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COMPOSITIONAL REPRESENTATION OF RATIONAL FUNCTIONS (*)

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Abstract. – *The rational functions are shown to coincide with the compositions of endmarkings, morphisms and inverses of injective morphisms. To represent a rational function* τ *we need one ednmarking* μ_m, *two morphisms* α_1, α_3 *and one inverse of an injective morphism* α_2 *and then* $\tau = \mu_m \alpha_1 \alpha_2^{-1} \alpha_3$.

Résumé. – *On montre que les fonctions rationnelles coïncident avec les compositions de marquages terminaux, morphismes et inverses de morphismes injectifs. Plus précisément, toute fonction rationnelle* τ *se représente sous la forme* $\tau = \mu_m \alpha_1 \alpha_2^{-1} \alpha_3$ *où* μ_m *est un marquage terminal,* α_1, α_3 *sont des morphismes et* α_2 *est un morphisme injectif.*

1. INTRODUCTION

In [5] and [9] it has been shown that a mapping τ is a rational transduction if and only if it has a factorization of the form $x\tau = (xm)\sigma$ for all x. Here m is an endmarker symbol and σ is a composition of morphisms and inverse morphisms. The endmarker can in general not be avoided, *see* [7] and [4]. The transductions realized by simple transducers can, however, be represented without the use of endmarkers. In [10] and in [7] it has been shown that they form precisely the class of mappings representable by a composition of morphisms and inverse morphisms. In addition it is known from [7] and [8] that such compositions can be assumed to be of length four. As a matter

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of fact length four is necessary and sufficient to characterize the whole class of these compositions.

In this paper we investigate the effect on the compositional representations when the class of rational transductions is restricted to the class of rational functions. We are in particular interested in obtaining an "injective" compositional characterization. It turns out that indeed the inverse morphisms can be replaced by inverses of injective morphisms. Moreover, in this case length three is necessary and sufficient to characterize the whole class of compositions of morphisms and inverses of injective morphisms. This class turns out to coincide with the rational functions realizable by simple transducers and having a free monoid as their domain.

In the case of rational functions that are realized by simple transducers we lack a characterization in terms of naturally arising compositions. In fact, these functions do not seem to possess such a characterization.

2. PRELIMINARIES

We refer to [1] and [3] for the preliminary results on automata theory. For convenience we give here some notations and terminology which come into use in this paper.

A *transducer* $T=(Q_T, \Sigma_T, \Delta_T, \delta_T, q_T, F_T)$ consists of a finite set Q_T of states, input alphabet Σ_T, output alphabet Δ_T, finite transition relation $\delta_T \subseteq Q_T \times \Sigma_T^* \times \Delta_T^* \times Q_T$, initial state q_T, and final state set F_T. The morphisms $I_T\colon \delta_T^* \to \Sigma_T^*$ and $W_T\colon \delta_T^* \to \Delta_T^*$ are defined by $(q, x, y, p) I_T = x$ and $(q, x, y, p) W_T = y$ for all $(q, x, y, p) \in \delta_T$. A sequence g of transitions (q_i, x_i, y_i, q_{i+1}), $i=1, 2, \ldots, k$, is a *computation* of T from q_1 to q_{k+1} and it produces the output $g W_T = y_1 y_2 \ldots y_k$ from the input $g I_T = x_1 x_2 \ldots x_k$. For $q, p \in Q_T$, $\mathscr{C}_T(q, p)$ denotes the set of all computations of T from q to p. By convention, the empty computation is an element of $\mathscr{C}_T(q, q)$ for all states q. The set of all computations of T forms a regular subset $\mathscr{C}_T$ of δ_T^*. Further, a computation $g \in \mathscr{C}_T$ is *accepting* if $g \in \mathscr{C}_T(q_T, p)$ for some $p \in F_T$. The set of all accepting computations of T forms a regular subset $\mathscr{A}_T$ of δ_T^*.

Let T be a transducer defined as above. T *realizes* the rational transduction $\tau_T \subseteq \Sigma_T^* \times \Delta_T^*$ defined by

$$\tau_T = \{(g I_T, g W_T) \mid g \in \mathscr{A}_T\}.$$

By $x \tau_T$ we denote the set $\{y \mid (x, y) \in \tau_T\}$. The *domain* of a rational transduction τ_T is the regular subset $\mathbf{dom}(\tau_T) = \{x \mid x \tau_T \neq \varnothing\}$ of Σ_T^*. Similarly,

$\mathbf{ran}(\tau_T)=\{y \mid y \in x\tau_T \text{ for some } x\}$. In this paper we consider only transducers with nonempty domains. This is not a restriction to our results.

