INFORMATIQUE THÉORIQUE ET APPLICATIONS

HOSAM M. MAHMOUD
REZA MODARRES
ROBERT T. SMYTHE

Analysis of quickselect : an algorithm for order statistics

Informatique théorique et applications, tome 29, nº 4 (1995), p. 255-276
<http://www.numdam.org/item?id=ITA_1995__29_4_255_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol. 29, n° 4, 1995, p. 255 à 276)

ANALYSIS OF QUICKSELECT: AN ALGORITHM FOR ORDER STATISTICS (*)

by Hosam M. MAHMOUD [1] †, Reza MODARRES [1]
and Robert T. SMYTHE [1] †

Communicated by P. FLAJOLET

Abstract. – *We study QUICKSELECT, a one-sided version of QUICKSORT suited for finding the order statistics of a sample. We identify procedures by which the moments of the number of comparisons can be found exactly under both assumptions that the order statistic in question is randomly chosen or fixed. The procedure is illustrated by finding the exact mean and variance for a randomly selected order statistic as well as the first few in the fixed case. The existence of an absolutely continuous infinitely divisible limit law with asymmetric left and right tails is demonstrated in the case of a randomly chosen order statistic. Some of these distributional properties carry over to the case of a very small fixed order statistic.*

Keywords: *Sorting, limit law, order statistics.*

Résumé. – *Nous étudions QUICKSELECT, une version latéralisée de QUICKSORT adaptée à la recherche de statistique de rang sur un échantillon. Nous identifions les procédures qui permettent de trouver exactement les moments du nombre de comparaisons, lorsque la statistique de rang en question est aléatoire ou qu'elle est fixe. La procédure est illustrée en trouvant la moyenne et la variance exactes pour une statistique aléatoire ainsi que les premiers moments dans le cas d'une statistique fixe. L'existence d'une loi limite absolument continue et infiniment divisible avec des queues gauche et droite asymétriques est démontrée dans le cas d'une statistique de rang aléatoire. Certaines de ces propriétés de distribution s'étendent au cas d'une statistique de rang fixe très petite.*

1. INTRODUCTION

QUICKSORT is the fastest known *in situ* sorting algorithm. The algorithm was invented by Hoare [10] in 1962. Since then the method has enjoyed high popularity and now appears in most standard textbooks on algorithms (*see* [1, 11, 21], for example), and several implementations are in operation as the sorting method of choice as in the UNIX operating system.

(*) Received January 1993; accepted September 1994.
† This author's research is supported in part by a grant from NSA: Contract no. MDA904-92-H3086.
[1] Department of Statistics, The George Washington University, Washington, D.C. 20052, U.S.A.
AMS Codes: Primary. *68 P 10, 68 Q 25*; secondary. 60 F 05.

Performing on n distinct keys forming a random permutation on $\{1, 2, \ldots, n\}$, QUICKSORT is known to possess a benign $O(n \log n)$ average time behavior with only $O(\log n)$ average extra space for a supporting stack. The algorithm and several of its variants have been thoroughly analyzed in [6, 8, 11, 16-20] under the random permutation model, where all permutations of $\{1, \ldots, n\}$ are considered equally likely input lists. The random permutation model represents a wide variety of real life situations as it is equivalent to sampling n keys from *any* continuous distribution ([14]; Section 2.3).

QUICKSORT is a divide-and-conquer algorithm that works as follows. A list of n distinct keys is given. We select an element, called the *pivot*, and locate its position in the final sorted list by comparing it to all the other elements in the list. In the process, the remaining $n - 1$ elements are classified into two groups: Those that are less than the pivot are moved to the left of the pivot's final position, and those that are greater than the pivot are moved to the right of the pivot's final position. The pivot itself is then moved between the two groups to its correct and final position. This stage of the algorithm is called the partitioning stage. QUICKSORT is then applied recursively to the left and right sublists until small lists of size 1 or less are reached; these are left intact as they are already sorted.

Simple modifications can be introduced to handle lists with key repetitions, a case that occurs with probability zero in a sample from a continuous distribution (*i.e.* when the random permutation model applies). We shall therefore assume for the rest of this paper that all n keys in the list are distinct and their actual values are assimilated by a random permutation of $\{1, \ldots, n\}$.

Obviously, the partitioning stage takes at least $n - 1$ comparisons; some implementations that actually take $n - 1$ comparisons exist (*see* [12]; p. 259). We shall assume a partitioning procedure implementation PARTITION (ℓ, u, k) that takes in ℓ and u, the lower and upper limits of the sublist being sorted, and returns k, the final position of the pivot.

Some authors prefer a fixed pivot, e. g. the first or the last in the list. Some implement QUICKSORT with a random choice of the pivot (all n keys in the list are equally likely). For a systematic development of recurrence relations, we shall assume that the pivot is always the first in the list. Also, the assumption that the partitioning stage preserves the randomness in the sublists is common in the analysis of QUICKSORT (*see* [8]) and we shall assume that the chosen implementation of PARTITION together with our choice of the pivot are consistent with this hypothesis.

A modified one-sided version of QUICKSORT, to be called QUICKSELECT (QS for short), may be used for finding the order statistics of a given list. This modification was introduced in Hoare [9] and several of its variants also appear in some books (e. g. [1, 7]). Obviously, if we are only interested in finding the mth order statistic in a list, we need not sort the two sublists as in QUICKSORT; we need only identify the sublist containing the mth order statistic and proceed recursively with that sublist. More precisely, QS operates as follows. It is a programming function that takes in the parameters ℓ and u identifying respectively the lower and upper limits of the sublist being considered and returns the actual value of the mth order statistic; the initial external call is, of course, $\mathrm{QS}\,(1,\ n)$. Within QS, m and the list itself are accessed globally. At the stage when the search has been narrowed down to the sublist extending between positions ℓ and u, QS first goes through ~~the partitioning process, PARTITION (ℓ, u, k), exactly as in QUICKSORT,~~ moving the chosen pivot to its final position k. If $k = m$, we are done; the element at position k is our mth order statistic. If $k > m$, the mth order statistic must be in the left sublist; we apply QS recursively on the left sublist, i.e. the situation is handled by the call $\mathrm{QS}\,(\ell,\ k-1)$; otherwise, the mth order statistic must be in the right sublist and it is now the $(m-k)$th smallest among the keys of the right sublist. This situation is handled by the call $\mathrm{QS}\,(k+1,\ u)$.

