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# AN ALGORITHM TO COMPUTE THE MÖBIUS FUNCTION OF THE ROTATION LATTICE OF BINARY TREES (*)

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Abstract. – *Though the rotation lattice of binary trees is not a modular lattice, we show that its Möbius function* $\mu$ *can be computed as for distributive lattices. If* $T$ *and* $T'$ *are binary trees with* $n$ *internal nodes, a* $O(n^{3/2})$ *time and* $O(n)$ *space algorithm is developed for computing* $\mu(T, T')$.

Résumé. – *Bien que le treillis de rotation des arbres binaires ne soit pas modulaire, on montre que sa fonction de Möbius* $\mu$ *se calcule de la même manière que pour les treillis distributifs. Si* $T$ *et* $T'$ *sont deux arbres binaires à* $n$ *nœuds internes, on exhibe un algorithme de complexité* $O(n^{3/2})$ *en temps et* $O(n)$ *en espace pour calculer* $\mu(T, T')$.

## 1. INTRODUCTION

A rotation in a binary tree is a simple, local, restructuring technique that changes the tree into another tree. Rotation is a very useful operation because in constant time it alters the depths of some of the nodes in the tree while maintaining the symmetric order of the items. Therefore it is used to rebalance binary trees in all search algorithms.

The combinatorial system of binary trees and their rotations is a fundamental one that is isomorphic to other natural combinatorial systems. Results concerning this system are of interest both from mathematical and practical points of view.

A system that is isomorphic to binary trees related by rotations is that of binary bracketings related by the semi-associative law shifting brackets, say, to the left. This system has been proved to be a lattice in [5] and [9]. Grätzer thinks that this result can dispel the false impression that the proof that a

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poset is a lattice is always trivial (*see* [6], p. 14). The graph of this lattice is neither bipartite nor homogeneous [1]. This lattice is meet-pseudocomplemented [13] and also complemented [3]. Recently it has been characterized via its ordered subsets of join and meet irreducible elements [1]. This result is based on the fact that any finite lattice can be recovered from its Markowsky poset of irreducibles defined above [10].

Another system that is isomorphic to binary trees related by rotations is that of triangulations of a polygon related by the diagonal flip operation. This is the operation that converts one triangulation of a polygon into another by removing a diagonal in the triangulation and adding the diagonal that subdivides the resulting quadrilateral in the opposite way. This system is studied in [17] using hyperbolic geometry.

Another combinatorial proof that the system of binary trees related by rotations is a lattice can be found in [2] and [11]. We call this system the rotation lattice of binary trees. A Hamilton path in the graph of this lattice has been exhibited in [15]. Unfortunately the rotation lattice is not modular. We hope that the study of its Möbius function as a combinatorial invariant will give useful informations about the structure of this lattice.

In this paper, we show that the Möbius function $\mu$ of the rotation lattice can be computed as for distributive lattices. This leads to a $O(n^{3/2})$ time and $O(n)$ space algorithm for computing $\mu(T, T')$ where $T$ and $T'$ are trees with $n$ internal nodes.

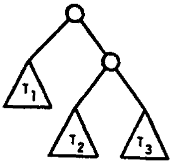
## 2. THE ROTATION LATTICE $B_n$

In a (rooted, ordered, unlabeled) binary tree, every node except the root has a parent. Every internal node ○ has a left and a right child (the order is significant) and each of these children is also a tree called subtree of this internal node. External nodes or leaves □ have no children.

The leaves of a (binary) tree $T$ are numbered by a preorder traversal of $T$ (*i. e.* visit the root and then the left and right subtrees recursively). The weight $|T|$ of a tree $T$ is the number of leaves of $T$. Let $B_n$ denote the set of binary trees with $n$ internal nodes (and thus $n+1$ leaves). The cardinality of $B_n$ is the $n$-th Catalan number $b_n=\binom{2n}{n}/(n+1)$.

Given $T \in B_n$, the weight sequence of $T$ is the integer sequence $w_T=(w_T(1), w_T(2), \ldots, w_T(n))$ where $w_T(i)$ is the weight of the largest subtree of $T$ whose last leaf is $i$ (*see* [11]).

Rotation is a transformation $\rightarrow$ on $B_n$ such that a subtree (*fig.* 1)




of a tree of $B_n$ is replaced by the subtree

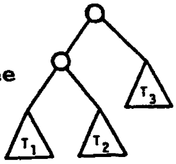



**Figure 1**

See some rotations and weight sequences of trees of $B_6$ illustrated in figure 2.