The transducer T is said to realize a *rational function* τ_T if τ_T is a partial function from Σ_T^* into Δ_T^*, that is, if $\mathbf{card}(x\tau_T) \leqq 1$ for all $x \in \Sigma_T^*$.

The transducer T is *unambiguous* if for all $x \in \Sigma_T^*$ there exists at most one accepting computation $g \in \mathscr{A}_T$ with $g I_T = x$. Clearly the rational transduction realized by an unambiguous transducer is a rational function.

A transducer T is called *simple* if $F_T=\{q_T\}$, that is, if the unique final state of T equals the initial state of T. If T is simple, then clearly $(x_1\tau_T)(x_2\tau_T) \subseteqq (x_1 x_2)\tau_T$. whenever $x_i \in \mathbf{dom}(\tau_T)$ for $i=1, 2$. Moreover, if T is simple and τ_T is a partial function then the domain of τ_T is a monoid. Also, in this case $\mathbf{1}\tau_T=\mathbf{1}$ and $(x_1 x_2)\tau_T=(x_1\tau_T)(x_2\tau_T)$ for all x_1, $x_2 \in \mathbf{dom}(\tau_T)$.

We use $\mathscr{T}$ ($\mathscr{F}$, respectively) to denote the family of all rational transductions (rational functions, respectively). $\mathscr{T}_*$ ($\mathscr{F}_*$, respectively) is the family of all rational transductions (rational functions, respectively) realizable by simple transducers. Further, $\mathscr{U}$ ($\mathscr{U}_*$, respectively) denotes the family of all rational transductions realizable by unambiguous (simple unambiguous, respectively) transducers.

A *finite automaton* A is identified with a transducer which has a transition relation $\delta_A \subseteqq Q_A \times \Sigma_A \times \{\mathbf{1}\} \times Q_A$.

3. NORMALIZED TRANSDUCERS

In this section we establish some normal form results for transducers and in particular for transducers which realize rational functions. Emphasis is on the elimination of transitions which read the empty word $\mathbf{1}$.

For an arbitrary transducer T we may assume without loss of generality that T satisfies the following five conditions.

(1) $\delta_T \subseteqq Q_T \times (\Sigma_T \cup \{\mathbf{1}\}) \times \Delta_T^* \times Q_T$,

(2) for all $q \in Q_T$, there is a computation $g_1 g_2 \in \mathscr{A}_T$ such that $g_1 \in \mathscr{C}_T(q_T, q)$ and $g_2 \in \mathscr{C}_T(q, p)$ for some $p \in F_T$,

(3) either T is simple or $(q, x, y, p) \in \delta_T$ implies that $p \neq q_T$,

(4) $(q, \mathbf{1}, y, p) \in \delta_T$ implies that either $y \neq \mathbf{1}$ or $q=q_T$ and $p \neq q_T$.

(5) F_T is a singleton and either T is simple or $(q, x, y, p) \in \delta_T$ implies that $q \notin F_T$.

These conditions are valid also for the unambiguous transducers. We omit here the straightforward proofs of these normal form results.

Let then T be a transducer which realizes a rational function. By the above T may be assumed to satisfy the conditions (1)-(5). Since τ_T is a partial function, there is a bound on the lengths of the computations $g \in \mathscr{C}_T$ for which $g I_T = \mathbf{1}$. This is because otherwise, by (4), there would be a $q \in Q_T$ and a computation $g_1 \in \mathscr{C}_T(q, q)$ such that $g_1 I_T = \mathbf{1}$ and $g_1 W_T \neq \mathbf{1}$. Hence, by (2), for some computations g_0, g_2 we would have $g_0 g_1 g_2$, $g_0 g_2 \in \mathscr{A}_T$ with $(g_0 g_1 g_2) W_T \neq (g_0 g_2) W_T$, which would contradict the functionality of τ_T. After this observation we can continue by standard methods and eliminate the transitions $(q, \mathbf{1}, y, p)$ of T for which $q \neq q_T$ or $p \notin F_T$ by combining them with the neighbouring transitions. Hence the transitions that read the empty word are of the form $(q_T, \mathbf{1}, y, p)$, where $p \in F_T - \{q_T\}$. The resulting transducer is equivalent to T and satisfies (1), (3), (4) and (5). Condition (2) is easily restored. If T is unambiguous (simple) then the new transducer is also unambiguous (simple).