One would not normally use QS for a particular order statistic as other algorithms are known to be more efficient for this task. For example, if we are interested in the first order statistic, QS consumes an average number of comparisons asymptotically equivalent to $2n$, whereas a simple linear scan of the list accomplishes the task in only $n-1$ comparisons. However, these more efficient algorithms are very specific to a particular order statistic and cannot be easily modified to handle other orders. Thus, QS is particularly useful when at different times we desire to compute different order statistics as is common in nonparametric statistics such as finding one-sided (semi-infinite) confidence intervals for distribution quantiles or such as finding distribution-free one-sided confidence intervals for the shift parameter in a shift model ([13], Chapter 2). Our analysis pertains to the situation where a single order statistic is to be computed when QS is applied only once to a random list. That is, by using the algorithm at diffferent times we mean starting with a fresh random list at each time.

The speed of QUICKSORT can be made insensitive to the distribution of the data, for data with no duplicates (as in sampling without replacement from a discrete distribution, for example), if the data are first subjected to an initial randomizing shuffle so that their ranks become a random permutation. An initial stage to randomize the data may serve as a way of guaranteeing the same uniform average speed for all possible distributions of lists with distinct items. Our analysis thus also applies to QUICKSELECT that performs this randomization prior to sorting, for data with no duplicates even if the ranks do not follow the random permutation probability model.

Let $C_n^{(m)}$ be the number of comparisons between list elements in QS when applied to a random list of size n to find the list's mth order statistic, $1 \leq m \leq n$. We shall consider the situation when the order statistic is a random variable that is *discrete uniform* $[1 \cdot \cdot n]$ (uniformly distributed over the set $\{1, \ldots, n\}$). When the order statistic is random, we shall refer to the situation as the case of *random selection* and denote QS's number of comparisons for it by $C_n^{(M_n)}$, and when the order statistic is fixed we shall refer to it as the case of *fixed selection* and denote QS's number of comparisons for it by $C_n^{(m)}$.

Ideally, we would like to analyze the distributional properties of $C_n^{(m)}$, both exactly and asymptotically as $n \to \infty$; with particular interest in the asymptotic case when $m/n \to \alpha$, $0 \leq \alpha \leq 1$, as $n \to \infty$. This appears to be a formidable problem, except for the case $m \ll n$, that is, $\alpha = 0$ in the above limit (*see* the discussion of Section 4).

A much more tractable problem is the analysis of $C_n^{(M_n)}$. The analysis of $C_n^{(M_n)}$ provides information about the number of comparisons involved in computing an "average" order statistic in random selection, and distributional properties of $C_n^{(M_n)}$ may then be regarded as an average measure of the performance of QS for all order statistics. An example is given in Section 5, where the "average second moment" is computed.

We identify a procedure for finding the moments of $C_n^{(M_n)}$ and use it to find the *exact* mean and variance of $C_n^{(M_n)}$. The weak convergence to a limit law for a normalized version of $C_n^{(M_n)}$ is established in the Wasserstein metric on the space of distribution functions with bounded second moments [2]. This is done by adapting an elegant technique due to Rösler [17]. Unlike the case of QUICKSORT where only the existence of the limit law was proved, we are fortunately able to explicitly characterize the limiting distribution of QS in the case of random selection. The limit law is shown to be absolutely continuous and infinitely divisible, and consequently with unbounded support, with asymmetric left and right tails.

As a by-product of our procedure for the moments of $C_n^{(M_n)}$, we also obtain a procedure for the moments of $C_n^{(m)}$. The procedure becomes computationally very tedious as m becomes larger, but can still be handled by symbolic computation. We illustrate our procedure for the exact mean and variance of $C_n^{(m)}$, for $m = 1, 2, 3$. Furthermore, several analyses for the limit of $C_n^{(M_n)}$ carry over to the case of a very small fixed order statistic, that is, the case $m = o(n)$, as $n \to \infty$, revealing similar distributional properties for the limit law of $C_n^{(m)}$ in this range of m.

For fixed m, Knuth ([11]; Exercise 5.2.2.32) analyzes the average number of comparisons in a version of QS that uses a partitioning stage with $n + 1$ comparisons. A QUICKSELECT that uses a partitioning state with only $n - 1$ comparisons "steals" two comparisons away from Knuth's average at each level of recursion. Since there are about $\log_2 n$ levels of recursion on average, we end up with an average that differs by about $2 \log_2 n$ from Knuth's. Devroye [6] identified some upper bounds on $\mathbf{E}\,[(C_n^{(m)})^p]$ for any m and for all $p \geq 1$. We shown by an example in Section 5 that the bounds for the average second moment are much smaller than the bounds in [6].

2. EXACT MOMENTS

We start with a recurrence for the probabilities. The probability distribution for $C_n^{(m)}$ (recalling that m is fixed in this notation) satisfies

$$P\,[C_n^{(m)} = j] = \begin{cases} 0, & j = 0,\,1, \ldots, n-2; \\ \dfrac{1}{n}, & j = n-1; \quad 1 < m < n; \\ \dfrac{2}{n}, & j = n-1; \quad m = 1 \text{ or } m = n; \\ \dfrac{1}{n}\left(\displaystyle\sum_{k=m+1}^{n} P\,[C_{k-1}^{(m)} = j - n + 1] \right. & \\ \qquad \left. + \displaystyle\sum_{k=1}^{m-1} P\,[C_{n-k}^{(m-k)} = j - n + 1] \right), & \text{otherwise,} \end{cases}$$

valid for $n \geq 1$, $j \geq 0$, and $1 \leq m \leq n$. The first relation follows from the fact that PARTITION $(1, n, k)$ takes $n - 1$ comparisons; the second from the fact that for $1 < m < n$, the first element in a random permutation on $\{1, \ldots, n\}$ could be $1, 2, \ldots, n$ with equal probability. The third follows from the same fact and the observation that if the pivot's final position is 2 (or $n-1$), the recursive application of QS consumes zero extra comparisons to find the first (last) order statistic (the resulting sublists have size 1). The last relation follows from the fact that PARTITION $(1, n, k)$ takes $n - 1$ comparisons and QS proceeds with either locating the mth smallest among the elements of the left sublist or the $(m-k)$th smallest among the elements of the right sublist, all positions being equally likely landing positions for the pivot.