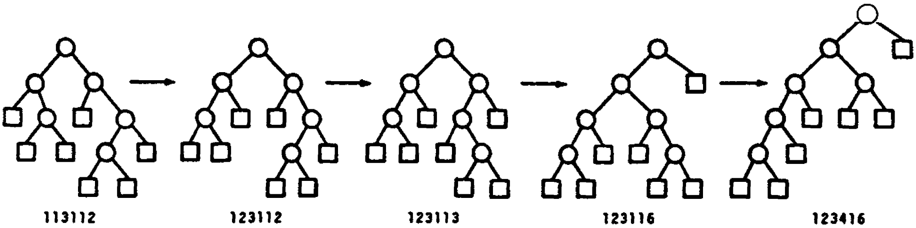



**Figure 2**

The following theorem characterizes the reflexive transitive closure $\overset{*}{\rightarrow}$ of $\rightarrow$:

THEOREM 2.1. [11]: *Given $T$ and $T' \in B_n$, we have $T \overset{*}{\rightarrow} T'$ iff $w_T(i) \leqq w_{T'}(i)$ for all $i \in [1, n]$ or in short $w_T \leqq w_{T'}$.*

COROLLARY 2.2. [11, 12]: *For all $n$, $(B_n, \overset{*}{\rightarrow})$ is a lattice with 0 and 1 whose weight sequences are $w_0 = (1, 1, \ldots, 1)$ and $w_1 = (1, 2, 3, \ldots, n)$. The weight sequence $w_{T \wedge T'}$ of the meet of $T$ and $T' \in B_n$ is obtained by*

$$w_{T \wedge T'}(i) = min(w_T(i), w_{T'}(i))$$

*for all $i \in [1, n]$ or in short $w_{T \wedge T'} = min(w_T, w_{T'})$. The weight sequence $w_{T \vee T'}$ of the join of $T$ and $T'$ can be computed as in* [14].

*Remark:* The rotation lattice $B_n$ is not modular since it contains at least the following pentagon (*fig.* 3):

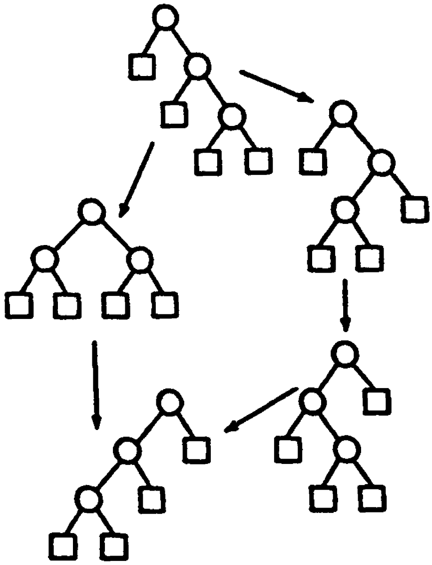


**Figure 3**

## 3. THE MÖBIUS FUNCTION OF A POSET

The Möbius function $\mu$ of a partially ordered set can be viewed as an enumerative tool, defined implicitly by the relations:

$$f(x)=\sum_{y \leqq x} g(y) \qquad \text{and} \qquad g(x)=\sum_{y \leqq x} \mu(y, x) f(y)$$

where $f$ and $g$ are arbitrary real valued functions on a poset $P$. $\mu$ is the unique integer valued function on $P * P$, depending only on $P$ (not on $f$ or $g$), defined recursively by the formulas:

$$\begin{aligned} &\mu(x, x)=1 && \text{for} \quad x \in P \\ &\mu(x, y)=0 && \text{if} \quad x \nleqq y \\ &\mu(x, y)=-\sum_{x<z \leqq y} \mu(z, y) && \text{if} \quad x<y. \end{aligned}$$

Implementing the above recursive formula needs to use an algorithm which lists all the weight sequences $w$ of trees in an interval $[T, T']$ of $B_n$, *i.e.* weight sequences $w$ such that $w_T<w<w_{T'}$. It is easy to exhibit such an algorithm. But computing $\mu$ with this recursive formula is highly impractical for $n$ large enough since $b_n \approx 4^n/((n+1)\sqrt{\pi n})$. In order to compute efficiently $\mu$, we use in the sequel two results of the theory of Möbius

functions: the principle of inclusion-exclusion ([6], p. 191, ex. 34) and the closure theorem ([16], p. 349, prop. 2). Proofs can be found in [7] and [16].