All this leads to the conclusion that a transducer T realizing a rational function can be assumed to satisfy the conditions (1)-(6), where

(6) if $(q, \mathbf{1}, y, p) \in \delta_T$, then $q = q_T$, $q_T \notin F_T$, and $F_T = \{p\}$.

Thus transducers realizing rational functions do not need more than one transition reading the empty word. In particular, simple transducers realizing rational functions can be assumed to have no transitions reading the empty word.

A transducer T realizing a rational function and satisfying the conditions (1)-(6) is called *normalized*. Our observations lead to the following conclusion.

LEMMA 3.1: *Let T be a (simple, unambiguous) transducer realizing a rational function. Then a normalized (simple, unambiguous) transducer realizing τ_T can be constructed.*

Unambiguous transducers realize rational functions. Conversely, in [1] and [3] it was shown by using the cross section theorem that each rational function can be realized by an unambiguous transducer. Our representation results are based on the structure of the transducers realizing rational functions. For this reason we reprove this unambiguity result by purely automata theoretic means.

LEMMA 3.2: *Let $\tau \in \mathscr{F}$. Then an unambiguous normalized transducer realizing τ can be constructed.*

Proof: Let T be a normalized transducer realizing $\tau: \Sigma^* \to \Delta^*$ and let $Q_T = \{q_0, \ldots, q_n\}$, where $q_T = q_0$. Define a new transducer V by letting $Q_V = Q_T \times \mathscr{P}$, where $\mathscr{P}$ is the family of all subsets of Q_T: for each $(q, S) \in Q_V$ and $a \in \Sigma \cup \{1\}$ let

$$((q, S), a, y, (q_i, R)) \in \delta_V$$

if and only if $(q, a, y, q_i) \in \delta_T$ and

$$R = \bigcup_{p \in S} \{r \mid (p, a, u, r) \in \delta_T \text{ for some } u \in \Delta^*\}$$
$$\cup \{q_j \mid (q, a, v, q_j) \in \delta_T \text{ for } j < i \text{ and for some } v \in \Delta^*\}.$$

Finally, let $q_V = (q_T, \varnothing)$ and $F_V = \{(p, R) \mid p \in F_T, R \cap F_T = \varnothing\}$.

Let $\rho: \delta_V^* \to \delta_T^*$ be a morphism defined by $((q, S), a, y, (p, R))\rho = (q, a, y, p)$ for all $((q, S), a, y, (p, R)) \in \delta_V$. Then $g\rho \in \mathscr{A}_T$ for all $g \in \mathscr{A}_V$ and thus $\tau_V \subseteq \tau_T$. On the other hand let $h \in \mathscr{A}_T$ be a minimal computation of T for a given input x and an output $y = x\tau_T$ according to the ordering of the sates q_i, $i = 0, 1, \ldots, n$, of T. By the construction of V there is a unique computation $g \in \mathscr{A}_V$ such that $g\rho = h$. Thus $\tau_T \subseteq \tau_V$. Clearly V is unambiguous.

Finally we normalize the transduceer V. This process preserves the unambiguity. In fact, we need only to normalize V with respect to the condition (2). □

Note that the construction given in the above proof does not preserve simplicity. Hence we have obtained (only) that $\mathscr{F} = \mathscr{U}$.

THEOREM 3.1: *Let $\tau \in \mathscr{T}$. Then $\tau \in \mathscr{F}$ if and only if it can be realized by an unambiguous normalized transducer.*

4. RATIONAL COMPOSITIONS AND RATIONAL FUNCTIONS

Let $\mathscr{H}$ ($\mathscr{H}_I$, respectively) be the family of all morphisms (injective morphisms, respectively) between finitely generated word monoids, and let $\mathscr{H}^{-1}$ ($\mathscr{H}_I^{-1}$, respectively) be the set of the inverse of morphisms from $\mathscr{H}$ ($\mathscr{H}_I$, respectively).

A *morphic composition* τ is a composition of morphisms and inverses of morphisms between word monoids, $\tau = \alpha_1^{\varepsilon_1} \ldots \alpha_n^{\varepsilon_n}$, where $\varepsilon_i = 1$ or -1 for all $i = 1, 2, \ldots, n$. The family of the morphic compositions is denoted by $(\mathscr{H} \cup \mathscr{H}^{-1})^*$ and the family of all morphic compositions with injective morphisms as inverses is denoted by $(\mathscr{H} \cup \mathscr{H}_I^{-1})^*$.

A *marker* μ_m is a mapping which sets a special symbol m at the end of each word, that is, $\mu_m : \Sigma^* \to (\Sigma \cup \{m\})^*$ is defined by $x\mu_m = xm$ for all $x \in \Sigma^*$. We denote by $\mathscr{M}$ the family of all markers.

