs of the upper bounds are based on majorations performed when considering the ordering of the vertices of the original graph with respect to the one of any of its quotient graphs. The proofs of the lower bounds are based on the almost-vertex- and edge-bisections, respectively.

THEOREM 1: *Let G be a graph, Q the quotient graph obtained from a partition Π of $V(G)$, φ_Q an ordering of Q achieving the bandwidth of Q. For all $\varphi_G \in \Phi_G(G, \Pi, \varphi_Q)$, we have*

$$\mathrm{Bd}(G) \leq \max(\varepsilon_i(\varphi_G), (\mathrm{Bd}(Q) - 1)\max_\pi + \varepsilon_e(\varphi_G))$$

where

$$\begin{cases} \varepsilon_i(\varphi_G) \stackrel{\text{def}}{=} \max\limits_{\pi \in \Pi} (\mathrm{dil}(\varphi_{G|\pi})), \\ \varepsilon_e(\varphi_G) \stackrel{\text{def}}{=} \max\limits_{\substack{\{v', v''\} \in E_E(\Pi) \\ \varphi_G(v') < \varphi_G(v'')}} ((P_{\varphi_Q(\pi(v'))} - \varphi_G(v')) \\ \quad + (\varphi_G(v'') - p_{\varphi_Q(\pi(v''))}) + 1). \end{cases}$$

Proof: We know that, for all φ_G in $\Phi_G(G, \Pi, \varphi_Q)$, $\mathrm{Bd}(G) \leq \mathrm{dil}(\varphi_G)$. In order to make parameters of Q appear within the expression of $\mathrm{dil}(\varphi_G)$, we rewrite this latter with respect to the blocks of the partition, splitting it into two terms accounting respectively for the edges of G internal to the blocks and for the other edges.

For all configurations $(\Pi, \varphi_Q, \varphi_G)$, $\mathrm{dil}(\varphi_G)$ rewrites into:

$$\mathrm{dil}(\varphi_G) = \max\Big(\max_{\{v', v''\} \in E_I(\Pi)} (|\varphi_G(v') - \varphi_G(v'')|), \max_{\{v', v''\} \in E_E(\Pi)} (|\varphi_G(v') - \varphi_G(v'')|)\Big).$$

Let us evaluate these two terms separately.

– Since φ_G is an ordering compatible with φ_Q, the vertices in each block are numbered in sequence, so

$$\begin{aligned}
&\max_{\{v', v''\} \in E_I(\Pi)} (|\varphi_G(v') - \varphi_G(v'')|) \\
&\quad = \max_{\pi \in \Pi} \Big(\max_{\{v', v''\} \in E_I(\pi)} (|\varphi_{G|\pi}(v') - \varphi_{G|\pi}(v'')|)\Big) \\
&\quad = \max_{\pi \in \Pi} (\mathrm{dil}(\varphi_{G|\pi})) \\
&\quad = \varepsilon_i(\varphi_G).
\end{aligned}$$

– When $E_E(\Pi)$ is not empty, $\mathrm{Bd}(Q) \geq 1$, and for all $\{v', v''\}$ in $E_E(\Pi)$, with $q' = \varphi_Q(\pi(v')) < \varphi_Q(\pi(v'')) = q''$, we have $q'' - q' \leq \mathrm{Bd}(Q)$ and

$$\begin{aligned}
&\varphi_G(v'') - \varphi_G(v') \\
&\quad = (\varphi_G(v'') - p_{q''} + 1) + (P_{q''-1} - p_{q'+1} + 1) + (P_{q'} - \varphi_G(v')) \\
&\quad = \sum_{q' < q < q''} |\varphi_Q^{-1}(q)| + (P_{q'} - \varphi_G(v')) + (\varphi_G(v'') - p_{q''}) + 1 \\
&\quad \leq (\mathrm{Bd}(Q) - 1) \max_\pi + (P_{q'} - \varphi_G(v')) + (\varphi_G(v'') - p_{q''}) + 1.
\end{aligned}$$

Therefore,

$$\max_{\{v', v''\} \in E_E(\Pi)} (|\varphi_G(v') - \varphi_G(v'')|) \leq (\mathrm{Bd}(Q) - 1) \max_\pi + \varepsilon_e(\varphi_G).$$

Combining these two results leads to the claimed result. □

Although the majorations performed within the proof seem rough, this theorem usually gives good results as most partitions are taken with blocks having the same size. If it is not the case, a more specific study has to be carried on.

From the above theorem, we can deduce the following straightforward corollary.

COROLLARY 2: *If G is a graph and Q is the quotient graph obtained from a partition Π of $V(G)$, then*

$$\mathrm{Bd}(G) \leq (\mathrm{Bd}(Q) + 1)\max_\pi - 1.$$

Proof: Notice that $\varepsilon_i(\varphi_G) \leq \max_\pi - 1$ and $\varepsilon_e(\varphi_G) \leq 2\max_\pi - 1$. □

PROPOSITION 3: *For all graphs G,*

$$\mathrm{Bd}(G) \geq \mathrm{bis}'_v(G).$$

Proof: Let φ_G be an ordering of G which achieves the bandwidth of G, *i.e.* such that $\max_{\{v', v''\} \in E(G)} (|\varphi_G(v') - \varphi_G(v'')|) = \mathrm{Bd}(G)$. Let us define

$$b_1 = \left\lfloor \frac{|V(G)| - \mathrm{Bd}(G)}{2} \right\rfloor \quad \text{and} \quad b_2 = \left\lfloor \frac{|V(G)| + \mathrm{Bd}(G)}{2} \right\rfloor,$$

and let $V_d = \{v \in V(G) / b_1 \leq \varphi_G(v) < b_2\}$. By what precedes, there is no edge $\{v', v''\}$ in $E(G)$ such that $\varphi_G(v') < b_1$ and $\varphi_G(v'') \geq b_2$, and V_d disconnects G into two subgraphs G_1 and G_2 such that $|V(G_1)| = |V(G_2)|$ or $|V(G_1)| = |V(G_2)| - 1$, so $\mathrm{bis}'_v(G) \leq |V_d| = \mathrm{Bd}(G)$.

Since the vertex-bisection is not known for many graphs, the most direct way to compute lower bounds for bandwith is by means of Chung's theorem.

