INFORMATIQUE THÉORIQUE ET APPLICATIONS

LEO CACCIARI

On the automorphism group of a toroidal hypermap

Informatique théorique et applications, tome 26, nº 6 (1992), p. 523-540

<http://www.numdam.org/item?id=ITA_1992__26_6_523_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol. 26, n° 6, 1992, p. 523 à 540)

ON THE AUTOMORPHISM GROUP OF A TOROIDAL HYPERMAP (*)

by Leo CACCIARI [1]

Communicated by R. CORI

Abstract. – *We prove that a finite group is (isomorphic to) an automorphism group of a toroidal hypermap if and only if it is abelian of rank at most two or it can be decomposed as the semi-direct product of a cyclic group of small order (namely 2, 3, 4 or 6) and such an abelian group. Moreover for any group arising this way we construct a toroidal hypermap having it as its full automorphism group.*

Résumé. – *Nous démontrons qu'un groupe fini est (isomorphe à) un groupe d'automorphisme d'une hypercarte toroïdale si et seulement si il est abelian de rang au plus deux ou bien il peut être décomposé dans le produit semi-direct d'un groupe cyclique d'ordre 2, 3, 4 ou 6 et d'un groupe abelian de rang au plus deux. De plus nous allons construire pour chacun de ces groupes une hypercarte toroïdale dont il est le groupe d'automorphisme.*

1. INTRODUCTION

In [4] it is proved that for any (finite) group G there exists a hypermap $\mathscr{H}$ such that $\mathrm{Aut}(\mathscr{H})$, the full automorphism group of $\mathscr{H}$, is (isomorphic to) G. It is then interestig to ask, for a given non-negative integer g which groups can arise as the full automorphism group of a hypermap of genus g and, more generaly, which groups can arise as automorphism group of some hypermap of genus g.

The case $g=0$ has a complete answer: the groups that can arise as the full automorphism group of a planar (*i. e.*, having genus 0) hypermap are precisely the polyhedral groups [5].

In the case $g \geqq 2$ an important result is the theorem of Hurwitz (*see* [11]) which states

(*) Received January 1991, accepted December 1991.
[1] L.I.T.A. Dép. Informatique, U.F.R. des Sciences et Techniques, Université de Pau et des Pays de l'Adour, avenue de l'Université, 64000 Pau, France.

FACT 1: *Let $\mathscr{H}$ be a hypermap of genus $g \geqq 2$ then:*

$$|\operatorname{Aut}(\mathscr{H})| \leqq 84(g-1)$$

In then follows that it is possible at least in principle to find all the groups that are the full automorphism group of some hypermap of genus $g \geqq 2$.

The case $g=1$, *i.e.* that of toroidal hypermaps, was settled by the author in [2]. Here we present a condensed form of these results. We will show that the groups that can be automorphism groups of a toroidal hypermap are either finite abelian groups of rank at most two or are decomposable as a semi-direct product of a cyclic group of small order (namely 2, 3, 4 or 6) and such an abelian group. Moreover we will show that each of these groups actually arise as the full automorphism group of some toroidal hypermap.

In proving our theorem we will use a technique introduced by Jones and Singerman [10] and based on Fuchsian groups. The needed facts about those groups will be recalled in the sequel but we will omit proofs.

2. HYPERMAPS AND AUTOMORPHISMS

A *hypermap* $\mathscr{H}$ is a triple (B, σ, α) where B is a finite set of *brins* and σ, α are two permutations of B such that the groupe $\langle \sigma, \alpha \rangle$ is transitive on B. In the special case where α is a fixed point free involution then $\mathscr{H}$ is a *map*. From the geometrical viewpoint it can be shown that maps and hypermaps give a representation of graphs and hypergraphs, respectively, in an orientable surface ([3], [6]).

This geometrical interpretation of hypermaps explains the reason for calling the cycles of α, σ and $\sigma\alpha$ the edges, vertex and faces of $\mathscr{H}$ respectively. Again from the geometrical interpretation of hypermaps we have the following definition

DEFINITION 1: *Let $\mathscr{H}=(B, \sigma, \alpha)$ be a hypermap. The* genus *of $\mathscr{H}$ is given by the following Euler's formula:*

$$g(\mathscr{H}) := |B| - z(\alpha) - z(\sigma) - z(\alpha\sigma), \tag{1}$$

where for any permutation γ, $z(\gamma)$ denotes the number of cycles of γ, including the trivial ones.

In can be proved that $g(\mathscr{H})$ is the genus of the surface in which $\mathscr{H}$ defines a embedding; it then follows that $g(\mathscr{H})$ is a non-negative integer. This last result can also be proved by combinatorial means [9]. A hypermap is said to be *planar* if $g(\mathscr{H})=0$ and to be *toroidal* if $g(\mathscr{H})=1$.

Geometrically an *automorphism* of a hypermap $\mathscr{H}=(B, \sigma, \alpha)$ can be defined as an automorphism of the underlying hypergraph preserving the embedding defined by $\mathscr{H}$. This can be proved to be equivalent to the following combinatorial definition.

DEFINITION 2: *An* automorphism *of the hypermap* $\mathscr{H}=(B, \sigma, \alpha)$ *is a permutation* φ *of* B *commuting with both* α *and* σ*:*

$$\varphi\sigma=\sigma\varphi, \qquad \varphi\alpha=\alpha\varphi$$

The full automorphism group of $\mathscr{H}$, denoted by $\mathrm{Aut}(\mathscr{H})$, is nothing else than the centralizer of $\langle \sigma, \alpha \rangle$ in $\mathrm{Sym}(B)$, the symmetric group over B. The transitivity of $\langle \sigma, \alpha \rangle$ implies that $\mathrm{Aut}(\mathscr{H})$ is semiregular. By an *automorphism group* of $\mathscr{H}$ we mean a subgroup of $\mathrm{Aut}(\mathscr{H})$.