## 4. THE MÖBIUS FUNCTION OF $B_n$

LEMMA 4.1: *Let $[T, T']$ be an interval of the lattice $B_n$. Let $T_1, T_2, \ldots, T_k$ be the k distinct dual atoms of the lattice $[T, T']$,* i.e. *the k trees $T_i$ which are covered by $T' : T \xrightarrow{*} T_i \to T'$. The sub-meet-semilattice S generated by $T_1, T_2, \ldots, T_k$ and including $T'$ is a Boolean lattice.*

*Proof:* Since $T_i \to T'$, the weight sequences of $T_i$ and $T'$ differ only by one integer. Since $w_{\tau \wedge \tau'} = \min(w_\tau, w_{\tau'})$, the lattice $S$ made up of $T'$ and of meets of subsets of $\{T_1, T_2, \ldots, T_k\}$ has cardinality $2^k$. Its zero is $T'' = T_1 \wedge T_2 \wedge \ldots \wedge T_k$. Its unit is $T'$. If $\tau \in S$, then $w_\tau(m)$ is either equal to $w_{T''}(m)$ or equal to $w_{T'}(m)$. Therefore $w_{\tau \vee \tau'} = \max(w_\tau, w_{\tau'})$. Every $\tau \in S$ has a unique complement $\tau^*$ whose weight sequence is obtained by: if $w_\tau(m) = w_{T''}(m)$ then $w_{\tau^*}(m) = w_{T'}(m)$ and if $w_\tau(m) = w_{T'}(m)$ then $w_{\tau^*}(m) = w_{T''}(m)$. Indeed we have $\tau \wedge \tau^* = T''$ and $\tau \vee \tau^* = T'$. Let us prove now distributivity, *i.e.* $(\tau \wedge \tau') \vee (\tau \wedge \tau'') = \tau \wedge (\tau' \vee \tau'')$ for all $\tau, \tau', \tau'' \in S$. Using weight sequences, this is equivalent to

$$\max(\min(w_\tau, w_{\tau'}), \min(w_\tau, w_{\tau''})) = \min(w_\tau, \max(w_{\tau'}, w_{\tau''})).$$

Since $\{w_\tau(m), w_{\tau'}(m), w_{\tau''}(m)\} = \{w_{T''}(m), w_{T'}(m)\}$ for all $m \in [1, n]$, the study of eight distinct cases shows that the equality holds.

THEOREM 4.2: *For two trees $T$ and $T' \in B_n$ such that $T \xrightarrow{*} T'$, we have $\mu(T, T') = 0$ if $T$ is not the meet of some of the trees which are covered by $T'$ and $\mu(T, T') = (-1)^k$ if $T$ is the meet of k distinct trees which are covered by $T'$. In particular we have $\mu(0, 1) = (-1)^{n-1}$ in $B_n$.*

*Proof*: In the lattice $[T, T']$ which is a sublattice of $B_n$, define $\bar{\tau}$ to be the meet of all trees which are covered by $T'$ and which dominate $\tau$. Then it is easy to see that $\tau \to \bar{\tau}$ is a closure relation with the property that $\bar{\tau} = T'$ only if $\tau = T'$. Furthermore, the set of closed elements is the Boolean lattice $S$ defined in lemma 4.1. Applying the closure theorem, then $\mu(T, T') = 0$ if $\bar{T} > T$, *i.e.* if $T$ is not the meet of trees which are covered by $T'$ (*see* also [8]) and $\mu(T, T') = \mu_s(T, T')$ if $\bar{T} = T$, *i.e.* if $T$ is the meet of trees which are covered by $T'$. Applying the principle of inclusion-exclusion, we have

$\mu_s(T, T')=(-1)^{h(T)-h(T')}$ where $h(T)$ is the height of $T$ in $S$. In $S$ we have $h(T)=0$ and $h(T')=k$ if there are $k$ distinct trees which are covered by $T'$.

*Remark*: The dual of theorem 4.2 using joins is also true. We choosed meets rather than joins because $w_{T \wedge T'}$ is more easy to compute that $w_{T \vee T'}$.