THEOREM 4: *Chung* [7]

$$\mathrm{Bd}(G) \geq \left\lceil \frac{n-1}{\mathrm{diam}(G)} \right\rceil.$$

The most interesting point of this bound is that it does not require a deep knowledge of the studied graph, while the (almost-) vertex-bisection is often hard to compute. As a matter of fact, the lower bounds of the bandwidth of the two chosen examples are computed using Chung's theorem.

Let us now deal with the cutwidth.

THEOREM 5: *If G is a graph, Q the quotient graph obtained from a partition Π of $V(G)$, and φ_Q an ordering of Q achieving the cutwidth of Q, then*

$$\mathrm{Cw}(G) \leq \left(\mathrm{Cw}(Q) - \left\lceil \frac{\delta(Q)}{2} \right\rceil\right) \cdot \max_\omega + \max_{\pi \in \Pi} (\mathrm{Cw}(G[\pi]) + |\omega(\pi)| - |\pi| \cdot \min(\delta^-_{\Pi, \varphi_Q}(\pi), \delta^+_{\Pi, \varphi_Q}(\pi))).$$

Proof: We know that, for all φ_G in $\Phi_G(G, \Pi, \varphi_Q)$, $\mathrm{Cw}(G) \leq \mathrm{cg}(\varphi_G)$. In order to make parameters of Q appear within the expression of $\mathrm{cg}(\varphi_G)$, we rewrite this latter with respect to the blocks of the given partition, splitting it into three terms accounting respectively for the edges of G external to a given block, the edges which have exactly one end in this block, and the edges internal to the block, as shown in Figure 1.

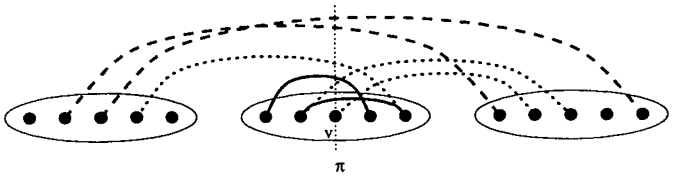


Figure 1. – External (dashed), cocycle (dotted), and internal (solid) edges with respect to block π.

Let

$$f_E(\pi) = \max_{v \in \pi} (|\{\{v', v''\} \in E_E(\pi) / \varphi_G(v') \leq \varphi_G(v) < \varphi_G(v'')\}|),$$
$$f_C(\pi) = \max_{v \in \pi} (|\{\{v', v''\} \in \omega(\pi) / \varphi_G(v') \leq \varphi_G(v) < \varphi_G(v'')\}|),$$

and

$$f_I(\pi) = \max_{v \in \pi} (|\{\{v', v''\} \in E_I(\pi) / \varphi_G(v') \leq \varphi_G(v) < \varphi_G(v'')\}|),$$

so that $\mathrm{cg}(\varphi_G) \leq \max_{\pi \in \Pi} (f_E(\pi) + f_C(\pi) + f_I(\pi))$. Let us study these three terms separately. Let π in Π and v in π.

– Since $f_E(\pi)$ accounts for the edges external to π, the choice of v in π is not significant. Let us then consider the quotient graph $Q = G_{/\Pi}$. Let π^- and π^+ be the vertices of Q such that $\varphi_Q(\pi^-) = \varphi_Q(\pi) - 1$ and $\varphi_Q(\pi^+) = \varphi_Q(\pi) + 1$, if they exist. The number of edges of Q whose end numbers strictly bracket $\varphi_Q(\pi)$, which we denote $g_E(\pi)$, can be written in two different ways:

$$g_E(\pi) = |\{\{w', w''\} \in E(Q) / \varphi_Q(w') \leq \varphi_Q(\pi^-) < \varphi_Q(w'')\}| - \delta^-_{\varphi_Q}(\pi),$$

and

$$g_E(\pi) = |\{\{w', w''\} \in E(Q)/\varphi_Q(w') \leq \varphi_Q(\pi) < \varphi_Q(w'')\}| - \delta^+_{\varphi_Q}(\pi).$$

Since the first term of each right member of the above equations is, by definition, bounded by $\mathrm{cg}(\varphi_Q)$, we have

$$g_E(\pi) \leq \mathrm{cg}(\varphi_Q) - \min(\delta^-_{\varphi_Q}(\pi), \delta^+_{\varphi_Q}(\pi)),$$

and since $\delta^-_{\varphi_Q}(\pi) + \delta^+_{\varphi_Q}(\pi) = \delta(\pi)$, the minimum of the partial degrees is always greater than or equal to $\left\lceil \frac{\delta(\pi)}{2} \right\rceil$, so $g_E(\pi) \leq \mathrm{cg}(\varphi_Q) - \left\lceil \frac{\delta(\pi)}{2} \right\rceil$. Since an edge of Q quotients at most $\max_\omega$ edges of G, $f_E(\pi) \leq \max_\omega \cdot \left(\mathrm{cg}(\varphi_Q) - \left\lceil \frac{\delta(\pi)}{2} \right\rceil\right)$.

– The second term is straightforwardly bounded by $|\omega(\pi)|$. However, from every vertex v' of π such that $\varphi_G(v') \geq \varphi_G(v)$, there exist at least $\delta^+_{\Pi,\varphi_Q}(\pi)$ edges of $\omega(\pi)$ incident to vertices v'' with $\varphi_G(v'') > \varphi_G(v)$, which must not be accounted for. Similarly, for each vertex v' of π such that $\varphi_G(v') < \varphi_G(v)$, at least $\delta^-_{\Pi,\varphi_Q}(\pi)$ edges must not be accounted for.

Therefore, $f_C(\pi) \leq |\omega(\pi)| - |\pi| \cdot \min(\delta^-_{\Pi,\varphi_Q}(\pi), \delta^+_{\Pi,\varphi_Q}(\pi))$.

– The third term is, by definition, equal to the congestion of $G[\pi]$ with the restriction of φ_G to π, *i.e.* $\mathrm{cg}(\varphi_{G|\pi})$.

By combining these three upper bounds, we obtain

$$\mathrm{Cw}(G) \leq \max_{\pi \in \Pi} \left(\left(\mathrm{cg}(\varphi_Q) - \left\lceil \frac{\delta(\pi)}{2} \right\rceil \right) \cdot \max_\omega + \mathrm{cg}(\varphi_{G_{|\pi}}) + (|\omega(\pi)| - |\pi| \cdot \min(\delta^-_{\Pi,\varphi_Q}(\pi), \delta^+_{\Pi,\varphi_Q}(\pi))) \right).$$

Since, in any block, the ordering of the vertices is continuous by definition, and independant from the orderings of other blocks, it is possible, once taken an ordering φ_Q which achieves the cutwidth of Q, to obtain a compatible ordering φ_G in $\Phi_G(G, \Pi, \varphi_Q)$ such that the restriction of φ_G to every block π in Π gives orderings which achieve the cutwidth of $G[\pi]$. Combining this ordering with the above equation yields the claimed result. □

This theorem leads to the more simple form stated in the next corollary.