PROPOSITION 1: *Let* $b_0 \in B$ *be a brin of the hypermap* $\mathscr{H}=(B, \sigma, \alpha)$ *and let* $G=\langle \sigma, \alpha \rangle$. *Then*

$$\mathrm{Aut}(\mathscr{H}) \cong \frac{\mathbf{N}_G(G_{b_0})}{G_{b_0}}$$

where $\mathbf{N}_G(G_{b_0})$ *is the normalizer in* G *of the stabilizer* G_{b_0} *of* $b_0 \in B$.

Proof: As G is transitive over B, for each $b \in B$ there exists $\gamma_b \in G$ such that $\gamma_b(b_0)=b$. Let φ be in $\mathbf{N}_G(G_{b_0})$. Then

$$\begin{aligned} \hat{\varphi}: \quad & B \to B \\ & b \to \gamma_b(\varphi^{-1}(b_0)) \end{aligned}$$

gives a well-defined permutation of B. Moreover $\hat{\varphi} \in \mathrm{Aut}(\mathscr{H})$ and $\widehat{\varphi\psi}=\hat{\varphi}\hat{\psi}$ so that

$$\begin{aligned} \wedge: \quad & \mathbf{N}_G(G_{b_0}) \to \mathrm{Aut}(\mathscr{H}) \\ & \varphi \quad \to \quad \hat{\varphi} \end{aligned}$$

defines a morphism. As $\mathrm{Aut}(\mathscr{H})$ is semi-regular the kernel of $\wedge$ is G_{b_0}. Conversely let $\tau \in \mathrm{Aut}(\mathscr{H})$ and let $\varphi \in G$ be such that $\varphi^{-1}(b_0)=\tau(b_0)$. We claim that $\varphi \in \mathbf{N}_G(G_{b_0})$ and that $\hat{\varphi}=\tau$. In fact if γ is in G_{b_0} we have

$$\varphi\gamma\varphi^{-1}(b_0)=\varphi\gamma(\tau(b_0))=\varphi\tau(\gamma(b_0))=\varphi(\tau(b_0))=\varphi(\varphi^{-1}(b_0))=b_0$$

proving that $\varphi\gamma\varphi^{-1} \in G_{b_0}$ and so that $\varphi \in \mathbf{N}_G(G_{b_0})$. Moreover for any brin b we have

$$\hat{\varphi}(b) = \gamma_b(\varphi^{-1}(b_0)) = \gamma_b(\tau(b_0)) = \tau(\gamma_b(b_0)) = \tau(b_0),$$

proving that $\hat{\varphi} = \tau$.

COROLLARY: *With the same notation as in Proposition* 1 *there is a bijection between the automorphism groups of* $\mathscr{H}$ *and the subgroups* N *of* G *such that* $G_{b_0} \trianglelefteq N$. *Namely if* $N \leqq G$ *is such that* $G_{b_0} \trianglelefteq N$ *then* N/G_{b_0} *is (isomorphic to) a subgroup of* $Aut(\mathscr{H})$ *and all subgroups of* $Aut(\mathscr{H})$ *arise this way.*

Let us recall the following standard result of group theory

FACT 2: *Let* $\pi : \Gamma \to G$ *be a group homomorphism of a group* Γ *onto a group* G. *Let* H *be a subgroup of* G *and* $T = \pi^{-1}(H)$ *is inverse image. Then there is a bijection between the subgroups of* Γ *containing* T *and those of* G *containing* H *given by*

$$K \to \pi(K)$$

Moreover $T \trianglelefteq K$ *if and only if* $H \trianglelefteq \pi(K)$ *and in this case*

$$\frac{K}{T} \cong \frac{\pi(K)}{H}.$$

Let now $\mathscr{H} = (B, \sigma, \alpha)$ be a hypermap and let A be a subgroup of $\mathrm{Aut}(\mathscr{H})$. Clearly $\langle \sigma, \alpha \rangle$ acts transitively on the set B/A of all A-class of brins (that is

$$B/A = \{[b] \,/\, b \in B\}$$

where

$$[b] = \{\varphi(b) \,/\, \varphi \in A\}$$

is the set of all brins A-equivalent to b). This means that the triple $\mathscr{H}' = (B/A, \sigma', \alpha')$ (where σ' and α' are defined by

$$\sigma'([b]) = [\sigma(b)],\ \alpha'([b]) = [\alpha(b)]$$

for all $[b] \in B/A$) is a hypermap. $\mathscr{H}'$ is called the *quotient hypermap of* $\mathscr{H}$ w.r.t. *the automorphism group* A and will the denoted by $\mathscr{H}/A$.

The following definition is a generalization of an idea of Jones and Singerman [10].

DEFINITION 3: *Let l, m and n by three integers such that the orders of α, σ, and $\sigma\alpha$ divides l, m and n respectively. Then the group*

$$(0;\, l,\, m,\, n) := \langle x, y, z;\; x^l = y^m = z^n = x^{-1} y^{-1} z^{-1} = 1 \rangle \tag{2}$$

is a covering of $\mathscr{H}$.