Compositions of markers and morphic compositions are called *rational compositions*. It is clear that every rational composition is a rational transduction. Also the converse holds, *see* [5], [9].

The following was proved in [7], [8], and [10].

THEOREM 4.1: *(a)* $\mathscr{T} = \mathscr{M}\mathscr{H}^{-1}\mathscr{H}\mathscr{H}^{-1}\mathscr{H} = (\mathscr{M} \cup \mathscr{H} \cup \mathscr{H}^{-1})^*$.

(b) $\mathscr{T}_* = \mathscr{H}^{-1}\mathscr{H}\mathscr{H}^{-1}\mathscr{H} = \mathscr{H}\mathscr{H}^{-1}\mathscr{H}\mathscr{H}^{-1} = (\mathscr{H} \cup \mathscr{H}^{-1})^*$.

We adapt now the general idea (*see* [7] or [10]) behind the construction of equivalent rational compositions for transducers in order to obtain an "injective" representation for rational functions.

LEMMA 4.1: *For each $\tau_T \in \mathscr{F}$ there is a $\tau_V \in \mathscr{F}$. such that $\tau_T = \mu_m . \tau_V$, where μ_m is a marker.*

Proof: The transducer T can be assumed to be unambiguous and normalized. Let m be a new symbol. Define $V = (Q_T, \Sigma_T \cup \{m\}, \Delta_T, \delta_V, q_T, \{q_T\})$, by $\delta_V = \delta_T \cup \{(p, m, \mathbf{1}, q_T)\}$, where $F_T = \{p\}$. Clearly, V is simple, normalized and unambiguous. Since in T either $q_T = p$ or q_T has no in-coming transitions and p has no out-going transitions, it follows that $\mathbf{dom}(\tau_V) = ((\mathbf{dom}(\tau_T)) . m)^*$, and $(xm)\tau_V = x\tau_T$ for all $x \in \mathbf{dom}(\tau_T)$. $\square$

Let T be a normalized simple transducer. Let $Q_T = \{q_0, \ldots, q_n\}$ with $q_0 = q_T$. Let $\Sigma = \Sigma_T \cup \{d\}$, where d is a new symbol. Define

(i) $\alpha_1 : \Sigma_T^* \to \Sigma^*$ by $a\alpha_1 = ad^{2^n - 1}$ for all $a \in \Sigma_T$;

(ii) $\alpha_2 : \delta_T^* \to \Sigma^*$ by $(q_i, a, y, q_j)\alpha_2 = d^{2^i - 1} a d^{2^n - 2^j}$ for all $(q_i, a, y, q_j) \in \delta_T$;

(iii) $\alpha_3 : \delta_T^* \to \Delta_T^*$ by $(q_i, a, y, q_j)\alpha_3 = y$ for all $(q_i, a, y, q_j) \in \delta_T$.

Since T has no transitions reading the empty word we immediately obtain

LEMMA 4.2: *Let T be a normalized simple transducer and let the morphisms α_i, $i = 1, 2, 3$, be defined for T as above. Then $\tau_T = \alpha_1 \alpha_2^{-1} \alpha_3$.*

Clearly, for every T the morphism α_1 is injective. In the case of a normalized unambiguous simple transducer also α_2 is injective.

LEMMA 4.3: *Let T be a normalized unambiguous simple transducer. Then the morphism α_2 defined above for T is injective.*

Proof: Let $w_1 \in \delta_T^*$ be a word of minimal length such that there exists another word $w_2 \in \delta_T^*$ with $w_1 \neq w_2$ and $w_1\alpha_2 = w_2\alpha_2$. Clearly $w_1\alpha_2 \neq \mathbf{1}$. Let $w_1 = e_1 e_2 \ldots e_t$ and $w_2 = f_1 f_2 \ldots f_k$ for some $t, k \geq 1$ and $e_i, f_i \in \delta_T$ with $e_1 \neq f_1$.