COROLLARY 6: *If G is a connected graph and Q is the quotient graph obtained from a partition Π of $V(G)$ with at least two blocks, then*

$$\mathrm{Cw}(G) \leq \mathrm{Cw}(Q)\,\Delta(G)\,\max_\pi.$$

Proof: If G is connected and Π is not restricted to a single block, then $\delta(Q) \geq 1$. For all blocks π, $\mathrm{Cw}(G[\pi]) + |\omega(\pi)|$ is bounded by the number of edges having at least an end in π and thus by $\Delta(G)\,|\pi|$, and $\max_\omega \leq \Delta(G)\max_\pi$. □

PROPOSITION 7: *For all graphs G,*

$$\mathrm{Cw}(G) \geq \mathrm{bis}_e(G).$$

Proof: Let φ_G be an ordering of G which achieves the cutwidth of G, and let $E_d = \left\{\{v', v''\} \in E(G)/\varphi_G(v') \leq \left\lfloor \frac{n-1}{2} \right\rfloor < \varphi_G(v'')\right\}$. By definition of the cutwidth, $|E_d| \leq \mathrm{Cw}(G)$, and E_d disconnects G into two subgraphs G_1 and G_2 such that $|V(G_1)| = \left\lfloor \frac{n-1}{2} \right\rfloor$ and $|V(G_2)| = \left\lceil \frac{n-1}{2} \right\rceil$, so $\mathrm{bis}_e(G) \leq |E_d| \leq \mathrm{Cw}(G)$. □

4. APPLICATIONS

4.1. Binary de Bruijn graph

We exclusively study the case of unoriented binary de Bruijn graphs, denoted $\mathrm{UB}(2, k)$. The vertices v of $\mathrm{UB}(2, k)$ are words of k letters taken in the $\{0, 1\}$ alphabet, and denoted $v = v_1 v_2 \ldots v_k$. There exists an edge between two distinct vertices v and v' if and only if the $(k-1)$ leftmost letters of one of the two vertices are equal to the $(k-1)$ rightmost letters of the other vertex, *i.e.* if $v' = v_2 v_3 \ldots v_k x$ or $v' = x v_1 \ldots v_{k-2} v_{k-1}$, with $x \in \{0, 1\}$.

We define the complement of v as $\overline{v} = \overline{v_1}\,\overline{v_2} \ldots \overline{v}_k$, where $\overline{v}_i = 1 - v_i$; by extension, the complement of an edge is the edge whose ends are the complements of the ends of this edge. In reference to language theory, we may denote x^k a letter x repeated k times within a word. We denote $\mathcal{H}(v)$ the Hamming weight of v, *i.e.* the number of "1"s in the word representation of v. If v is a vertex to which is associated a word on a binary alphabet, we define as $[v]$ the value of this word taken as a binary number representation.

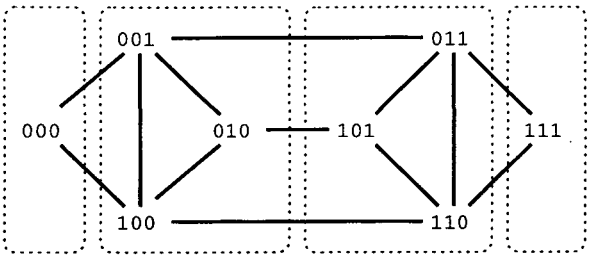


Figure 2. – UB(2, 3) and its partition by Hamming weights.

PROPOSITION 8:

$$\mathrm{Bd}\,(\mathrm{UB}\,(2,\,k)) \leq \left(\begin{matrix} k \\ \left\lceil \frac{k}{2} \right\rceil \end{matrix} \right).$$

Proof: In order to obtain a quotient graph which is a path, we quotient $\mathrm{UB}\,(2,\,k)$ using the Hamming weight of its vertices, since the difference of the Hamming weight of any two vertices linked by an edge is at most 1 by definition. Therefore, we define the following configuration:

– $\Pi = \{\pi_0, \pi_1, ..., \pi_k\}$, where $\pi_q = \{v \in V\,(\mathrm{UB}\,(2,\,k))/\mathcal{H}\,(v) = q\}$;

– $\varphi_Q\,(\pi_q) = q$;

– φ_G, defined as:

• $\varphi_G\,(0^k) = 0$,

• $\forall\, v',\, v'' \in V\,(\mathrm{UB}\,(2,\,k))$,

$$(\varphi_G\,(v') > \varphi_G\,(v'')) \quad \Leftrightarrow \quad ((\varphi_Q\,(\pi\,(v')) > \varphi_Q\,(\pi\,(v''))) \text{ or } ((\pi\,(v') = \pi\,(v'')) \text{ and } ([v'] < [v''])))).$$

φ_G, which is compatible with φ_Q, amounts, in each block, to numbering the vertices v in decreasing order with respect to $[v]$; it is therefore equivalent to the ordering defined by Harper for the hypercube [15].

This ordering has several interesting properties:

– $\mathrm{UB}\,(2,\,\mathrm{k})_{/\Pi}$ is isomorphic to the path graph $P\,(k+1)$.

– $|\pi_q| = \binom{k}{q} \leq \left(\begin{matrix} k \\ \left\lceil \frac{k}{2} \right\rceil \end{matrix} \right)$.

– $(v \in \pi_q) \Leftrightarrow (\overline{v} \in \pi_{k-q})$.

$- \varphi_G(\overline{v}) = 2^k - 1 - \varphi_G(v)$.

Since $\mathrm{UB}(2, k)_{/\Pi}$ is isomorphic to the path graph $\mathrm{P}(k+1)$, then $\mathrm{Bd}(\mathrm{UB}(2, k)_{/\Pi}) = 1$ and $\mathrm{Bd}(\mathrm{UB}(2, k)) = \max(\varepsilon_i(\varphi_G), \varepsilon_e(\varphi_G))$. Moreover, for each edge $\{v', v''\}$ of $E_E(\Pi)$ such that $\varphi_G(v') < \varphi_G(v'')$, we have $P_{\varphi_Q(\pi(v'))} = p_{\varphi_Q(\pi(v''))} - 1$, and thus $\varepsilon_e(\varphi_G) = \max_{\{v', v''\} \in E_E(\Pi)} (|\varphi_G(v'') - \varphi_G(v')|)$.