Let $\Gamma = (0;\, l,\, m,\, n)$ be a covering of $\mathscr{H}$. Then from the definition of covering and from elementary properties of free groups (*see* [12], p. 15) it follows that there exists a (unique) group homomorphism π of Γ onto $\langle \sigma, \alpha \rangle$ defined by

$$\pi(x) = \alpha. \qquad \pi(y) = \sigma, \qquad \pi(z) = (\sigma\alpha)^{-1}. \tag{3}$$

The projection π defined by (3) will be referred to as the *natural projection* of the covering Γ.

Let now b be a brin of $\mathscr{H}$ and let G_b be the stabilizer of b in $\langle \sigma, \alpha \rangle$

$$G_b = \{\gamma \in \langle \alpha, \sigma \rangle / \gamma(b) = b\}.$$

The subgroup $H = \pi^{-1}(G_b)$ will be called the *group of $\mathscr{H}$* w.r.t. the covering Γ and the base point b, and denoted by $H = \Gamma_{\mathscr{H},\, b}$ or simply by $H = \Gamma_b$.

By Fact 2 and the proof of Proposition 1 we obtain the following result

PROPOSITION 2: *Let $\mathscr{H}$ be a hypermap, b one of its brins and Γ be a covering of $\mathscr{H}$. Let A be an automorphism group of $\mathscr{H}$. Then there exists $N \leqq \Gamma$ such that $\Gamma_b \trianglelefteq N$ and*

$$A \cong \frac{N}{\Gamma_b}.$$

Moreover Γ is also a covering of $\mathscr{H}' = \mathscr{H}/A$ and $N = \Gamma_{[b],\, \mathscr{H}'}$.

As a special case of Proposition 2 we have:

$$\mathrm{Aut}(\mathscr{H}) \cong \frac{N_\Gamma(\Gamma_b)}{\Gamma_b}.$$

3. FUCHSIAN GROUPS

A group like $(0;\, l,\, m,\, n)$ is called a *triangular group*. It is a special case of a *Fuchsian group*.

DEFINITION 4: *A* Fuchsian group *of genus* $g \geqq 0$ *and of periods* $m_1, m_2, \ldots, m_t$ ($t \geqq 0$, $m_i \geqq 1$ *for* $i = 1, 2, \ldots, t$) *is a group having a presentation of the form*

$$\langle x_1, \ldots, x_t, a_1, b_1, \ldots, a_g, b_g; \\ x_1^{m_1}, \ldots, x_t^{m_t}, x_1^{-1} \ldots x_t^{-1} [a_1, b_1] \ldots [a_g, b_g] \rangle \quad (4)$$

where $[a, b]$ *is the commutator* $a^{-1} b^{-1} ab$.

Various facts are known about Fuchsian groups. We recall here the most significant for our work. A nice combinatorial proof of Facts 3 to 5 below can be found in ([7, 8]).

FACT 3: *Any element of finite order in a Fuchsian group is conjugated to a power of a generator of finite order.*

FACT 4: *Two Fuchsian groups are isomorphics if and only if they have the same genus and the same periods.*

Fact 4 allows us to write $(g; m_1, \ldots, m_t)$ to denote the Fuchsian group of genus g and periods m_1 to m_t. Moreover it tells us that the following defines an invariant of a Fuchsian group.

DEFINITION 5: *Let* $\Gamma = (g; m_1, \ldots, m_t)$ *be a Fuchsian group. The* measure *of* Γ *is defined by:*

$$\mu(\Gamma) = 2g - \sum_{i=1}^{t} \left(1 - \frac{1}{m_i}\right)$$

FACT 5: *Let* Γ *be as above and let* H *be a subgroup of* Γ *having finite index in* Γ. *Then the following hold:*

1. H *is itself a Fuchsian group* $(g'; m_{1,1}, \ldots, m_{1,s_1}, \ldots, m_{t,1}, \ldots, m_{t,s_t})$;
2. *the* Riemann-Hurwitz formula:

$$\mu(H) = [\Gamma : H] \mu(\Gamma). \quad (5)$$

holds;

3. H *as a presentation*

$$\langle x_{i,j} (i = 1, \ldots, t, j = 1, \ldots, s_i), a_1, b_1, \ldots, a_{g'}, b_{g'}; \\ x_{i,j}^{m_{ij}}, x_{1,1}^{-1} x_{1,2}^{-1} \ldots x_{t,s_t}^{-1} [a_1, b_1] \ldots [a_{g'}, b_{g'}] \rangle$$

where

$$x_{i,j}=k^{-1}\,x_i^{m_i/m_{ij}}\,k.$$

Note that 2, implies that $g' \leqq g$.

The following result, which is due to Singerman [13], gives the relationship between Fuchsian groups and permutation groups.

PROPOSITION 3: *Let* Γ *be the Fuchsian group* $(g;\, m_1, \ldots, m_t)$. *Then* Γ *has the Fuchsian group* $H=(g';\, m_{1,1}, \ldots, m_{1,s_1}, \ldots, m_{t,1}, \ldots m_{t,s_t})$ *as a subgroup of finite index if and only if* (5) *holds and there exists a transitive permutation group* G *of degre* n *and a homomorphism* $\pi:\Gamma \rightarrow G$ *such that the permutation* $\xi_i=\pi(x_i)$ *has exactly* s_i *non-trivial cycles of length less than* m_i, *these lengths being*

$$\frac{m_i}{m_{i,1}},\ \frac{m_i}{m_{i,2}},\ \ldots,\ \frac{m_i}{m_{i,s_i}}.$$

Sketch of the proof (see [13] for details): If Γ has a subgroup like H, let act Γ on the set Γ/H of the coset modulo H. This give us the group we are looking for in the form Γ/K (where K is the intersection of all subgroup of Γ conjugate to H).