Let $e_i=(q_{g_1(i)}, a_i, y_i, q_{g_2(i)})$ and $f_i=(q_{h_1(i)}, b_i, z_i, q_{h_2(i)})$. It follows immediately that $t=k$ and $a_i=b_i$ for all $1 \leqq i \leqq t$. Further, $q_{g_1(1)}=q_{h_1(1)}$. Consider now the matched word between the letters a_i, a_{i+1} in $e_i e_{i+1} \alpha_2$ and $f_i f_{i+1} \alpha_2$. We obtain that $2^n-2^{g_2(i)}+2^{g_1(i+1)}-1=2^n-2^{h_2(i)}+2^{h_1(i+1)}-1$, from which it follows that either $q_{g_2(i)}=q_{h_2(i)}$ and $q_{g_1(i+1)}=q_{h_1(i+1)}$ or $q_{g_2(i)}=q_{g_1(i+1)}$ and $q_{h_2(i)}=q_{h_1(i+1)}$. The first of these alternatives would imply, however, that $(e_1 e_2 \ldots e_i) \alpha_2=(f_1 f_2 \ldots f_i) \alpha_2$ contradicting the minimality of w_1. Hence we may assume that for all $i<t$, $q_{g_2(i)}=q_{g_1(i+1)}$ and $q_{h_2(i)}=q_{h_1(i+1)}$. This means that w_1 and w_2 are computations of T from $q_{g_1(1)}$ to $q_{g_2(t)}$. Since T is unambiguous and satisfies the condition (2), it must be that $w_1=w_2$. This contradiction completes the proof. $\square$

On the other hand if τ is a rational composition with injective morphisms as its inverses then clearly τ is a rational function. Thus we have obtained

THEOREM 4.2: $\mathscr{F}=\mathscr{U}=\mathscr{M}\mathscr{H}_I\mathscr{H}_I^{-1}\mathscr{H}=(\mathscr{M}\cup\mathscr{H}\cup\mathscr{H}_I^{-1})^*$.

In contrast to the general case described in Theorem 4.1, the above theorem does not state that the absence/presence of markers characterizes the difference between the compositional representation of $\mathscr{F}$ and the compositional representation of $\mathscr{F}_*$. In the next section we investige $(\mathscr{H}\cup\mathscr{H}_I^{-1})^*$ and its relationship with $\mathscr{F}_*$.

5. MORPHIC COMPOSITIONS AND RATIONAL FUNCTIONS

From the observations in the previous section it follows that $\mathscr{U}_* \subseteqq (\mathscr{H}\cup\mathscr{H}_I^{-1})^* \subseteqq \mathscr{F}_*$. In Section 3 the inclusion $\mathscr{F} \subseteqq \mathscr{U}$ has been proved, but the construction there does not preserve simplicity. Indeed, not every rational function realizable by a simple transducer can be realized by an unambiguous simple transducer. It turns out that $\mathscr{U}_*=(\mathscr{H}\cup\mathscr{H}_I^{-1})^*$, which is precisely the class of simple rational functions with a free monoid as their domain. Since every regular set of the form R^* is a domain of a transducer from $\mathscr{F}_*$, $\mathscr{F}_*$ strictly includes $\mathscr{U}_*$.

From [2], p. 188, it follows that the functions from $\mathscr{U}_*$ have a free monoid as their domain. We shall now prove that also the functions from $(\mathscr{H}\cup\mathscr{H}_I^{-1})^*$ enjoy this property. In fact we prove a more general result based on the notion of an unambiguous composition.

Let $\tau=\alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\ldots\alpha_n^{\varepsilon_n}$ $(\varepsilon_i=\pm 1)$ be a morphic composition and let us denote $\tau_i=\alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\ldots\alpha_i^{\varepsilon_i}$ for $i=1, 2, \ldots, n$. We shall say that τ is *unambiguous* if $\mathscr{U}\tau_i \cap \mathrm{dom}(\alpha_{i+1}^{\varepsilon_{i+1}}\ldots\alpha_n^{\varepsilon_n})$ is a singleton for all $u \in \mathbf{dom}(\tau)$, $i=1, 2, \ldots, n$. (Here an empty composition is identified with the identity function.) Clearly, every unambiguous morphic composition is a rational function.

LEMMA 5.1: *Let τ be an unambiguous morphic composition. Then* **dom** (τ) *is a free monoid.*

Proof: Let $\tau=\alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\ldots\alpha_n^{\varepsilon_n}$, $\varepsilon_i=\pm 1$, be an unambiguous morphic composition. In order to prove that the domain is free we need to show that the condition $uw=wv\in\mathbf{dom}(\tau)$ for $u, v\in\mathbf{dom}(\tau)$ implies $w\in\mathbf{dom}(\tau)$, *see* e. g. [6], p. 106. Let

$$u\tau_i\cap\mathbf{dom}(\alpha_{i+1}^{\varepsilon_{i+1}}\ldots\alpha_n^{\varepsilon_n})=\{u_i\},\qquad v\tau_i\cap\mathbf{dom}(\alpha_{i+1}^{\varepsilon_{i+1}}\ldots\alpha_n^{\varepsilon_n})=\{v_i\},$$
$$(uw)\tau_i\cap\mathbf{dom}(\alpha_{i+1}^{\varepsilon_{i+1}}\ldots\alpha_n^{\varepsilon_n})=\{z_i\}=(wv)\tau_i\cap\mathbf{dom}(\alpha_{i+1}^{\varepsilon_{i+1}}\ldots\alpha_n^{\varepsilon_n})$$

for $i=0, 1, \ldots, n$, where $u_0=u$, $v_0=v$ and $uw=z_0=wv$. Hence we have for $i=0, 1, \ldots, n-1$ either