Let us consider all the edges $\{v', v''\}$ in $E(\mathrm{UB}(2, k))$, assuming without loss of generality that $q' = \varphi_Q(\pi(v')) \leq \varphi_Q(\pi(v'')) = q''$.

- If $q'' = q'$, then $|\varphi_G(v') - \varphi_G(v'')| \leq |\pi_{q'}| - 1 \leq \binom{k}{q'} - 1$, and then $\varepsilon_i(\varphi_G) \leq \binom{k}{\lceil \frac{k}{2} \rceil} - 1$.

- If $q'' = q' + 1$, then v'' is obtained from v by deletion of a letter "0" and insertion of a letter "1".

• If $v' = 0\,m$ and $v'' = m\,1$, let us consider vertex $v'_\perp = 0^{k-q}\,1^q = 0\,m_\perp$ in $\pi_{q'}$ and its neighbor $v''_\perp = m_\perp\,1$ in $\pi_{q''}$. By definition, for all m, $[m_\perp] \leq [m]$, and $([0\,m] - [0\,m_\perp]) \leq ([m\,1] - [m_\perp\,1])$. Since $(\varphi_G(v'_\perp) - \varphi_G(v')) = ([0\,m] - [0\,m_\perp])$ and $(\varphi_G(v''_\perp) - \varphi_G(v'')) = ([m\,1] - [m_\perp\,1])$, we have $(\varphi_G(v'') - \varphi_G(v')) \leq (\varphi_G(v''_\perp) - \varphi_G(v'_\perp)) = |\pi_{q''}| \leq \binom{k}{\lceil \frac{k}{2} \rceil}$.

• If $v' = m\,0$ and $v'' = 1\,m$, then $\overline{v}' = \overline{m}\,1$ and $\overline{v}'' = 0\,\overline{m}$, which amounts to considering the first case since $|\varphi_G(v'') - \varphi_G(v')| = |\varphi_G(\overline{v}'') - \varphi_G(\overline{v}')|$.

By the above, $\varepsilon_e(\varphi_G) = \binom{k}{\lceil \frac{k}{2} \rceil}$.

By Theorem 1, taking the maximum of $\varepsilon_i(\varphi_G)$ and $\varepsilon_e(\varphi_G)$ yields the claimed result. □

Remark:

$$\frac{2^k - 1}{k} \leq \mathrm{Bd}(\mathrm{UB}(2, k)).$$

Proof: This derives straightforwardly from Theorem 4, with $|V(\mathrm{UB}(2, k))| = 2^k$ and $\mathrm{diam}(\mathrm{UB}(2, k)) = k$.

PROPOSITION 9:

$$\frac{1}{2} \cdot \frac{2^k}{k} \leq \mathrm{Cw}\,(\mathrm{UB}\,(2,\,k)) \leq 2 \left(\begin{matrix} k \\ \left\lceil \frac{k}{2} \right\rceil \end{matrix} \right) + 2.$$

Proof: To prove the upper bound of this proposition, we use a configuration based on the same Π and φ_Q as above. Since $\mathrm{UB}\,(2,\,k)_{/\Pi}$ is isomorphic to $\mathrm{P}\,(k+1)$, $\mathrm{Cw}\,(\mathrm{UB}\,(2,\,k)_{/\Pi}) = 1$ and $\delta\,(\mathrm{UB}\,(2,\,k)_{/\Pi}) = 1$. Moreover, $|\omega\,(\pi_q)| \leq 2\,|\pi_q|$, as from each vertex in any π exit at most 2 edges to other π's.

The edges $\{v',\,v''\}$ internal to any block π link vertices of same weight, so if $v' = v_1\,v_2 \ldots v_k$, then $v'' = v_k\,v_1 \ldots v_{k-1}$ or $v'' = v_2 \ldots v_k\,v_1$. Therefore, the connected components of $\mathrm{UB}\,(2,\,k)\,[\pi]$ are single vertices, single edges, or cycles (named Etzion cycles) and, for all π, $\mathrm{Cw}\,(\mathrm{UB}\,(2,\,k)\,[\pi]) = 2$.

For $k \geq 3$ and for any block π_q, it is possible to find a vertex not linked to either a vertex of π_{q-1} or π_{q+1} (*i.e.* a vertex whose first and last letters are identical) so, for all π, $\min\,(\delta^-_{\Pi,\varphi_Q}\,(\pi),\,\delta^+_{\Pi,\varphi_Q}\,(\pi)) = 0$.

According to the above, Theorem 5 yields

$$\begin{aligned}
\mathrm{Cw}\,(\mathrm{UB}\,(2,\,k)) &\leq \left(\mathrm{Cw}\,(\mathrm{UB}\,(2,\,k)_{/\Pi}) - \left\lceil \frac{\delta\,(\mathrm{UB}\,(2,\,k)_{/\Pi})}{2} \right\rceil \right) \cdot \max_\omega \\
&\quad + \max_{\pi \in \Pi} (\mathrm{Cw}\,(\mathrm{UB}\,(2,\,k)\,[\pi]) + |\omega\,(\pi)| \\
&\qquad - |\pi| \,.\, \min\,(\delta^-_{\Pi,\varphi_Q}\,(\pi),\,\delta^+_{\Pi,\varphi_Q}\,(\pi))) \\
&\leq \max_{\pi \in \Pi} (\mathrm{Cw}\,(\mathrm{UB}\,(2,\,k)\,[\pi]) + |\omega\,(\pi)|) \\
&\leq 2 + 2 \left(\begin{matrix} k \\ \left\lceil \frac{k}{2} \right\rceil \end{matrix} \right).
\end{aligned}$$

It is easy to see that $\mathrm{Cw}\,(\mathrm{UB}\,(2,\,1)) = 1$ and $\mathrm{Cw}\,(\mathrm{UB}\,(2,\,2)) = 3$, which extends the upper bound to all $k \geq 1$.