Conversely let H be the inverse image by π of G_1; one can then shown that H has the prescribed presentation.

COROLLARY: *A triangular group* $(0;\, l, m, n)$ *has a subgroup of finite index like* $(g;\, l_1, \ldots, l_p, m_1, \ldots, m_q, n_1, \ldots, n_r)$ *if and only if there exists some hypermap* $\mathscr{H}=(B, \sigma, \alpha)$ *of genus* g *such that* α *(resp.* σ, $\sigma\alpha$*) has* p *(resp.* q, r*) cycles of length less than* l *(resp. less than* m, *less than* n*) these lengths being* $l/l_1, \ldots, l/l_p$ *(resp.* $m/m_1, \ldots, m/m_q$; $n/n_1, \ldots, n/n_r$*).*

Proof: The proof is an application of Proposition 3. A straightforward calculation proves that $g=g(\mathscr{H})$.

We shall examine now a special case of Proposition 3 which will be needed in the sequel.

Let $H \trianglelefteq \Gamma$. As Γ acts regularly on the quotient group Γ/H we have, for $i=1, 2, \ldots, t$,

$$m_{i,1}=m_{i,2}=\ldots=m_{i,s_i}=m'_i$$

and, if $\pi : \Gamma \rightarrow \Gamma/H$ is the natural projection, the order k_i of $\xi_i = \pi(x_i)$ is given by

$$k_i = \frac{m_i}{m'_i}.$$

Moreover, we have

$$s_i = \frac{[\Gamma : H]}{k_i}.$$

With the above notations we obtain

$$\mu(H) = 2g' - 2 - [\Gamma : H] \left(\sum_{i=1}^{t} \left(1 - \frac{1}{k_i}\right) - \sum_{i=1}^{t} \left(1 - \frac{1}{m_i}\right)\right) \tag{6}$$

and substituting (6) in (5) we get

$$2g' - 2 = [\Gamma : H] \left(2g - 2 + \sum_{i=1}^{t} \left(1 - \frac{1}{k_i}\right)\right) \tag{7}$$

As a final remark, if Γ and H are as above then

$$\frac{\Gamma}{H} \cong \frac{\Gamma'}{H'}$$

where Γ' is the Fuchsian group $(g;\ k_1, \ldots, k_t)$ and H' has no elements of finite order [it then follows from Facts 3 and 5 that H is the surface group $(g';\ -)$].

4. THE MAIN THEOREM

Before stating and proving our theorem we need same more result about the relationship between Fuchsian group and hypermaps.

THEOREM 1: *Let $\mathscr{H} = (B, \sigma, \alpha)$ be a hypermap and let $A \leq \mathrm{Aut}(\mathscr{H})$ be an automorphism group of $\mathscr{H}$. If $g(\mathscr{H}) = g$ and $g(\mathscr{H}/A) = g'$ then A can be expressed as a quotient Γ/H where*

$$\Gamma = \langle x_1, \ldots, x_t, a_1, b_1, \ldots, a_{g'}, b_{g'};\ x_i^{k_i}\ (i = 1, \ldots, t),\ x_1^{-1} \ldots x_t^{-1} [a_1, b_1] \ldots [a_{g'}, b_{g'}] \rangle$$

and H is a (normal) subgroup of Γ in the form

$$\langle c_1, d_1, \ldots, c_g, d_g;\ [c_1, d_1] \ldots [c_g, d_g] \rangle.$$

Moreover the following holds

$$2g-2=[\Gamma:\mathrm{H}]\left(2g'-2+\sum_{i=1}^{t}\left(1-\frac{1}{k_i}\right)\right) \tag{8}$$

It should be remarked that formula (8) looks like the formula given in [11]. This is not a coincidence. It can be proved that formula (8) is nothing else that the one in [11] obtained by other means.

Let now $\mathscr{H}=(B, \sigma, \alpha)$ be toroidal so that in Theorem 1 we have $g=1$ and $g'=0$ or 1. From (8) we get that $\mu(\Gamma)$ must be 0. So the first step towards the main theorem is to find all Fuchsian group Γ such that $\mu(\Gamma)=0$.

PROPOSITION 4: *Let Γ be a fuchsian group such that $\mu(\Gamma)=0$. Then Γ is one of the following fuchsian groups* (see *above for notation*)*:* $(1; -)$, $(0; 2, 2, 2, 2)$, $(0; 2, 4, 4)$, $(0; 2, 3, 6)$ *or* $(0; 3, 3, 3)$.

Proof: Straightforward calculation.

THEOREM 2: *If $\mathscr{H}=(B, \sigma, \alpha)$ is a toroidal map and A an automorphism group of $\mathscr{H}$ such that $\mathscr{H}/A$ is still toroidal, then A is isomorphic to the following group*

$$G(h, q)=C_h \times C_{qh},$$

where h and q are integers $\geqq 1$.

Proof: We known that A is isomorphic to a quotient of $(1; -)$ so that A is a finite abelian group of rank at most two. The proposition then follows from the structure theorem for finite abelian groups.