We next try to solve the recurrence using generating functions. Differencing a version of the recurrence with n and j from a version with $n + 1$ and $j + 1$ replacing n and j, respectively, gives

$$\begin{aligned}(n+1)\, P\,[C_{n+1}^{(m)} = j+1] - nP\,[C_n^{(m)} = j] &= P\,[C_n^{(m)} = j-n+1] \\ &\quad + \sum_{k=1}^{m-1} P\,[C_{n-k+1}^{(m-k)} = j-n+1] \\ &\quad - \sum_{k=1}^{m-1} P\,[C_{n-k}^{(m-k)} = j-n+1], \end{aligned} \tag{2.1}$$

valid for $j \geq 0$ and $n \geq m \geq 1$. Let us rewrite this recurrence as $a - b = c + d - e$. Define

$$A_m(x, y) = \sum_{\substack{j \geq 0 \\ n \geq m}} P\,[C_n^{(m)} = j]\, x^n\, y^j,$$

and

$$A(x, y, z) = \sum_{\substack{n \geq 1 \\ j \geq 0}} \sum_{m=1}^{n} P\,[C_n^{(m)} = j]\, x^n\, y^j\, z^m = \sum_{m \geq 1} A_m(x, y)\, z^m.$$

Note that

$$m!A_m(x, y) = \frac{\partial^m A(x, y, 0)}{\partial z^m}.$$

Multiplying (2.1) by $x^n y^j z^m$ and summing over $n \geq 1$, $j \geq 0$, $1 \leq m \leq n$, the five terms involved in (2.1) yield the following. First

$$a = \frac{1}{y}\left[\frac{\partial A(x, y, z)}{\partial x} - \frac{\partial A_1(xz, y)}{\partial x}\right],$$

where the term $\partial A_1(xz, y)/\partial x$ appears to adjust for the boundary conditions; similarly

$$b = x\frac{\partial A(x, y, z)}{\partial x};$$
$$c = y\, A(xy, y, z).$$

For the term d, one obtains

$$\begin{aligned} d &= \sum_{\substack{n\geq 1\\ j\geq 0}} \sum_{m=1}^{n} \sum_{k=1}^{m-1} P\,[C_{n-k+1}^{(m-k)} = j - n + 1]\, x^n\, y^j\, z^m \\ &= \frac{1}{y} \sum_{n\geq 1} \sum_{j\geq 0} \sum_{k=1}^{n-1} (xyz)^k \sum_{m=k+1}^{n} P\,[C_{n-k+1}^{(m-k)} = j]\,(xy)^{n-k}\, y^j\, z^{m-k} \\ &= \frac{1}{xy^2} \sum_{k=1}^{\infty} (xyz)^k \sum_{\substack{n\geq 2\\ j\geq 0}} \sum_{r=1}^{n-1} P\,[C_n^{(r)} = j]\,(xy)^n\, y^j\, z^r \\ &= \frac{1}{xy^2}\left(\sum_{k=1}^{\infty} (xyz)^k \sum_{\substack{n\geq 1\\ j\geq 0}} \sum_{r=1}^{n} (P\,[C_n^{(r)} = j]\,(xy)^n\, y^j\, z^r \right. \\ &\qquad \left. - P\,[C_n^{(n)} = j]\,(xy)^n\, y^j\, z^n)\vphantom{\sum_{k=1}^{\infty}}\right) \\ &= \frac{z}{y}\left(\frac{A(xy, y, z) - A_1(xyz, y)}{1 - xyz}\right), \end{aligned}$$

where we used the symmetry between $C_n^{(n)}$ and $C_n^{(1)}$. Similarly

$$e = \frac{xzA(xy, y, z)}{1 - xyz},$$

Putting these terms together and simplifying, one obtains the partial differential equation

$$\frac{\partial A(x, y, z)}{\partial x} = \frac{1 + z - 2xyz}{(1 - xy)(1 - xyz)} A(xy, y, z) + \left(\frac{1}{1 - xy}\right) \frac{\partial A_1(xz, y)}{\partial x} - \frac{z}{(1 - xy)(1 - xyz)} A_1(xyz, y). \tag{2.2}$$

Our concern is to find the moments of $C_n^{(M_n)}$. As we shall see shortly, differential equations for the moments of $C_n^{(M_n)}$ involve generating functions $G_k^{(1)}(x)$ that are the generating functions of the kth factorial moments of the first order statistic, *i.e.* they are defined by

$$G_k^{(1)}(x) = \sum_{n=1}^{\infty} \mathbf{E}\left[C_n^{(1)}(C_n^{(1)} - 1)\dots(C_n^{(1)} - k + 1)\right] x^n.$$

Thus, we need to develop the latter generating functions first. In what follows H_n and $H_n^{(2)}$ denote the first- and second-order harmonic numbers of order n.

LEMMA 1: *The generating function for the mean and the second factorial moments of* $C_n^{(1)}$ *are:*

$$G_1^{(1)}(x) = \frac{2}{(1-x)^2} - \frac{2}{1-x} - \frac{2}{1-x}\ln\left(\frac{1}{1-x}\right),$$

$$G_2^{(1)}(x) = \frac{18}{1-x}\ln\left(\frac{1}{1-x}\right) - \frac{28}{(1-x)^2} + \frac{9}{(1-x)^3} + \frac{4}{1-x}\ln^2\left(\frac{1}{1-x}\right) - \frac{8}{(1-x)^2}\ln\left(\frac{1}{1-x}\right) + \frac{19}{1-x}.$$

Proof: Observe that

$$A_1(x, y) = \frac{\partial A(x, y, 0)}{\partial z},$$

and that

$$\frac{\partial^k A_1(x, 1)}{\partial y^k} = G_k^{(1)}(x).$$

Differentiate (2.2) once with respect to z at $z = 0$ to obtain

$$\frac{\partial A_1(x, y)}{\partial x} = \frac{A_1(xy, y)}{1 - xy} + \frac{1}{1 - xy}. \tag{2.3}$$

Now differentiate (2.3) once with respect to y at $y = 1$ to obtain

$$\frac{dG_1^{(1)}(x)}{dx} = \frac{G_1^{(1)}(x)}{1 - x} + \frac{2x}{(1 - x)^3}.$$