For the lower bound, it is known from [19, p. 480] that $\mathrm{bis}_e\,(\mathrm{SE}\,(k)) \geq \dfrac{2^{k-1}}{k}$. Since $\mathrm{SE}\,(k)$ is a spanning subgraph of $\mathrm{UB}\,(2,\,k)$,

$$\mathrm{Cw}\,(\mathrm{UB}\,(2,\,k)) \geq \mathrm{bis}_e\,(\mathrm{UB}\,(2,\,k)) \geq \mathrm{bis}_e\,(\mathrm{SE}\,(k)) \geq \frac{1}{2} \cdot \frac{2^k}{k}. \quad \square$$

Remark: It is known that, for all $k \geq 1$, $\left(\left\lceil \frac{k}{\frac{k}{2}} \right\rceil \right) \leq 2^k \sqrt{\frac{2}{k\,\pi}}$. Hence, our upper bounds of the bandwidth and the cutwidth of the binary de Bruijn graphs are both in $O\left(\frac{2^k}{\sqrt{k}}\right)$.

4.2. Butterfly graph

The vertices of $\mathrm{BF}(k)$ are pairs of integer numbers $(l; m)$, where $0 \leq l < k$ and $0 \leq m < 2^k$, and each vertex $(l; m)$ of $\mathrm{BF}(k)$ is linked to vertices $((l-1) \bmod k;\ m)$, $((l-1) \bmod k;\ m \oplus 2^k)$, $((l+1) \bmod k;\ m)$, and $((l+1) \bmod k;\ m \oplus 2^{(l+1) \bmod k})$. Vertices $(l; m)$ of same l value are said to belong to the same level l.

PROPOSITION 10:

$$\mathrm{Bd}(\mathrm{BF}(k)) \leq 9 \cdot 2^{k-2}.$$

Proof: The first part of the proof is devoted to the definition of a suitable configuration. The φ_G numbering that we obtain is then studied to achieve a better upper bound.

A natural way to partition Butterfly graphs is to quotient them by levels, as illustrated in Figure 3. We will thus define Π, φ_Q, and φ_G as follows:

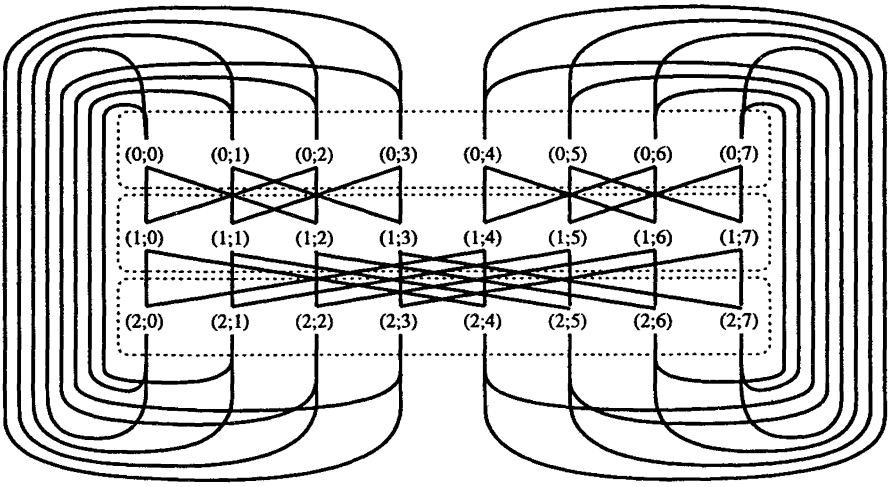


Figure 3. – BF (3) and its partition by levels.

- $\Pi = \{\pi_0, \pi_1, ..., \pi_{k-1}\}$, where $\pi_q = \{(l; m) \in V(\mathrm{BF}(k))/l = q\}$
- $\varphi_Q(\pi_q) = \begin{cases} 2(q+1) & \text{if } 0 \leq q < \left\lfloor \frac{k-1}{2} \right\rfloor \\ 2(k-(q+1))-1 & \text{if } \left\lfloor \frac{k-1}{2} \right\rfloor \leq q < k-1 \\ 0 & \text{if } q = k-1 \end{cases}$
- $\varphi_G(l; m) = \varphi_Q(\pi_l) \cdot 2^k + m$.

By definition of Π, it is clear that, for all q, $|\pi_q| = 2^k$, and that φ_G is compatible with φ_Q. Since no edge links vertices belonging to the same block, $\varepsilon_i(\varphi_G) = 0$. By definition, $\mathrm{BF}(k)_{/\Pi}$ is isomorphic to the cycle graph $C(k)$, so $\mathrm{Bd}(\mathrm{BF}(k)_{/\Pi}) = 2$. Moreover, the above definition of φ_Q is an ordering which achieves this bandwidth on the cycle.

For all $v = (l; m) \in V(\mathrm{BF}(k))$, v is linked to $v' = ((l+1) \bmod k; m)$ and to $v'' = ((l+1) \bmod k; m \oplus 2^{(l+1) \bmod k})$. Thus,

$$\varepsilon_e(\varphi_G) = 2^k + 2^{k-1}.$$

However, with the chosen φ_Q, all the edges which maximize $\varepsilon_e(\varphi_G)$ link block π_{k-2} to block π_{k-1}, and $(\varphi_Q(\pi_{k-2}) - \varphi_Q(\pi_{k-1}))$ is by construction always equal to 1 (and not to 2, which is the bandwidth of Q). Therefore, the maximum dilation of the external edges is in fact less than what would be obtained by using Theorem 1. As a matter of fact,

$$\begin{aligned} \mathrm{Bd}(\mathrm{BF}(k)) &\leq \max(2^k \cdot |\varphi_Q(\pi_{k-1}) - \varphi_Q(\pi_{k-2})| + 2^{k-1}, \\ &\qquad 2^k \cdot |\varphi_Q(\pi_{k-2}) - \varphi_Q(\pi_{k-3})| + 2^{k-2}) \\ &\leq \max(1 \cdot 2^k + 2^{k-1}, 2 \cdot 2^k + 2^{k-2}) \\ &\leq 9 \cdot 2^{k-2}. \quad \square \end{aligned}$$

Remark:

$$\left\lfloor \frac{8}{3} \cdot 2^{k-2} \right\rfloor \leq \mathrm{Bd}(\mathrm{BF}(k))$$