Let $\Gamma_1=\langle a, b; [a, b]\rangle$ be a presentation of $(1; -)$ and let h_1, h_2, k_1 and k_2 be integers. By basic properties of free abelian groups there is a unique endomorphism $\xi:\Gamma_1 \rightarrow \Gamma_1$ induced by

$$\xi(a)=a^{h_1} b^{k_1}; \qquad \xi(b)=a^{h_2} b^{k_2}.$$

Moreover ξ is an automorphism if and only if

$$\begin{vmatrix} h_1 & k_1 \\ h_2 & k_2 \end{vmatrix} = \pm 1.$$

This means that we can identify $\mathrm{Aut}(\Gamma_1)$ with $\mathrm{GL}(2, \mathbb{Z})$ (*see* [12], p. 169).

The next step is to prove that the groups $(0; 2, 2, 2, 2)$, $(0; 3, 3, 3)$, $(0; 2, 4, 4,)$ and $(0; 2, 3, 6)$ can be written as a semi-direct product $\mathrm{C} \times_{\varphi} \Gamma_1$ where C is a cyclic group of order 2, 3, 4 and 6 respectively. It is well known that in $\mathrm{Aut}(\Gamma_1)$ the only elements of finite order have order 2, 3, 4 or 6 so these are the only semi-direct product whith cyclic finite group one can form.

Let

$$\xi_2=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix};\quad \xi_3=\begin{pmatrix}-1 & -1\\ 0 & 0\end{pmatrix};$$

$$\xi_4=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix};\quad \xi_6=\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix};$$

It is immediate to verify that under the identification given above ξ_n gives an automorphisms of Γ_1 of order $n\,(n=2, 3, 4, 6)$.

The following theorem can be found, in a slaightly different form, in the classical work of Burnside ([1], p. 410 ff).

Proposition 5: *Let* Γ_2, Γ_3, Γ_4 *and* Γ_6 *the groups* (0; 2, 2, 2, 2), (0; 3, 3, 3), (0; 2, 4, 4) *and* (0; 2, 3, 6) *respectively. Then*

$$\Gamma_n \cong C_n \times_{\Phi_n} \Gamma_1 \tag{9}$$

where $C_n=\langle x;\, x^n\rangle$ *and*

$$\Phi_n:\quad C_n \to \mathrm{Aut}(\Gamma_1)$$
$$x \to \xi_n$$

for $n=2, 3, 4, 6$.

Proof: Let us start with the case of Γ_2. This group has a presentation of the form

$$\langle x_1, x_2, x_3, x_4;\, x_i^2,\, i=1, \ldots, 4,\, x_1 x_2 x_3 x_4\rangle. \tag{10}$$

As all the relators in (10) have even length we may divide the elements of Γ_2 into two subset: those that can be write as the product of an even number of generators and those which need an odd number of generators. The elements of even length form a subgroup we will denote by E. As $[\Gamma_2 : E]=2$, E is characteristic in Γ_2. It can be proved that E is generated by the products $x_1 x_2$ and $x_2 x_3$ and that

$$\psi(a)=x_1 x_2;\qquad \psi(b)=x_2 x_3$$

gives an isomorphism of Γ_1 onto E. To prove this it is enough to remark that

$$[x_1 x_2,\, x_2 x_3]=(x_1 x_2)^{-1}(x_2 x_3)^{-1}(x_1 x_2)(x_2 x_3)=1_{\Gamma_2}$$

proving that E is an abelian Fuchsian group. As E is of infinite order it must be isomorphic to Γ_1. It can then be proved that Γ_2 splits in E and C, where C is the cyclic group of order two generated by x_2. Moreover as

$$x_2(x_1 x_2)x_2 = x_2 x_1 = (x_1 x_2)^{-1}; \qquad x_2(x_2 x_3)x_2 = x_3 x_2 = (x_2 x_3)^{-1}$$

we get (9) for $n=2$. (We remark that E contains all the elements of infinite order of Γ_2. In fact if $f \notin E$ then $f = x_2 h$, some $h \in E$, but then

$$f^2 = (x_2 h)(x_2 h) = h^{-1} h = 1_{\Gamma_2}$$

and f has order 2).

Let us take for Γ_3, Γ_4 and Γ_6 the presentation given by (2) and let T_n $(n=2, 3, 6)$ be the subgroup of Γ_n whose generators a and b are given in Table I below. By a straightforward calculation it can be proved that T_n is isomorphic to Γ_1 under the obvious isomorphism.

TABLE I

n	a	b
3	xy^{-1}	yxy
4	xy^2	yxy
6	$y^{-1}z^2$	$zy^{-1}z$

To prove that $T_n \trianglelefteq \Gamma_n$ one can show that T_n is formed by all the elements of infinite order of Γ_n and so it is characteristic in Γ_n. To finish the proof of (9) one shows that Γ_n split in T_n and the cyclic group of order n generated by y for $n=3$, 4 and by z for $n=6$.

Note that, as it follows from the proof of Proposition 5, an element γ of $\Gamma_n \cong C_n \times_{\Phi_n} \Gamma_1$ is of infinite order if and only if $\gamma \in \Gamma_1$. As we are looking for quotients of Γ_n by some normal subgroup H isomorphic to Γ_1, and so having only elements of infinite order, we are led to look for the subgroups of Γ_1 that are

1. of finite index in Γ_1,
2. ξ_n-invariant
3. isomorphics to Γ_1.

If $H \leqq \Gamma_1$ satisfies 1 to 3 above then we can form the quotient Γ_n/H which turns out to be

$$C_n \times_\varphi \frac{\Gamma_1}{H}$$

where φ is obtained from Φ_n in the obvious way.