Solving this differential equation under the obvious initial condition $G_1^{(1)}(0) = 0$ we get $G_1^{(1)}(x)$ as in the lemma. Similarly, differentiating (2.3) twice with respect to y at $y = 1$, we obtain $G_2^{(1)}(x)$ as in the lemma after some lengthy calculations. □

Extracting the coefficients of x^n from $G_1^{(1)}(x)$ and $G_2^{(1)}(x)$ leads to:

THEOREM 1: *The mean and variance of the number of comparisons in QS for finding the minimum (or the maximum, by symmetry) in a list are given by:*

$$\begin{aligned}\mathbf{E}[C_n^{(1)}] &= 2n - 2H_n \\ &\sim 2n, \quad \text{as } n \to \infty,\end{aligned}$$

and

$$\begin{aligned}\mathbf{Var}[C_n^{(1)}] &= \frac{n(n-9)}{2} + 8H_n - 4H_n^{(2)} \\ &\sim \frac{1}{2}n^2, \quad \text{as } n \to \infty.\end{aligned}$$

We now return to analyzing $C_n^{(M_n)}$. Toward this end, let $B(x, y)$ be the generating function

$$B(x, y) = \sum_{\substack{n \geq 1 \\ k \geq 0}} P[C_n^{(M_n)} = k]\, x^n\, y^k.$$

This generating function is related to $A(x, y, z)$ as follows:

$$\begin{aligned}P[C_n^{(M_n)} = k] &= \sum_{m=1}^{n} P[C_n^{(M_n)} = k | M_n = m]\, P[M_n = m] \\ &= \frac{1}{n} \sum_{m=1}^{n} P[C_n^{(m)} = k],\end{aligned}$$

or

$$x \frac{\partial B(x, y)}{\partial x} = A(x, y, 1). \tag{2.4}$$

Using (2.4), one can determine, from (2.2), a partial differential equation for $B(x, y)$. First evaluate (2.2) at $z = 1$, then use (2.4) to obtain

$$\begin{aligned} x \frac{\partial^2 B(x, y)}{\partial x^2} &+ \frac{\partial B(x, y)}{\partial x} \\ &= \left(\frac{2x}{1 - xy}\right) \frac{\partial B(xy, y)}{\partial x} + \left(\frac{1}{1 - xy}\right) \frac{\partial A_1(x, y)}{\partial x} - \frac{A_1(xy, y)}{(1 - xy)^2}. \end{aligned} \tag{2.5}$$

The last partial differential equation does not seem to be tractable but we may develop tractable ordinary differential equation for the moments as follows. Let

$$L_j(x) \stackrel{\text{def}}{=} \sum_{n \geq 0} \mathbf{E}\,[C_n^{(M_n)}(C_n^{(M_n)} - 1) \ldots (C_n^{(M_n)} - j + 1)]\, x^n.$$

Note that $L_j(x)$ is a generating function for the jth factorial moments of $C_n^{(M_n)}$. Clearly,

$$L_j(x) = \frac{\partial^j B(x, 1)}{\partial y^j}.$$

Differentiating (2.5) once with respect to y at $y = 1$, we obtain

$$\begin{aligned} x L_1''(x) + \frac{1 - 3x}{1 - x} L_1'(x) &= \frac{2x(1 + x)}{(1 - x)^4} \\ &\quad + \left(\frac{1}{1 - x}\right) \frac{dG_1^{(1)}(x)}{dx} - \frac{G_1^{(1)}(x)}{(1 - x)^2}; \end{aligned}$$

the generating function $G_1^{(1)}(x)$ is given in Lemma 1. This is an ordinary differential equation whose solution under the obvious boundary conditions $L_1(0) = 0$ and $L_1'(0) = \mathbf{E}\,[C_1^{(M_1)}] = 0$ is

$$\begin{aligned} L_1(x) = \frac{3}{(1 - x)^2} &+ \frac{10}{1 - x} - 13 - \frac{8}{1 - x} \ln\left(\frac{1}{1 - x}\right) \\ &- 4 \ln^2\left(\frac{1}{1 - x}\right)^2 - 8\, \mathrm{dilog}(x), \end{aligned}$$

where dilog is the dilogarithm function defined by

$$\operatorname{dilog}(x) = \int_0^x \frac{1}{u} \ln\left(\frac{1}{1-u}\right) du.$$

For $\mathbf{Var}\,[C_n^{(M_n)}]$ we first need to develop a differential equation for $L_2(x)$. Differentiating (2.5) twice with respect to y at $y = 1$, one obtains

$$\begin{aligned} & x L_2''(x) + \frac{1-3x}{1-x} L_2'(x) \\ & \quad = \frac{10}{(1-x)^2} + \frac{100}{(1-x)^3} - \frac{230}{(1-x)^4} + \frac{120}{(1-x)^5} \\ & \qquad + \frac{96}{(1-x)^3} \ln\left(\frac{1}{1-x}\right) - \frac{96}{(1-x)^4} \ln\left(\frac{1}{1-x}\right). \end{aligned}$$

Solving this differential equation we obtain $L_2(x)$. The coefficients in the generating functions $L_1(x)$ and $L_2(x)$ provide us with exact expressions for the mean and variance of $C_n^{(M_n)}$.

THEOREM 2: *The mean and variance of QS when it performs a random selection are given by:*

$$\begin{aligned} \mathbf{E}\,[C_n^{(M_n)}] &= 3n - 8H_n + 13 - \frac{8H_n}{n} \\ &\sim 3n, \quad \text{as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned} \mathbf{Var}\,[C_n^{(M_n)}] &= n^2 - 10n - 16H_n^2 + 108H_n - 47 - 48H_n^{(2)} - 80\frac{H_n^2}{n} \\ &\quad + \frac{204H_n}{n} - \frac{48H_n^{(2)}}{n} - \frac{64H_n^2}{n^2} \\ &\sim n^2, \quad \text{as} \quad n \to \infty. \end{aligned}$$

We conclude this section with an illustration of the use of the above in finding the exact means and variances for the fixed order statistics. The generating functions for factorial moments of $C_n^{(m)}$, for different values of m, satisfy a recurrence relation.