(1) $\varepsilon_{i+1}=+1$ and $u_i\alpha_{i+1}=u_{i+1}$, $v_i\alpha_{i+1}=v_{i+1}$ and $z_i\alpha_{i+1}=z_{i+1}$, or

(2) $\varepsilon_{i+1}=-1$ and $u_{i+1}\alpha_{i+1}=u_i$, $v_{i+1}\alpha_{i+1}=v_i$ and $z_{i+1}\alpha_{i+1}=z_i$.

By unambiguity, $(uwv)\tau_i\cap\mathbf{dom}(\alpha_{i+1}^{\varepsilon_{i+1}}\ldots\alpha_n^{\varepsilon_n})=\{u_i z_i\}=\{z_i v_i\}$ for all i, and thus there are words r_i, s_i such that $u_i=r_i s_i$, $v_i=s_i r_i$ and $z_i=(r_i s_i)^{k_i} r_i$ for some integer $k_i\geqq 0$.

In case (1), $((r_i s_i)^{k_i} r_i)\alpha_{i+1}=(r_{i+1} s_{i+1})^{k_{i+1}} r_{i+1}$, where $(r_i s_i)\alpha_{i+1}=r_{i+1} s_{i+1}$, and thus $(r_{i+1} s_{i+1})^{k_{i+1}} r_{i+1}=(r_{i+1} s_{i+1})^{k_i}(r_i\alpha_{i+1})$. It follows that $r_i\alpha_{i+1}=r_{i+1}$ and further that $s_i\alpha_{i+1}=s_{i+1}$.

In case (2) a similar argumentation yields that

$$r_{i+1}\alpha_{i+1}=r_i \qquad\text{and}\qquad s_{i+1}\alpha_{i+1}=s_i.$$

Now, $w=(r_0 s_0)^{k_0-1} r_0$ and thus $w\in\mathbf{dom}(\tau)$. $\square$

From this lemma we obtain immediately

LEMMA 5.2: *If* $\tau\in(\mathscr{H}\cup\mathscr{H}_I^{-1})^*$, *then* **dom** (τ) *is a free monoid.*

In the other direction we have

LEMMA 5.3: *Let* $\tau\in\mathscr{F}_*$ *be such that* **dom** (τ) *is a free monoid. Then* $\tau\in\mathscr{U}_*$.

Proof: Let C be a code such that $\mathbf{dom}(\tau)=C^*$, and let T be a normalized simple transducer realizing τ. Since T is simple, it has no transitions reading **1**. From the cross section theorem it follows that there exists a regular set $R\subseteq CI_T^{-1}\cap\mathscr{A}_T$ such that I_T is injective on R and

$$RI_T=(CI_T^{-1}\cap\mathscr{A}_T)I_T=C\cap\mathscr{A}_T I_T=C.$$

Since C is a code and I_T is injective on R it follows that R is a code. Furthermore, I_T preserves the lengths of the words and thus I_T is injective on R^*.

Let D be an unambiguous simple finite automaton with $\Sigma_D = \delta_T$ that recognizes R^*. We refer to [2], p. 187, for the existence of such a finite automaton. From D a new transducer $V = (Q_D, \Sigma_T, \Delta_T, \delta_V, q_D, \{q_D\})$ is obtained by setting $\delta_V = \{(p, e I_T, e W_T, q) \mid (p, e, \mathbf{1}, q) \in \delta_D\}$.

Now, $\mathbf{dom}(\tau_V) = R^* I_T = C^*$ and hence $\mathbf{dom}(\tau_V) = \mathbf{dom}(\tau_T)$. Moreover, for all $x \in \mathbf{dom}(\tau_V)$ we have $x \tau_V \subseteqq x \tau_T$, and because τ_T is a function, this implies that $\tau_V = \tau_T$.