LEMMA 1: *Let T be a subgroup of finite index in $\Gamma_1 = \langle a, b; [a, b] \rangle$. Let n be the least positive integer such that $b^n \in T$ and h be the least positive integer such that $a^h b^k \in T$, some $k \in \mathbb{Z}$. Then T is generated by b^n and $a^h b^k$.*

Proof: Let $b^m \in T$, then $m = qn$. Let $m = qn + r$, with $0 \leqq r < n$, we have

$$b^r = b^m (b^n)^{-q} \in T$$

which is impossible by the choice of n unless $r = 0$.

Let now $a^x b^y \in T$ and let $x = qh + r$ with $0 \leqq r < h$. Then

$$a^r b^{y-qk} = (a^x b^y)(a^h b^k)^{-q} \in \mathrm{T}$$

which implies $r = 0$, so that we get $b^{y-qk} \in T$ and so $y - qk = q' n$ which in turn gives us

$$a^x b^y = (a^h b^k)^{-q} (b^n)^{q'}$$

proving that $a^h b^k$ and b^n generate T.

COROLLARY: *If T is a finite index subgroup of Γ_1 then T is isomorphic to Γ_1.*

This means that condition 3 above is redundant.

LEMMA 2: *Let T be a subgroup of finite index of Γ_1, then T is ξ_2-invariant.*

Proof: This is straightforward as ξ_2 sends any element of Γ_1 into its inverse.

LEMMA 3: *Let T a subgroup of finite index of Γ_1 and let T be generated by $a^h b^k$ and b^n as in Lemma 1. Let T be ξ-invariant where ξ is one of ξ_3, ξ_4 or ξ_6. Then there exist $q, q' \in \mathbb{Z}$ such that $k = qh$ and $n = q' h$.*

Proof: Let us prove the lemma for $\xi = \xi_3$, the proof for the other cases being similar.

Suppose that $k = qh + r$, where $0 \leqq r < h$. As T is ξ_3-invariant we have $\xi_3 (a^h b^k) \in T$ which means that

$$a^r b^{-h-(q-1)k} = (\xi_3 (a^h b^k))(a^h b^k)^{-(q-1)} \in T$$

This can only happen if $r = 0$, that is $k = qh$.

Let now $n = q' h + r$, $0 \leqq r < h$. We have

$$a^r b^{-q' k} = (\xi_3 (b^n))(a^h b^k)^{-q'} \in T$$

which implies $r = 0$ and so $n = q' h$.

By Lemma 1 we have that is T is of finite index in $\Gamma_1 = \langle a, b; [a, b] \rangle$ then $G = \Gamma_1/T$ has the following presentation

$$G = \langle a, b; a^h b^k, b^n, [a, b] \rangle.$$

If now T is ξ-invariant for ξ one of ξ_3, ξ_4 or ξ_6 we have that

$$G = \langle b, c; c^h, b^{q'h}, [b, c] \rangle = C_h \times C_{q'h}.$$

where $c = ab^q$. Now T is ξ-invariant ($\xi = \xi_3$, ξ_4 or ξ_6) if and only if ξ induces the indentity on G. Table 2 gives the effect of ξ_3, ξ_4 and ξ_6 on the generators c and b of G.

TABLE II

ξ	$\xi(c)$	$\xi(b)$
ξ_3	$c^{q-1} b^{-(q-1)^2-q}$	cb^{-q}
ξ_4	$c^q b^{-(q^2+1)}$	cb^{-q}
ξ_6	b	cb^{-q^1}

Then it is simply a matter of calculation to prove that:

T is ξ_3-invariant if and only if $q' \mid (q^2+1)$,

T is ξ_4-invariant if and only if $q' \mid (q^2-q+1)$,

T is ξ_6-invariant if and only if $q' \mid (q^2+q+1)$,

Putting together the results obtained so far we have

THEOREM 3: *Let T be a subgroup of Γ_n ($n=2$, 3, 4 or 6) having finite index in Γ_n and being isomorphic to Γ_1. Let G be the quotient group Γ_n/H. Then*

1. *If $n=2$ then G is isomorphic to*

$$G(2, h, q) = C_2 \times_{\varphi 2} (C_h \times C_{qh}),$$

where $h, q \geqq 1$ and φ_2 sends the generator of C_2 into the isomorphism ξ_2 of $C_h \times C_{qh}$ defined by

$$\xi_2(x) = x^{-1}$$

for all $x \in C_h \times C_{qh}$.

2. *If $n=3$ then G is isomorphic to*

$$G(3, h, q, q') = C_3 \times_{\varphi_{3,q}} (C_h \times C_{q'h}),$$

where $q' \mid (q^2+1)$ and $\varphi_{3,q}$ sends the generator of C_3 into the automorphism of $C_h \times C_{q'h}$ defined by first row of Table I.

3. *If $n=4$ G is isomorphic to*

$$G(4, h, q, q')=C_4 \times_{\varphi_{4,q}} (C_h \times C_{q'h}),$$

where $q' \mid (q^2-q+1)$ and $\varphi_{4,q}$ sends the generators of C_4 into the automorphism of $C_h \times C_{q'h}$ defined by second row of Table I.

4. *Finally, if $n=6$ G is isomorphic to*

$$G(6, h, q, q')=C_6 \times_{\varphi_{6,q}} (C_h \times C_{q'h}),$$

where $q' \mid (q^2+q+1)$ and $\varphi_{6,q}$ sends the generators of C_6 into the automorphism of $C_h \times C_{q'h}$ defined by third row of Table I.