Differentiating (2.2) first m times with respect to z at $z = 0$ and then j times with respect to y, at $y = 1$, yields the recurrence

$$\begin{aligned}\frac{dG_1^{(m)}}{dx} &= \frac{1}{1-x}\, G_1^{(m)}(x) + \sum_{j=1}^{m-1} x^{j-1}\, G_1^{(m-j)}(x) + \frac{x^m\,((2-m)\,x+m)}{(1-x)^3}\\ &\quad - \frac{x^{m-1}\,((3m^2-11m+6)\,x+(m-1)\,(m-2))}{2\,(1-x)^2}\\ &\quad + \frac{(m-1)\,(m-2)\,x^{m-1}}{1-x} + \frac{mx^{m-1}}{1-x}\,\mathbf{E}\,[C_m^{(1)}]\\ &\quad + \frac{1}{1-x}\sum_{j=1}^{m-1}\mathbf{E}\,[C_{m-j}^{(1)}].\end{aligned}$$

It is interesting to note that the last differential equation bears some resemblance to differential equations connected with QUICKSORT [8] and m-ary search trees [14-15]. Thus, for example,

$$G_1^{(2)}(x) = \frac{2}{(1-x)^2} - \frac{6}{1-x} + 2\ln\left(\frac{1}{1-x}\right) + 4,$$

from which it follows that

$$\begin{aligned}\mathbf{E}\,[C_n^{(2)}] &= 2n - 4 + \frac{2}{n}\\ &\sim 2n, \quad \text{as } n \to \infty,\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}G_1^{(3)}(x) &= \frac{2}{1-x}\ln\left(\frac{1}{1-x}\right) + \frac{2}{(1-x)^2}\\ &\quad - 2\,(1-x)\ln\left(\frac{1}{1-x}\right) - \frac{25}{3\,(1-x)} + \frac{25}{3} + \frac{13x}{3} - \frac{2x^2}{3},\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\,[C_n^{(3)}] &= 2n + 2H_n - \frac{25}{3} + \frac{2}{n-1}\\ &\sim 2n, \quad \text{as } n \to \infty.\end{aligned}$$

Also,

$$\begin{aligned} G_2^{(2)}(x) = {} & \frac{9}{(1-x)^3} - \frac{31}{(1-x)^2} - \frac{2}{1-x}\ln\left(\frac{1}{1-x}\right) \\ & + \frac{63}{1-x} - 26\ln\left(\frac{1}{1-x}\right) \\ & - \frac{4}{1-x}\ln^2\left(\frac{1}{1-x}\right) - 4\ln^2\left(\frac{1}{1-x}\right) - 41, \end{aligned}$$

from which, together with (2.6) we have

$$\begin{aligned} \mathbf{Var}\,[C_n^{(2)}] &= \frac{n^2}{2} + \frac{n}{2} - 4H_n^2 - 2H_n + 13 + 4H_n^{(2)} - \frac{8H_n}{n} - \frac{8}{n} + \frac{4}{n^2} \\ &\sim \frac{1}{2}n^2, \quad \text{as } n \to \infty, \end{aligned}$$

and similarly we obtain

$$\begin{aligned} \mathbf{Var}\,[C_n^{(3)}] &= \frac{n^2}{2} + \frac{11}{2}n - 12H_n^2 + \left(\frac{20}{3} + \frac{8}{n} - \frac{16}{n-1}\right)H_n + 8H_n^{(2)} \\ &\quad + \frac{85}{3} - \frac{172}{3(n-1)} + \frac{4}{(n-1)^2} - \frac{6}{n} - \frac{8}{n^2} \\ &\sim \frac{1}{2}n^2, \quad \text{as } n \to \infty. \end{aligned}$$

It should be clear that we can continue in this fashion to obtain exact higher order moments for $m = 1, 2, 3$ as well as exact moments for other fixed order statistics. But is should also be clear that we have an explosion of computational complexity. However, the procedure may still be useful with a symbolic manipulation system.

3. THE LIMIT LAW FOR RANDOM SELECTION

In this section we shall use the following standard notation from probability theory. The symbol $\stackrel{\mathcal{L}}{=}$ will denote equality in law, whereas the symbol $\stackrel{\mathcal{L}}{\to}$ will denote convergence in law. The law of a random variable X will be denoted by $\mathcal{L}(X)$. The indicator $\mathbf{1}_{\mathcal{E}}$ of an event $\mathcal{E}$ assumes the value 1 if the event $\mathcal{E}$ occurs and is zero otherwise.

Following Rösler [17] we can develop a functional relation for the limit law of

$$Y_n = \frac{C_n^{(M_n)} - \mathbf{E}\,[C_n^{(M_n)}]}{n}.$$

The functional for QUICKSORT in [17] was not tractable and was used only in an existential proof of a limit law. Fortunately the functional we obtain below for random selection is tractable and the limit law is characterized explicitly.

Let Z_n be the position to which the chosen pivot moves. As the starting list is assimilated to a random permutation, Z_n is discrete uniform $[1 \cdot \cdot n]$. We have the conditional behavior (given Z_n and M_n):

$$C_n^{(M_n)} = \begin{cases} n - 1 + C_{Z_n - 1}^{(M_{Z_n - 1})}, & \text{if } \ Z_n > M_n; \\ n - 1, & \text{if } \ Z_n = M_n; \\ n - 1 + C_{n - Z_n}^{(M_{n - Z_n})}, & \text{if } \ Z_n < M_n. \end{cases}$$

Thus Y_n satisfies the following conditional relation (given Z_n):

$$\begin{aligned} Y_n \stackrel{\mathcal{L}}{=} {} & \mathbf{1}_{\{Z_n > M_n\}} \frac{Z_n - 1}{n} Y_{Z_n - 1} + \mathbf{1}_{\{Z_n < M_n\}} \frac{n - Z_n}{n} Y^*_{n - Z_n} \\ & + \frac{n-1}{n} - \frac{\mathbf{E}\,[C_n^{(M_n)}]}{n} \\ & + \frac{1}{n} \mathbf{1}_{\{Z_n > M_n\}} \mathbf{E}\,[C_{Z_n - 1}^{(M_{Z_n - 1})} | Z_n] \\ & + \frac{1}{n} \mathbf{1}_{\{Z_n < M_n\}} \mathbf{E}\,[C_{n - Z_n}^{(M_{n - Z_n})} | Z_n], \end{aligned} \tag{3.1}$$

where $Y_i^* \stackrel{\mathcal{L}}{=} Y_i$. Note that Y_i, Y_i^*, Z_n, and M_n are mutually independent, $1 \leq i < n$. Theorem 2 states that

$$\mathbf{E}\,[C_n^{(M_n)}] = 3n - 8H_n + O\,(1),$$

so that

$$\mathbf{E}\,[C_{Z_n - 1}^{(M_{Z_n - 1})} | Z_n] = 3\,(Z_n - 1) - 8H_{Z_n - 1} + O\,(1),$$

and

$$\mathbf{E}\,[C_{n-Z_n}^{(M_{n-z_n})}|Z_n] = 3\,(n - Z_n) - 8H_{n-Z_n} + O\,(1),$$

and we may write

$$Y_n \stackrel{\mathcal{L}}{=} \mathbf{1}_{\{Z_n > M_n\}}\,\frac{Z_n - 1}{n}\,(Y_{Z_n - 1} + 3)$$
$$+ \mathbf{1}_{\{Z_n < M_n\}}\,\frac{n - Z_n}{n}\,(Y^*_{n-Z_n} + 3) + O\,(1).$$