Proof: As we know by [19] that $\mathrm{diam}(\mathrm{BF}(k)) = \left\lfloor \frac{3}{2} k \right\rfloor$, we have by Theorem 4

$$\begin{aligned} \mathrm{Bd}(\mathrm{BF}(k)) &\geq \left\lceil \frac{k \cdot 2^k - 1}{\left\lfloor \frac{3}{2} k \right\rfloor} \right\rceil \geq \left\lceil \frac{k \cdot 2^k - 1}{\frac{3}{2} k} \right\rceil \\ &= \left\lceil \frac{8}{3} \cdot 2^{k-2} - \frac{2}{3k} \right\rceil \geq \left\lfloor \frac{8}{3} \cdot 2^{k-2} \right\rfloor, \end{aligned}$$

as the contents of the ceiling operator are never integer. □

PROPOSITION 11:

$$\left\lfloor \frac{1}{5} \cdot 2^{k+1} \right\rfloor \leq \mathrm{Cw}\left(\mathrm{BF}\left(k\right)\right) \leq 3 \cdot 2^{k}.$$

Proof: Let us begin with the upper bound. An interesting characteristic of Butterfly graphs is that their construction is recursive. As a matter of fact, let us define the $\overline{\mathrm{BF}}$ graphs as BF graphs without their wrap-around edges (*i.e.* edges linking level $k-1$ to level 0). It is easy to see that $\mathrm{BF}\left(k\right)$ can be obtained by putting two copies of $\overline{\mathrm{BF}}\left(k-1\right)$ atop a level of 2^k vertices and linking them so that the added level becomes level k, as illustrated in figure 4. This decomposition is straightforward for $\overline{\mathrm{BF}}\left(k\right)$, as illustrated in figure 5.

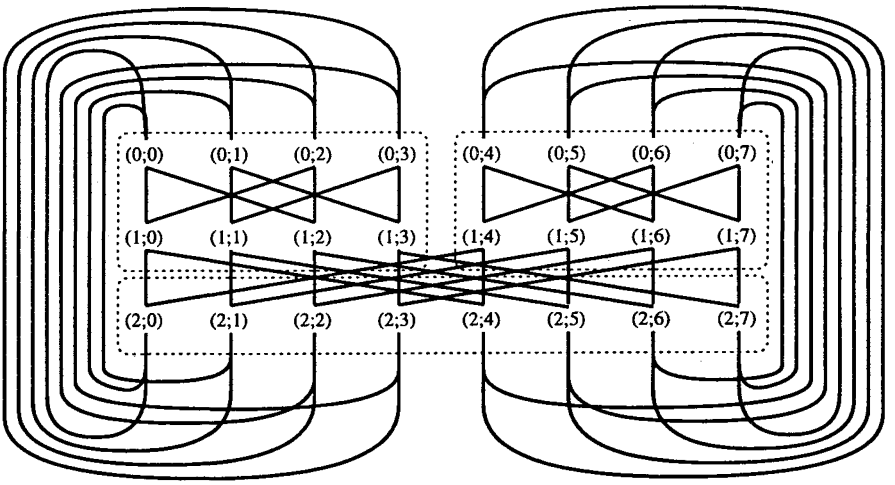
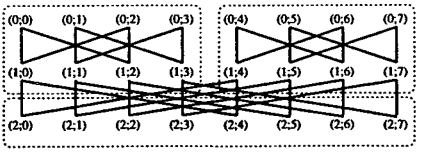
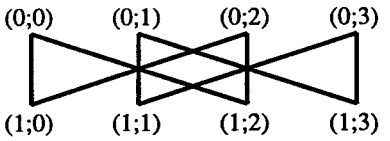


Figure 4. – BF (3) and its partition: two $\overline{\mathrm{BF}}$ (2) and a level of vertices.



a. $\overline{\mathrm{BF}}(3)$ and its partition: two $\overline{\mathrm{BF}}(2)$ and a level of vertices.



b. $\overline{\mathrm{BF}}(2)$ is isomorphic to two $C(4)$

Figure 5. – Recursive structure of $\overline{\mathrm{BF}}$ graphs.

To prove the upper bound, we use this two-step recursive construction of $\mathrm{BF}(k)$. The first step, involving the decomposition of $\mathrm{BF}(k)$, leads to the following Π and φ_Q:

- $\Pi = \{\pi_0, \pi_1, \pi_2\}$, where

$$\begin{cases} \pi_0 = \{(l; m) \in V(\mathrm{BF}(k))/l < k-1,\ m < 2^{k-1}\} \\ \pi_1 = \{(l; m) \in V(\mathrm{BF}(k))/l = k-1\} \\ \pi_2 = \{(l; m) \in V(\mathrm{BF}(k))/l < k-1,\ m \geq 2^{k-1}\} \end{cases}$$

- $\varphi_Q(\pi_q) = q$.

By definition, π_0 and π_2 are isomorphic to $\overline{\mathrm{BF}}(k-1)$ graphs, and the quotient graph $\mathrm{BF}(k)_{/\Pi}$ is a path, so $\mathrm{Cw}(\mathrm{BF}(k)_{/\Pi}) = 1$, $\delta(\mathrm{BF}(k)_{/\Pi}) = 1$, and $\mathrm{BF}(k)[\pi_1]$ is a set of independant vertices, so $\mathrm{Cw}(\mathrm{BF}(k)[\pi_1]) = 0$. This structure yields the following values:

$$\begin{aligned} &|\pi_0| = (k-1)\,2^{k-1}, \quad |\omega(\pi_0)| = 4 \cdot 2^{k-1}, \mathrm{Cw}(\mathrm{BF}(k)[\pi_0]) = \mathrm{Cw}(\overline{\mathrm{BF}}(k-1)), \\ &|\pi_1| = 2^k, \qquad\quad |\omega(\pi_1)| = 4 \cdot 2^k, \quad \mathrm{Cw}(\mathrm{BF}(k)[\pi_1]) = 0, \\ &|\pi_2| = (k-1)\,2^{k-1}, \quad |\omega(\pi_2)| = 4 \cdot 2^{k-1}, \mathrm{Cw}(\mathrm{BF}(k)[\pi_2]) = \mathrm{Cw}(\overline{\mathrm{BF}}(k-1)), \\ &\qquad \delta^-_{\Pi,\varphi_Q}(\pi_0) = 0, \qquad \delta^+_{\Pi,\varphi_Q}(\pi_0) = 0, \\ &\qquad \delta^-_{\Pi,\varphi_Q}(\pi_1) = 1, \qquad \delta^+_{\Pi,\varphi_Q}(\pi_1) = 1, \\ &\qquad \delta^-_{\Pi,\varphi_Q}(\pi_2) = 0, \qquad \delta^+_{\Pi,\varphi_Q}(\pi_2) = 0. \end{aligned}$$