THEOREM 4: *Let $\mathscr{H}$ be a toroidal hypermap and $A \leq \mathrm{Aut}(\mathscr{H})$. Then A is (isomorphic to) one of $G(h, q)$, $G(2, h, q)$ $G(3, h, q, q')$, $G(4, h, q, q')$ or $G(6, h, q, q')$.*

Proof: This follows immediately from Theorems 1, 2 and 3 and Proposition 2 above.

Notice that Theorem 4 only gives a necessary condition for a finite group to be the full automorphism group of a toroidal hypermap. We still have to prove that these groups actually arise as the full automorphism group of some toroidal hypermap. This will be done in the following section.

5. THE HYPERMAPS

In the present section we will show that for each of the groups of Theorem 4 it can be constructed a hypermap having it as its full automorphism group. This will complete the proof of the following theorem:

THEOREM 5: *Let A be a finite group. There exists a toroidal hypermap $\mathscr{H}$ such that A is (isomorphic to) a subgroup of* Aut$(\mathscr{H})$ *if and only if A is (isomorphic to) one of $G(h, q)$, $G(2, h, q)$, $G(3, h, q, q')$, $G(4, h, q, q')$ or $G(6, h, q, q')$. Moreover is A is one of the groups listed above then there exists a toroidal hypermap whose full automorphism group is (isomorphic to) A.*

Proof: First of all note that Γ_3, Γ_4 and Γ_6 all are triangular group and that $G(n, h, q, q')$ is obtained as a quotient of Γ_n by a normal subgroup H of finite index isomorphic to Γ_1. It follows from Proposition 2 and from the corollary to Proposition 3 that setting $B=\Gamma_n/H$ and letting α (resp. σ) be the permutation of B induced by the generator x (resp. y) of Γ_n gives us a hypermap $\mathscr{H}(n, h, q, q')$ such that

$$\mathrm{Aut}(\mathscr{H}(n, h, q, q')) = \frac{\Gamma_n}{H} = G(n, h, q, q').$$

setting the problem for the last three groups.

The proof of the theorem for the remaining groups is a straigthforward consequence of Lemmas 5 and 6 below.

LEMMA 4: *Let $\mathscr{H} = (B, \sigma, \alpha)$ be a hypermap and A an automorphism group of $\mathscr{H}$. The cardinality of B is a multiple of that of A.*

Proof: This is obvious since A acts semi-regularly on B with orbit lenght $|A|$.

PROPOSITION 6: *With the same notations as in Lemma* 4 *if the ratio $|B|/|A|$ is a prime number then either $A = \mathrm{Aut}(\mathscr{H})$ or $\mathrm{Aut}(\mathscr{H})$ is transitive on B.*

For any $h, q \geqq 1$ define $\mathscr{H}(h, q)$ *as follows: the set of brins is*

$$\mathrm{B} = \mathbb{Z}_h \times \mathbb{Z}_{qh} \times \mathbb{Z}_5;$$

the vertices are given by

$$\sigma((i, j, k)) = (i, j, k+1)$$

and the edges by

$$\alpha((i, j, k)) = \begin{cases} (i, j, 1) & \text{if } k=0, \\ (i, j+1, 3) & \text{if } k=1, \\ (i+1, j, 4) & \text{if } k=2, \\ (i, j-1, 0) & \text{if } k=3, \\ (i, -1, j, 2) & \text{if } k=4. \end{cases}$$

LEMMA 5: *For any $h, q \geqq 1$ we have*

$$\mathrm{Aut}(\mathscr{H}(h, q)) \cong G(h, q).$$

Proof: Let τ and ρ be defined as follows for any $(i, j, k) \in B$:

$$\rho((i, j, k)) = (i+1, j, k), \qquad \tau((i, j, k)) = (i, j+1, k). \tag{11}$$

It's easy to see that $\tau, \rho \in \mathrm{Aut}(\mathscr{H}(h, q))$ and that $[\tau, \rho] = \mathrm{id}_B$. Moreover for any x, y $(0 \leqq x < h, 0 \leqq y < qh)$ we have

$$\rho^x \tau^y((i, j, k)) = (i+x, j+y, k) \neq (i, j, k),$$

proving that $\rho^x \tau^y \neq \mathrm{id}_B$. Also the order of ρ is h and that of τ is qh so that $a \to \rho$, $b \to \tau$ gives an isomorphism of $G(h, q) = \langle a, b;\, a^h, b^{qh}, [a, b] \rangle$ into $\mathrm{Aut}(\mathscr{H}(h, q))$. To prove that this isomorphism is onto let A be the subgroup of $\operatorname{Aut}(\mathscr{H})$ generated by ρ and τ. We have $|B|/|A|=5$ so that, by Proposition 6 either A is all of $\operatorname{Aut}(\mathscr{H}(h, q))$ or $\operatorname{Aut}(\mathscr{H}(h, q))$ acts transitively on B. Now $\mathscr{H}(h, q)$ has $h^2 q$ edges of length 3 and the same number of edges of length 2 and no automorphism can send a brin belonging to an edge of length 2 into a brin belonging to an edge of length 3. So $\operatorname{Aut}(\mathscr{H}(h, q))$ can't be transitive on B and the lemma is proved.