THEOREM 3:

$$Y_n \xrightarrow{\mathcal{L}} Y,$$

where Y *satisfies the relation*

$$Y \stackrel{\mathcal{L}}{=} X\,(Y + 3) - 2,$$

where X *and* Y *are independent and* X *has the density*

$$f\,(x) = \begin{cases} 2x, & 0 < x < 1; \\ 0, & elsewhere. \end{cases}$$

Before proving the theorem, we specify the limit Y in terms of its characteristic function:

LEMMA 2: *If*

$$Y \stackrel{\mathcal{L}}{=} X\,(Y + 3) - 2,$$

with X *and* Y *as in Theorem 3, the characteristic function of* Y *is*

$$\phi_Y\,(t) = \exp\left(2\int_0^t \frac{e^{iu} - 1 - iu}{u}\,du\right).$$

Proof: Let $\tilde{Y} = Y + 2$, so that $\tilde{Y} \stackrel{\mathcal{L}}{=} X\,(\tilde{Y} + 1)$. The characteristic function $\phi_{\tilde{Y}}$ of $\tilde{Y}$ satisfies

$$\phi_{\tilde{Y}}\,(t) = \int_0^1 2xe^{itx}\,\phi_{\tilde{Y}}\,(tx)\,dx = \int_0^t \frac{2ue^{iu}\,\phi_{\tilde{Y}}\,(u)}{t^2}\,du.$$

Differentiating (which is permissible in view of Theorem 3) gives the equation

$$t^2 \frac{d\phi_{\tilde{Y}}(t)}{dt} = 2t\,(e^{it} - 1)\,\phi_{\tilde{Y}}(t),$$

with the solution

$$\phi_{\tilde{Y}}(t) = \exp\left(2 \int_0^t \frac{e^{iu} - 1}{u}\, du\right).$$

Hence Y has the given characteristic function. □

Proof of Theorem 3: We proceed in the manner of Rösler [17] to show that $d_2\,[\mathcal{L}(Y_n),\, \mathcal{L}(Y)]$ converges to 0 as $n \to \infty$, where d_2 is the Wasserstein metric on the space of zero-mean distribution functions (*see* [2] for a discussion of this metric):

$$d_2(F,\, G) = \inf \|X - Y\|^2,$$

where $\|\cdot\|^2$ denotes the L_2-norm and the infimum is over all random variables X with distribution function F and all random variables Y with distribution function G.

Let Y and Y^* be independent with the distribution of Lemma 2. Let Q_i and Q_i^* be independent copies of a random variable with law $\mathcal{L}(Y_i)$, $1 \le i \le n-1$, with

$$\mathbf{E}\,[(Q_i - Y)^2] \le d_2\,[\mathcal{L}(Q_i),\, \mathcal{L}(Y)] + \frac{1}{i},$$
$$\mathbf{E}\,[(Q_i^* - Y^*)^2] \le d_2\,[\mathcal{L}(Q_i^*),\, \mathcal{L}(Y^*)] + \frac{1}{i}.$$

Let Ω_n be the sample space of Y_n (that is the set of all permutations of $\{1, \dots, n\}$). Let $V,\, \overline{V} : \Omega_n \times [0,\, 1) \to \mathbb{R}$, with $V(\cdot,\, x) = V_x$, and $\overline{V}(\cdot,\, x) = \overline{V}_x$, be given by

$$V_x = \sum_{i=1}^{n} \mathbf{1}_{\{\frac{i-1}{n} < x \le \frac{i}{n}\}}\,(Y_{i-1} + 3),$$

and

$$\overline{V}_x = \sum_{i=1}^{n} \mathbf{1}_{\{\frac{i-1}{n} < x \le \frac{i}{n}\}}\,(Y_{n-i}^* + 3).$$

Writing $\overline{Q}_i = Q_i + 3$, $\overline{Y} = Y + 3$, and using (3.1), one has

$$Q_n \stackrel{\mathcal{L}}{=} \mathbf{1}_{\{Z_n > M_n\}} \frac{Z_n - 1}{n} \overline{Q}_{Z_n - 1} + \mathbf{1}_{\{Z_n < M_n\}} \frac{n - Z_n}{n} \overline{Q}^*_{n - Z_n} - 2 + o_p(1),$$

where $o_p(1)$ denotes a quantity tending to zero in probability. For purposes of convergence we can, and therefore will, ignore the $o_p(1)$ term.

To get a suitable coupling, let T and W be independent uniform on (0, 1), and independent of Q_i, Q_i^*, $0 \leq i \leq n-1$. It is easily checked that

$$Q_n \stackrel{\mathcal{L}}{=} \mathbf{1}_{\{T > W\}} \left(\frac{\lceil nT \rceil - 1}{n} \right) V_T + \mathbf{1}_{\{T < W\}} \left(\frac{n - \lceil nT \rceil}{n} \right) \overline{V}_T - 2.$$

Also, from the integral representation of the characteristic function of Y in Lemma 2, it is straightforward to show that

$$Y \stackrel{\mathcal{L}}{=} \mathbf{1}_{\{T > W\}} T\overline{Y} + \mathbf{1}_{\{T < W\}} (1 - T) \overline{Y}^* - 2.$$