Using Theorem 5, we obtain

$$\begin{aligned} \mathrm{Cw}(\mathrm{BF}(k)) &\leq \max_{\pi \in \{\pi_0, \pi_1, \pi_2\}} (\mathrm{Cw}(\mathrm{BF}(k)[\pi]) \\ &\quad + |\omega(\pi)| - |\pi| \cdot \min(\delta^-_{\Pi,\varphi_Q}(\pi), \delta^+_{\Pi,\varphi_Q}(\pi))) \\ &\leq \max(\mathrm{Cw}(\overline{\mathrm{BF}}(k-1)) + 2^{k+1}, 3 \cdot 2^k). \end{aligned} \tag{1}$$

In the second step, an analogous study on $\overline{\mathrm{BF}}(k)$ based on the same partition leads to:

$$\begin{aligned} \mathrm{Cw}(\overline{\mathrm{BF}}(k)) &\leq \max(\mathrm{Cw}(\overline{\mathrm{BF}}(k-1)) + 2^k, 2^k) \\ &\leq \mathrm{Cw}(\overline{\mathrm{BF}}(k-1)) + 2^k. \end{aligned} \tag{2}$$

Since $\overline{\mathrm{BF}}(2)$ is isomorphic to two $C(4)$, $\mathrm{Cw}(\overline{\mathrm{BF}}(2)) = 2$. By injecting this value into (2), and studying the induced recurrence, we obtain

$$\mathrm{Cw}(\overline{\mathrm{BF}}(k)) \leq 2^{k+1} - 6$$

which, injected itself into (1), yields the claimed result.

Let us now prove the lower bound. The Cube-Connected Cycles graph of dimension k, denoted $\mathrm{CCC}(k)$, is the graph obtained by replacing the vertices of a k-dimensional hypercube by k-node cycles [26]. In [27] has been given a lower bound of the edge-bisection of a graph with respect to the congestion of the embedding of the graph into the complete graph with double edges. In particular, for the CCC graph, this approach yields

$$\mathrm{bis}_e(\mathrm{CCC}(k)) \geq \left\lfloor \frac{1}{5} \cdot 2^{k+1} \right\rfloor.$$

Since it has been proven in [8] that $\mathrm{CCC}(k)$ is a spanning subgraph of $\mathrm{BF}(k)$, $\mathrm{Cw}(\mathrm{CCC}(k)) \leq \mathrm{Cw}(\mathrm{BF}(k))$, and by Proposition 7

$$\mathrm{Cw}(\mathrm{BF}(k)) \geq \left\lfloor \frac{1}{5} \cdot 2^{k+1} \right\rfloor. \quad \square$$

4.3. Summary

Using the same techniques as above, our method has been successfully applied to other classes of graphs [2]. All these results are summarized in the following table.

Remark: The upper bound that we have found for $\mathrm{Cw}(\mathrm{SE}(k))$ is in $O\left(\frac{2^k}{\sqrt{k}}\right)$. However, by combining the constant-congestion embedding of $\mathrm{SE}(k)$ into $\mathrm{BF}(k+2-\log(k))$ described in [16, p. 237] with the embedding that we use to compute the upper bound of $\mathrm{Cw}(\mathrm{BF})$, one can prove that $\mathrm{Cw}(\mathrm{SE}(k))$ is in $O\left(\frac{2^k}{k}\right)$. Since the lower bound that we give is also of this order of magnitude, then $\mathrm{Cw}(\mathrm{SE}(k))$ is in $\Theta\left(\frac{2^k}{k}\right)$.

5. CONCLUSION

We have presented in this paper a method for determining upper bounds of the bandwidth and the cutwidth of a graph from the bandwidth and the

Graph	Bandwidth		Cutwidth	
	lower	upper	lower	upper
H(k)	$\Theta\left(\left(\lceil\frac{k}{\frac{k}{2}}\rceil\right)\right)$ [17]		$\lfloor\frac{1}{3}2^{k+1}\rfloor$ [3, 4, 21]	
UB(2, k)	$\frac{2^k-1}{k}$	$\left(\lceil\frac{k}{\frac{k}{2}}\rceil\right)$	$\frac{1}{2}\cdot\frac{2^k}{k}$	$2\left(\lceil\frac{k}{\frac{k}{2}}\rceil\right)+2$
SE(k)	$\frac{2^k-1}{2k-1}$	$2\left(\lceil\frac{k-2}{\frac{k-2}{2}}\rceil\right)$	$\frac{1}{2}\cdot\frac{2^k}{k}$ †	$\left(\lceil\frac{k}{\frac{k}{2}}\rceil\right)+2$ †
FFT(k)	2^{k-1}	$3\cdot 2^{k-1}$	$\lfloor\frac{1}{5}\cdot 2^{k+1}\rfloor$	$2^{k+1}-2$
BF(k)	$\lfloor\frac{8}{3}\cdot 2^{k-2}\rfloor$	$9\cdot 2^{k-2}$	$\lfloor\frac{1}{5}\cdot 2^{k+1}\rfloor$	$3\cdot 2^k$
CCC(k)	$\lfloor\frac{8}{5}\cdot 2^{k-2}\rfloor$	$9\cdot 2^{k-2}$	$\lfloor\frac{1}{5}\cdot 2^{k+1}\rfloor$	$2^{k+1}+1$

† Cw(SE(k)) is in fact in $\Theta\left(\frac{2^k}{k}\right)$ (see remark above).

cutwidth of the quotient graph and induced subgraphs issued from any of its partitions; lower bounds of these parameters have also been given, based on vertex- and edge-bisection notions.

The quotienting method is well suited for graphs which can be quotiented, possibly recursively, into graphs the bandwidth or cutwidth of which are known or for which accurate upper bounds are available. However, building suitable quotient graphs is not easy, since it inherits the complexity of the problem. Quotientings must be simple so that their parameters can be easily computed, and must be judiciously chosen in order to capture in the quotient graph the topological properties of the studied graphs.

When the graph structure is well-known, it is often worth fine-tuning the ordering of the vertices induced by the quotienting, in order to directly compute more accurate upper bounds. In that way, our results can be seen as a methodological framework for finding good orderings rather than directly applicable theorems.