Let h, q still denote two positive integers. We can define a hypermap $\mathscr{H}(2, h, q)$, whose automorphism group will turn out to be $G(2, h, q)$, as follows: the brin set is

$$B=\mathbb{Z}_h \times \mathbb{Z}_{qh} \times \mathbb{Z}_6;$$

the vertices are given by

$$\sigma((i, j, k))=(i, j, k+1)$$

and the edges by

$$\alpha((i, j, k))=\begin{cases} (i, j, 1) & \text{if } k=0, \\ (i, j+1, 3) & \text{if } k=1, \\ (i+1, j, 5) & \text{if } k=2, \\ (i, j, 4) & \text{if } k=3, \\ (i, j-1, 0) & \text{if } k=4, \\ (i, j, 2) & \text{if } k=5. \end{cases}$$

LEMMA 6: *For any $h, q \geqq 1$ we have*

$$\operatorname{Aut}(\mathscr{H}(2, h, q)) \cong G(2, h, q).$$

Proof: As in the proof of Lemma 5 the permutations of B defined by (11) form a subgroup of $\operatorname{Aut}(\mathscr{H}(2, h, q))$ isomorphic to $G(h, q)$. Let now ξ be defined by

$$\xi((i, j, k))=(-i, -j, k+3).$$

Then ξ is an automorphism of $\mathscr{H}(2, h, q)$ of order two. Moreover if x, y are integers such that $0 \leqq x<h$ and $0 \leqq qh$ we have

$$\xi \rho^x \tau^y((i, j, k))=(-(i+x), -(j+y), k+3) \neq (i, j, k).$$

Finaly $\xi\rho\xi=\rho^{-1}$ and $\xi\tau\xi=\tau^{-1}$ proving that $a \to \rho$, $b \to \tau$ and $x \to \xi$ define an isomorphism of $G(2, h, q)=\langle x, a, b;\, x^2, a^h, b^{qh}, [a, b], (xa)^2, (xb)^2 \rangle$ into $\operatorname{Aut}(\mathscr{H}(2, h, q))$. To prove that this isomorphism is onto let A be the subgroup of $\operatorname{Aut}(\mathscr{H}(2, h, q))$ generated by ξ, ρ and τ, then $|B|/|A|=3$ so that we are left with two possibility: either A is all of $\operatorname{Aut}(\mathscr{H}(2, h, q))$ or

Aut$(\mathscr{H}(2, h, q))$ is transitive on B. Now if $h \neq 2$ there cannot exists an automorphism carrying $(i, j, 0)$ into $(i, j, 2)$ as the edge containing the formed as length 4 and the one containing the latter has length $2h \neq 4$. On the other side if $h = 2$ then it is impossible to find an automorphism carrying $(i, j, 0)$ into $(i, j, 1)$ as the face containing the former as length 4 and the one containing the latter has length 2.

6. CONCLUSIONS

We have proved that the finite groups that can arise as the automorphism group of some toroidal hypermap are of a really simple structure. Moreover there exist infinitely many such groups, like in the planar case and unlike the case of genus greater than 1 (*see* Fact 1).

Some work is left. For instance can it be proved that the five families of groups described above are disjoint (exept for triviality like $G(2, 1, 1)$ being isomorphic to $G(1, 2)$)?

Another interesting question is to find a proof of our theorem by elementary means, that is by arguments like the ones used in [5] to settle the planar case.

ACKNOWLEDGEMENTS

This paper was partialy done as a part of collaboration program of the Mathematics dep. of University of Rome "La Sapienza" and the Computer Sciences dep. of the University of Bordeaux-I.

We also wish to thanks Professor A. Machì who sugested us this work and Professor Cori who supervised our thesis [2].

REFERENCES

1. W. Burnside, *Theory of Groups of Finite Order*, 2nd Ed., 1911.
2. L. Cacciari, Hypercartes Combinatoires : Automorphismes et Systèmes d'Équations Algébriques, *Thèse de doctorat*, Université Bordeaux-I, Bordeaux, 1989.
3. R. Cori, Un code pour les cartes planaires et ses applications, *Astérisque*, 1975, *27*, Paris.
4. R. Cori and A. Machì, Construction of Maps with Prescribed Automorphism Group, *T.C.S.*, 1982, *21*, pp. 91-98.
5. R. Cori, A. Machì, J. G. Penaud and B. Vauquelin, On the Automorphism Group of a Planar Hypermap, *E. J. Combinatorics*, 1981, *2*, pp. 331-334.
6. J. Edmonds, A Combinatorial Representation for Polyedral Surfaces, *Notices Amer. Math. Soc.*, 1960, *7*, p. 646.

7. A. H. M. Hoare, A. Karras and D. Solitar, Subgroups of Finite Index of Fuchsian Group, *Math. Z.*, 1971, *120*, pp. 289-298.
8. A. H. M. Hoare, A. Karras and D. Solitar, Subgroups of Infinite Index of Fuchsian Group, *Math. Z.*, 1972, *125*, pp. 59-69.
9. A. Jacques, Constellations et propriétés algébriques des graphes topologiques, *Ph. D. Thesis*, 1969, Paris.
10. G. A. Jones and D. Singerman, Theory of Maps on Orientable Surfaces, *Proc. London Math. Soc.*, (3), 1978, *37*, pp. 273-307.
11. A. Machì, The Riemann-Hurwitz formula for the centralizer of a pair of permutations, *Arch. Math.*, 1984, *42*, pp. 280-288.
12. W. Magnus, A. Karras and D. Solitar, Combinatorial group Theory 2nd Ed., *Dover Pub. Inc.* New York, 1976.
13. D. Singerman, Subgroups of Fuchsian Group and Finite Permutations Groups, *Bull. Lond. Math. Soc.*, 1970, *2*, pp. 319-323.