Thus

$$\begin{aligned}
&d_2 [\mathcal{L}(Q_n), \mathcal{L}(Y)] \\
&\quad \leq \mathbf{E} \left[\left\{ \mathbf{1}_{\{T > W\}} \left(\left(\frac{\lceil nT \rceil - 1}{n} \right) V_T - T\overline{Y} \right) \right.\right. \\
&\qquad \left.\left. + \mathbf{1}_{\{T < W\}} \left(\left(\frac{n - \lceil nT \rceil}{n} \right) \overline{V}_T - (1 - T) \overline{Y}^* \right) \right\}^2 \right] \\
&\quad = \mathbf{E} \left[\mathbf{1}_{\{T > W\}} \left(\left(\frac{\lceil nT \rceil - 1}{n} \right) V_T - T\overline{Y} \right)^2 \right] \\
&\qquad + \mathbf{E} \left[\mathbf{1}_{\{T < W\}} \left(\left(\frac{n - \lceil nT \rceil}{n} \right) \overline{V}_T - (1 - T) \overline{Y}^* \right)^2 \right] \\
&\quad = \mathbf{E} \left[\left\{ \sum_{i=1}^{n} \mathbf{1}_{\{\frac{i-1}{n} < T \leq \frac{i}{n}\}} \mathbf{1}_{\{T > W\}} \left(\frac{i-1}{n} \overline{Q}_i - T\overline{Y} \right) \right\}^2 \right] \\
&\qquad + \mathbf{E} \left[\left\{ \sum_{i=1}^{n} \mathbf{1}_{\{\frac{i-1}{n} < T \leq \frac{i}{n}\}} \mathbf{1}_{\{T < W\}} \left(\frac{n-i}{n} \overline{Q}^*_{n-i} - (1 - T) \overline{Y}^* \right) \right\}^2 \right]
\end{aligned}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}\left[\mathbf{1}_{\{T>W\}} \left(\frac{i-1}{n} \overline{Q}_i - \frac{i-1}{n} \overline{Y}\right)^2\right]$$
$$+ \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}\left[\mathbf{1}_{\{T<W\}} \left(\frac{n-i}{n} \overline{Q}^*_{n-i} - \frac{n-i}{n} \overline{Y}^*\right)^2\right] + o(1)$$
$$= \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \mathbf{E}\,[\overline{Q}_i - Y)^2] + o(1).$$

Letting $a_i = d_2 [\mathcal{L}(Q_i), \mathcal{L}(Y)]$, we thus have

$$a_n \leq \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 a_i + o(1).$$

It follows as in proposition 3.3 of Rösler [17] that $a_n \to 0$. □

4. PROPERTIES OF THE LIMIT LAW

Some of the properties of the limiting variable Y may be deduced immediately from its characteristic function, given in Lemma 2.

LEMMA 3: *Y has an infinitely divisible distribution;* i.e., *for every positive integer k, there exist i.i.d. random variables $S_{1,k}, \ldots, S_{k,k}$ such that $Y \overset{\mathcal{L}}{=} S_{1,k} + \ldots + S_{k,k}$.*

Proof: The characteristic function $\phi_Y(t)$, by an obvious change of variable, may be written

$$\phi_Y(t) = \exp\left(2 \int_0^1 \frac{e^{itx} - 1 - itx}{x}\, dx\right),$$

which is in the form of an infinitely divisible distribution with finite variance (*see* Chow and Teicher [4], p. 420]). □

COROLLARY 1: *The support of Y is unbounded.*

Proof: This is true of any infinitely divisible distribution (*see* Chow and Teicher [4], p. 413]). □

LEMMA 4: *The distribution of Y is absolutely continuous.*

Proof: It is not difficult to show that $|\phi_Y(t)| \sim |t|^{-1}$, as $|t| \to \infty$; thus ϕ_Y is square integrable, and the result follows from Plancherel's theorem (*see* Chung [5], p. 159]). $\square$

Information on the tail behavior of Y can be derived from its moment generating function, using Chernoff's approach [3]. An argument similar to that of Lemma 4 shows that the moment generating function of Y is

$$f(t) = \exp\left(2\int_0^t \left(\frac{e^u - 1 - u}{u}\right) du\right).$$

An easy argument based on a Taylor expansion shows that for $t > 0$,

$$f(t) < \exp(e^t - t).$$

With the aid of the last inequality, a large deviation result for Y can be easily derived.

PROPOSITION 1: *For* $\lambda > 0$,

$$P[Y > \lambda] \leq \exp\left(\frac{e}{1+\lambda}\right)^{1+\lambda}.$$

Proof:

$$\begin{aligned} P[Y > \lambda] &= P[tY > t\lambda], \qquad \text{for} \quad t > 0 \\ &\leq \frac{\mathbf{E}[e^{tY}]}{e^{\lambda t}} \\ &\leq \exp(e^t - t - \lambda t). \end{aligned}$$

Minimizing the exponent in t, we get the result. $\square$

Using the bound

$$\frac{e^{-v} - (1 - v)}{v} \leq \frac{v}{2}, \qquad \text{for} \quad v > 0, \tag{4.1}$$

we have the following result for the left tail.

PROPOSITION 2: *For* $\lambda > 0$,

$$P[Y < -\lambda] \leq \exp\left(-\frac{\lambda^2}{2}\right).$$

Proof:

$$\begin{aligned} P[Y < -\lambda] &= P[-tY > \lambda t], \qquad \text{for} \quad t > 0 \\ &\le \frac{\mathbf{E}[e^{-tY}]}{e^{\lambda t}} \\ &\le \exp\left(\frac{t^2}{2} - \lambda t\right). \end{aligned}$$

Minimizing the exponent, again, gives the result. □

According to Lemma 3, $\phi_Y(t)$ has an nth root for any positive integer n. The next theorem reveals the interesting result that the square root, *i.e.*

$$\exp\left(\int_0^t \frac{e^{iu} - 1 - iu}{u}\, du\right), \tag{4.2}$$

is the characteristic function of the limit of $Y_n^{(m)}$, the normalized number of comparisons needed to find the mth order statistic using QS, for m fixed.