Applications of the method to the binary de Bruijn and Butterfly graphs have been exposed in the paper. Studies of other classes of interconnection networks (Shuffle-Exchange, FFT, CCC), and generalizations of these results to d-ary graphs have also been carried out in related articles [1, 2, 23, 24].

ACKNOWLEDGEMENTS

Thanks the referees for their very useful comments and suggestions, in particular for the proof of optimality of the cutwidth of the Shuffle-Exchange graph.

REFERENCES

1. D. BARTH, Réseaux d'interconnexion : Structures et Communications, Thèse de Doctorat, LaBRI, Université Bordeaux-I, 351 cours de la Libération, 33405 Talence, France, janvier 1994.
2. D. BARTH, F. PELLEGRINI, A. RASPAUD and J. ROMAN, *On Bandwidth, Cutwidth, and Quotient graphs,* Research report 814-94, LaBRI, Université Bordeaux-I, 351 cours de la Libération, 33405 Talence, France, mars 1994.
3. A. BEL HALA, Congestion optimale du plongement de l'hypercube $H(n)$ dans la chaîne $P(2^n)$, *Rairo Informatique Théorique et Applications*, 1993, *27* (4), pp. 1-17.
4. A. BEL HALA, Communications dans les réseaux d'interconnexion : plongements optimaux de l'hypercube. Thèse de Doctorat, LaBRI, Université Bordeaux-I, 351 cours de la Libération, 33405 Talence, France, janvier 1995.
5. C. BERGE, *Graphs and Hypergraphs,* North Holland Publishing, 1973.
6. P. Z. CHINN, J. CHVÀTALOVÀ, A. K. DEWDNEY and N. E. GIBBS, The Bandwidth Problem for Graphs and Matrices, A survey, *Journal of Graph Theory,* 1982, *6*, pp. 223-254.
7. F. R. K. CHUNG, *The Theory and Applications of Graphs,* chapter Some Problems and Results in Labelling of Graphs, pp. 255-263, John Wiley, 1981.
8. R. FELDMANN and W. UNGER, The Cube-Connected Cycles network is a subgraph of the Butterfly network, *Parallel Processing Letters*, 1992, *2* (1), pp. 13-19.
9. A. FELLER, Automatic layout of low-cost quick-turnaround random-logic customs LSI devices, In *Proc. 13th Design Automation Conf.,* pp. 79-85, San Francisco, 1976.
10. M. R. GAREY and D. S. JOHNSON, *Computers and Intractability: A Guide to the Theory of NP-completeness,* W. H. Freeman, San Francisco, 1979.
11. F. GAVRIL, Some NP-complete problems on graphs, In *Proc. 11th conference on information sciences and systems,* pp. 91-95, John Hopkins University, Baltimore.
12. N. E. GIBBS, W. G. POOLE and P. K. STOCKMEYER, A comparison of several bandwidth and profile reduction algorithms, *ACM Trans. Math. Software*, 1976, *2*, pp. 322-330.
13. E. M. GURARI and I. H. SUDBOROUGH, Improved dynamic algorithms for bandwidth minimization and the mincut linear arrangement problem, *Journal of Algorithms,* 1984, *5*, pp. 531-546.
14. L. H. HARPER, Optimal assignement of number of vertices, *J. SIAM,* 1964, *12*, pp. 131-135.
15. L. H. HARPER, Optimal numberings and isoperimetric problems on graphs, *Journal of Combinatorial Theory,* 1966, *1*, pp. 385-393.
16. R. KOCK, T. LEIGHTON, B. MAGGS, S. RAO and A. ROSENBERG, Work-preserving emulations of fixed-connection networks, In *Proc. 21st Annual Symposium on Theory of Computing,* pp. 227-240, ACM, 1989.
17. T. H. LAI and A. S. SPRAGUE, *Placement of the processors of a hypercube*, Technical report, 1988.
18. F. T. LEIGHTON, Complexity issues in VLSI. Optimal layouts for the shuffle exchange graph and other networks, in *Foundations of Computing Series*, The MIT press, 1983.

19. F. T. LEIGHTON, *Introduction to Parallel Algorithms and Architectures: arrays, trees, hypercubes,* Morgan Kaufman Publisher, 1992.
20. B. MONIEN and H. SUDBOROUGH, Embedding one interconnection network in another, *Computing Supplements*, 1990, *7*, pp. 257-282.
21. K. NAKANO, W. CHEN, T. MASUZAWA, K. HAGIHARA and N. TOKURA, Cut width and bisection width of hypercube graph, *IEICE Transactions, J73-A*, (4), pp. 856-862, april 1990, In Japanese.
22. C. H. PAPADIMITRIOU, The NP-completeness of the bandwidth minimization problem, *Computing*, 1976, *16*, pp. 263-270.
23. F. PELLEGRINI, Bounds for the bandwidth of the d-ary de Bruijn graph, *Parallel Processing Letters. Special Number on Interconnection Networks,* 1993, *3* (4), pp. 431-443.
24. F. PELLEGRINI, Application de méthodes de partition à la résolution de problèmes de graphes issus du parallélisme, Thèse de Doctorat, LaBRI, Université Bordeaux-I, 351 cours de la Libération, 33405 Talence, France, janvier 1995.
25. G. PERSKY, D. DEUTSCH and D. SCHWEIKERT, LTX-A minicomputer-based system for automated LSI layout, *J. Design Automation and Fault Tolerant Computing,* 1977, *1*, pp. 217-255.
26. F. PREPARATA and J. VUILLEMIN, The Cube-Connected Cycles: a versatile network for parallel computation, *Communications of the ACM,* May 1981, *24* (5), pp. 300-309.
27. F. SHAHROKHI and L. A. SZEKELY, *An algebraic approach to the uniform concurrent multicommodity flow. Theory and applications,* Technical Report CRPDC-91-4, University of North Texas, 1991.
28. W. F. SMITH and I. ARANY, Another algorithm for reducing bandwidth and profile of a sparse matrix, In *Proc. AFIPS 1976 NCC*, pp. 341-352, Montvale, New Jersey, 1976, AFIP Press.
29. S. TRIMBERG, Automating chip layout, *IEEE Spectrum,* pp. 38-45, 1982.
30. H. YOSHIZAWA, H. KAWANISHI and K. KANI, A heuristic procedure for ordering MOS arrays, In *Proc. of Design Automation Conference,* pp. 384-393, 1975.