It remains to show that V is unambiguous. For this let $g_1, g_2 \in \mathscr{A}_V$ be such that $g_1 I_V = g_2 I_V$. By the definition of V there are $h_1, h_2 \in \mathscr{A}_D$ such that $(h_i I_D) I_T = g_i I_V$ and $(h_i I_D) W_T = g_i W_V$ for $i = 1, 2$. By the injectivity of I_T on R^*, $h_1 I_D = h_2 I_D$. Since D is unambiguous, $h_1 = h_2$, and thus $g_1 = g_2$ follows from the definition of δ_V. $\square$

By combining the above results we obtain

THEOREM 5.1: $\mathscr{U}_* = \mathscr{H}_I \mathscr{H}_I^{-1} \mathscr{H} = (\mathscr{H} \cup \mathscr{H}_I^{-1})^* = \{\tau \in \mathscr{F}_* \mid \mathbf{dom}(\tau)$ *a free monoid*$\}$.

Note that the condition $\tau \in \mathscr{F}_*$ in the above statement is necessary. For, even if a rational function τ with a free monoid as its domain has the property that $\mathbf{1}\tau \doteq \mathbf{1}$, in general it is not realizable by a simple transducer. As an example consider the nonsimple transducer T with transitions

$$\delta_T = \{(q_T, a, a, q_1), (q_1, b, a, q_1), (q_1, a, a, q_T), \\ (q_T, b, b, q_2), (q_2, a, b, q_2), (q_2, b, b, q_T)\},$$

and final states q_T, q_1, q_2. Here $\tau_T \in \mathscr{F}$ and $\mathbf{dom}(\tau_T) = \{a, b\}^*$. However, the conditions $a\tau = a$, $b\tau = b$, $ab\tau = aa$ imply that $\tau \notin \mathscr{F}_*$.

In conclusion $\mathscr{U}_* \subsetneqq \mathscr{F}_*$ but $\mathscr{U} = \mathscr{F} = \mathscr{M}\mathscr{U}_* = \mathscr{M}\mathscr{F}_*$.

6. HIERARCHIC RESULTS

In this chapter we present a hierarchy for the morphic compositions with injective morphisms as inverses according to their compositional length. We start with a general result which connects these compositions to the general morphic compositions for which a hierarchy was given in [7].

LEMMA 6.1: *For every transducer T there exist a morphism α and a rational function σ such that $\tau_T = \alpha^{-1}\sigma$. Moreover, if T is simple then σ is a morphic composition.*

Proof: By Nivat's Theorem each rational transduction τ_T can be represented as $I_T^{-1}(\bigcap \mathscr{A}_T)W_T$. The transduction $\sigma = (\bigcap \mathscr{A}_T)W_T$ is a rational function which can be realized by a simple transducer if T is simple. Hence by choosing $\alpha = I_T$ the claim follows. □

Using this result we prove

LEMMA 6.2: *$\mathscr{H}^{-1}\mathscr{H}\mathscr{H}^{-1}$ and $\mathscr{H}\mathscr{H}_I^{-1}\mathscr{H}$ are incomparable.*

Proof: Clearly, $\mathscr{H}^{-1}\mathscr{H}\mathscr{H}^{-1}$ is not contained in $\mathscr{H}\mathscr{H}_I^{-1}\mathscr{H}$, because the latter class consists of functions only.

It was shown in [7] that there is a morphic composition τ_T which is not in $\mathscr{H}^{-1}\mathscr{H}\mathscr{H}^{-1}$. By Lemma 6.1 there exist a morphism α and a rational function $\sigma \in (\mathscr{H} \cup \mathscr{H}^{-1})^*$ such that $\tau_T = \alpha^{-1}\sigma$. If $\sigma = \alpha_1^{-1}\alpha_2\alpha_3^{-1}$ for some morphisms α_i, $i = 1, 2, 3$, then $\tau_T = (\alpha_1\alpha)^{-1}\alpha_2\alpha_3^{-1}$ would be in $\mathscr{H}^{-1}\mathscr{H}\mathscr{H}^{-1}$ contradicting the assumption made for τ_T. Thus $\sigma \notin \mathscr{H}^{-1}\mathscr{H}\mathscr{H}^{-1}$. □

In particular, because $\mathscr{H}_I^{-1}\mathscr{H}\mathscr{H}_I^{-1} \subseteq \mathscr{H}\mathscr{H}_I^{-1}\mathscr{H}$ by Theorem 5.1, we obtain the following theorem.