## 5. COMPUTING THE MÖBIUS FUNCTION OF $B_n$

Given $T$ and $T' \in B_n$ and applying theorem 2.1, $T \overset{*}{\rightarrow} T'$ does not hold if there exists an $i \in [1, n]$ such that $w_T(i) > w_{T'}(i)$. In this case $\mu(T, T')=0$. If $T \overset{*}{\rightarrow} T'$ holds, *i.e.* if $w_T \leqq w_{T'}$, the number of trees which are covered by $T'$ is the number of integers $w_{T'}(i)$ which are different from 1. For such an integer $w_{T'}(i) \neq 1$ we can compute the weight sequence $w$ of the corresponding tree which is covered by $T'$ in the following way:

$$w := w_{T'}$$
$$j := i - w(i) + 1$$
$$q := \max\{p \in [j, i-1] \mid j = p - w(p) + 1\}$$
$$w(i) := w(i) - q + j - 1$$

We only retain the $k$ trees $T_j$ such that $T \overset{*}{\rightarrow} T_j$, *i.e.* $w_T \leqq w_{T_j}$. If $T = T_1 \wedge T_2 \wedge \ldots \wedge T_k$, *i.e.* $w_T = \min(w_{T_1}, w_{T_2}, \ldots, w_{T_k})$, then

$$\mu(T, T')=(-1)^k \text{ else } \mu(T, T')=0.$$

## 6. COMPLEXITY OF THE ALGORITHM

The work done for computing $\mu(T, T')$ is proportional to the number of integer comparisons. In the average case, there are $n/2$ integers $i$ such that $w_{T'}(i) \neq 1$. In the worst case where $T'=1$, there are $n-1$ integers $i$ such that $w_{T'}(i) \neq 1$. For each integer $i$ such that $w_{T'}(i) \neq 1$, we need $w_{T'}(i)-2$ comparisons to compute $w_\tau(i) = w_{T'}(i) - q + j - 1$ where $\tau$ is covered by $T'$ since

$$q = \max\{p \in [i - w_{T'}(i) + 1, i-1] \mid i - w_{T'}(i) = p - w_{T'}(p)\}.$$

Compute the average value $a_n$ of $w_{T'}(i) \neq 1$ for $T' \in B_n$. The $b_n$ weight sequences of length $n$ have $(n+1) b_n/2$ elements equal to 1 and $(n-1) b_n/2$ elements different from 1. Let us define $c_n(k)$ be the number of elements equal to $k$ $(2 \leqq k \leqq n)$ in all these sequences. Then we have $a_n = p_n/((n-1)\, b_n/2)$ where

$$p_n = \sum_{i=2}^{n} i c_n(i).$$

Let $T_1$ be a tree of $B_n$ whose weight sequence has its $k$-th element equal to $i (2 \leqq i \leqq k \leqq n)$. If we delete in $T_1$ the subtree of weight $i$ whose last leaf is $k$, we obtain a tree $T_2$ of $B_{n-i+1}$ such that $w_{T_2}(k-i+1)=1$. There are $b_{i-1}$ trees of weight $i$. In the $b_{n-i+1}$ weight sequences of length $n-i+1$, there are $(n-i+2)\, b_{n-i+1}/2$ elements equal to 1. Therefore we get $c_n(i) = (n-i+2)\, b_{n-i+1}\, b_{i-1}/2$ and

$$p_n = 1/2 \sum_{i=2}^{n} i(n-i+2)\, b_{n-i+1}\, b_{i-1} = 1/2 \sum_{i=1}^{n-1} \binom{2i}{i} \binom{2n-2i}{n-i}.$$

The equality $\sum_{i=0}^{n} \binom{2i}{i} \binom{2n-2i}{n-i} = 2^{2n}$ holds ([14], p. 86). Thus $a_n = (2^{2n} - 2(n+1)\, b_n)/((n-1)\, b_n)$. Since $b_n \approx 4^n/((n+1)\sqrt{\pi n})$, $a_n$ asymptotically behaves as $\sqrt{\pi n}$. Computing $w_\tau(i)$ requires $O(n^{1/2})$ comparisons in average and $n-2$ comparisons in the worst case. For each $\tau$ among $O(n)$ trees which are covered by $T'$, one must verify if the equality $w_T \leqq w_\tau$ holds. For this, $n$ comparisons are required in general. However here, only one comparison is needed since $T \xrightarrow{*} T'$ and $w_\tau$ is known to differ $w_{T'}$ in exactly one known place, which is one of the integers $i$ such that $w_{T'}(i) \neq 1$. It suffices to check this index to make the necessary comparison. Therefore the average time complexity for computing $\mu(T, T')$ is $O(n^{3/2})$ and $O(n^2)$ in the worst case. The space complexity is clearly $O(n)$.

## 7. CONCLUSION

We have presented an efficient algorithm for computing the Möbius function of the rotation lattice $B_n$ with values in $[-1, 0, +1]$. Though $B_n$ is not a modular lattice, it is a surprising fact that its Möbius function can be evaluated as if $B_n$ was distributive.

Rotation has been generalized to regular $k$-ary trees [2]. But for $k \geqq 3$, the poset obtained is not a lattice. It seems very difficult to compute its Möbius function.

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