THEOREM 4:

$$Y_n^{(m)} \xrightarrow{\mathcal{L}} Y^{(m)},$$

where $Y^{(m)} \stackrel{\mathcal{L}}{=} U(Y^{(m)} + 2) - 1$, *with* U *independent of* $Y^{(m)}$ *and uniform on* $(0, 1)$. *The characteristic function of* $Y^{(m)}$ *is given in* (4.2) *above.*

Proof: The proof is very similar to that of Theorem 3, noting that from (3.1) and Knuth ([11]; Exercise 5.2.2.32)

$$Y_n^{(m)} \stackrel{\mathcal{L}}{=} \frac{Z_n - 1}{n}(Y_{Z_n - 1}^{(m)} + 2) - 1 + o_p(1). \quad \square$$

The characteristic function (4.2) gives the curious result that Y of Theorem 3 (the limit of the variable $(C_n^{(M_n)} - \mathbf{E}[C_n^{(M_n)}])/n$) is stochastically equal to the sum of two independent copies of $Y^{(m)}$, the limit of $(C_n^{(m)} - \mathbf{E}[C_n^{(m)}])/n$).

Clearly, the distribution of $Y^{(m)}$ is also infinitely divisible of unbounded support, and absolutely continuous. Bounds on the tails of $Y^{(m)}$ can be derived in the same fashion as those for Y; they are slightly tighter, since the moment generating function of $Y^{(m)}$ has no factor of 2 multiplying the integral.

5. USING RANDOM SELECTION AS AN AVERAGE MEASURE

We mentioned in the introduction that the analysis of random selection may be used to provide average measures for fixed selection. We show in this section an example of this by deriving an upper bound for the average second moment of $C_n^{(m)}$, $1 \leq m \leq n$, when $n \geq 3$.

To illustrate the use of $C_n^{(M_n)}$ as a measure of average performance, note that

$$\begin{aligned}\mathbf{E}\,[(C_n^{(M_n)})^2] &= \sum_{m=1}^{n} \mathbf{E}\,[(C_n^{(M_n)})^2 | M_n = m]\, P\,[M_n = m] \\ &= \frac{1}{n} \sum_{m=1}^{n} \mathbf{E}\,[(C_n^{(m)})^2],\end{aligned}$$

which is the average of the second moments for the random variables $C_n^{(m)}$, $1 \leq m \leq n$.

Then trivially,

$$\begin{aligned}\mathbf{Var}\,[Y_n^{(M_n)}] &= \frac{1}{n^2}\,\mathbf{E}\,[(C_n^{(M_n)})^2] - \frac{1}{n^2}\,\mathbf{E}^2\,[C_n^{(M_n)}] \\ &= \frac{1}{n^3} \sum_{m=1}^{n} \mathbf{E}\,[(C_n^{(m)})^2] - \frac{1}{n^2}\,\mathbf{E}^2\,[C_n^{(M_n)}].\end{aligned}$$

Since $\mathbf{E}\,[C_n^{(M_n)}]/n \to 3$, as $n \to \infty$, and $\mathbf{Var}\,[Y_n^{(M_n)}] \to 1$ (*cf.* Theorem 2),

$$\frac{1}{n^3} \sum_{m=1}^{n} \mathbf{E}\,[(C_n^{(m)})^2] \to 10.$$

In fact, it is easily seen from Theorem 2 that, for all $n \geq 3$,

$$\mathbf{E}\,[C_n^{(M_n)}] \leq 3n, \qquad \mathbf{Var}\,[C_n^{(M_n)}] \leq n^2.$$

Thus, for $n \geq 3$, the average value of $\mathbf{E}\,[(C_n^{(m)})^2]/n^2$, $1 \leq m \leq n$ is ≤ 10 and hence considerably smaller than the bound $32/(3 \ln(4/3)) \approx 37.078$ given by Devroye [6].

REFERENCES

1. G. Brassard and P. Bratley, *Algorithms: Theory and Practice*, Prentice-Hall, Englewoord Cliffs, New Jersey, 1988.
2. S. Cambanis, G. Simons and W. Stout, Inequalities for the expected value of $K(x, y)$ when the marginals are fixed, *Zeit. Wahrsch. Verw. Geb.*, 1976, *36*, pp. 285-294.
3. H. Chernoff, A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations, *Annals of Mathematical Statistics*, 1952, *23*, pp. 493-507.
4. Y. Chow and H. Teicher, *Probability Theory*, Springer Verlag, New York, 1978.
5. K. Chung, *A Course in Probability Theory*, Second Edition, Academic Press, Orlando, Florida, 1974.
6. L. Devroye, Exponential bounds for the running time of a selection algorithm, *Journal of Computer and System Sciences*, 1984, *29*, pp. 1-7.
7. G. Gonnet and R. Baeza-Yates, *Handbook of Algorithms and Data Structures in Pascal and C*, Second Edition, Addison-Wesley, Reading, Massachusetts, 1991.
8. P. Hennequin, Combinatorial Analysis of Quicksort Algorithm, *RAIRO, Theoretical Informatics and Applications*, 1989, *23*, pp. 317-333.
9. C. Hoare, Find (Algorithm 65), *Communications of the ACM*, 1961, *4*, pp. 321-322.
10. C. Hoare, Quicksort, *The Computer Journal*, 1962, *5*, pp. 10-15.
11. D. Knuth, *The Art of Computer Programming, 3: Sorting and Searching*, Addison-Wesley, Reading, Massachussetts, 1973.
12. R. Kruise, *Data Structures and Program Design*, Second Edition, Prentice-Hall, Englewood Cliffs, New Jersey, 1987.
13. E. Lehmann, *Nonparametrics: Statistical Methods Based on Ranks*, Holden-Day, San Francisco/McGraw-Hill, New York, 1975.
14. H. Mahmoud, *Evolution of Random Search Trees*, John Wiley, New York, 1992.
15. H. Mahmoud, A law of large numbers for path lengths in search trees, Chapter in *Random Graphs: Vol. II*, A. Frieze and Tomasz Łuczak, eds., John Wiley, New York, 1992.
16. M. Régnier, A limiting distribution for Quicksort, *RAIRO, Theoretical Informatics and Applications*, 1989, *23*, pp. 335-343.
17. U. Rösler, A limit theorem for "QUICKSORT", *RAIRO, Theoretical Informatics and Applications*, 1991, *25*, pp. 85-100.
18. R. Sedgewick, Quicksort with equal keys, *SIAM J. on Computing*, 1977, *6*, pp. 240-267.
19. R. Sedgewick, The analysis of quicksort programs, *Acta Informatica*, 1977, *7*, pp. 327-355.
20. R. Sedgewick, *Quicksort*, Garland, New York, 1980.
21. R. Sedgewick, *Algorithms*, Second edition, Addison-Wesley, Reading, Massachusetts, 1988.