THEOREM 6.1: *$\mathscr{H}_I^{-1}\mathscr{H}\mathscr{H}_I^{-1}$ is properly contained in $\mathscr{H}\mathscr{H}_I^{-1}\mathscr{H}$.*

It immediately follows from the preceding theorem that the sets $\mathscr{H}_I^{-1}\mathscr{H}$ and $\mathscr{H}\mathscr{H}_I^{-1}$ are incomparable. The conclusions are collected in Figure 6.1, where elements without a connecting line are incomparable as sets and a connecting line indicates that the lower class is properly included in the upper one.

In addition we have the following characterizations for four of the above classes.

THEOREM 6.2: (*a*) $\mathscr{H} = \{\tau \in \mathscr{F}_* \mid \mathbf{dom}(\tau) = \Sigma^*$ *for an alphabet* $\Sigma\}$.

(*b*) $\mathscr{H}_I^{-1} = \{\tau \in \mathscr{F}_* \mid \mathbf{ran}(\tau) = \Sigma^*, \mathbf{dom}(\tau) = C^*$ *with C a code such that* $\mathbf{card}(C) = \mathbf{card}(\Sigma)\}$.

(*c*) $\mathscr{H}_I^{-1}\mathscr{H} = \{\tau \in \mathscr{F}_* \mid \mathbf{dom}(\tau) = C^*$ *for a finite code* $C\}$.

(*d*) $\mathscr{H}\mathscr{H}_I^{-1}\mathscr{H} = \{\tau \in \mathscr{F}_* \mid \mathbf{dom}(\tau)$ *is a free monoid*$\}$.

Proof: (*a*) Immediate from the observations that $\mathbf{1}\tau = \mathbf{1}$ and $(uv)\tau = (u\tau)(v\tau)$ for all $u, v \in \Sigma^*$, whenever $\tau \in \mathscr{F}_*$ and $\mathbf{dom}(\tau) = \Sigma^*$.

(*b*) Immediate.

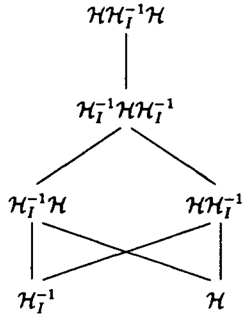

Figure 6.1

(*c*) Let $\tau=\alpha^{-1}\beta$ for an injective morphism $\alpha:\Gamma^*\to\Sigma^*$ and a morphism $\beta:\Gamma^*\to\Delta^*$. Then $\mathbf{dom}(\tau)=\Gamma^*\alpha=(\Gamma\alpha)^*$. Since α is injective, $\Gamma\alpha$ is a finite code.

Conversely, assume that $\mathbf{dom}(\tau)=C^*$ for a finite code

$$C=\{w_1, \ldots, w_n\}\subseteq\Sigma^*$$

and let $\Gamma=\{a_1, \ldots, a_n\}$ be an alphabet. Let $\alpha:\Gamma^*\to\Sigma^*$ be the morphism defined by $a_i\alpha=w_i$ for $i=1, \ldots, n$. Since C is a code, α is injective. Now, $\mathbf{dom}(\alpha\tau)=\Gamma^*$ which implies that $\alpha\tau\in\mathcal{H}$. Consequently, $\tau=\alpha^{-1}\alpha\tau\in\mathcal{H}_I^{-1}\mathcal{H}$.

(*d*) Follows immediately from Theorem 5.1. □

7. ON SIMPLE FUNCTIONS

What we are notably missing in the previous sections is a characterization of the simple rational functions $\mathcal{F}_*$ in terms of morphic compositions. Notice that we do have

$$\mathcal{F}_*=\mathcal{M}\mathcal{H}\mathcal{H}_I^{-1}\mathcal{H}\cap\mathcal{H}\mathcal{H}^{-1}\mathcal{H}=(\mathcal{M}\cup\mathcal{H}\cup\mathcal{H}_I^{-1})^*\cap(\mathcal{H}\cup\mathcal{H}^{-1})^*$$

by the previous results. We conjecture that there does not exist any "natural" morphic representation of $\mathcal{F}_*$. More precisely

CONJECTURE: $\mathcal{F}_*$ cannot be represented in the form $\mathcal{H}_1^{\varepsilon_1}\mathcal{H}_2^{\varepsilon_2}\ldots\mathcal{H}_k^{\varepsilon_k}$, where the morphism classes $\mathcal{H}_i$, $i=1, 2, \ldots, k$, are closed under renamings.

A *renaming* is an injective morphism $\alpha:\Sigma^*\to\Delta^*$ such that $a\alpha\in\Delta$ for all letters $a\in\Sigma$. Let $\mathcal{H}_r$ denote the class of renamings. A morphism class $\mathcal{H}$ is closed under renamings if $\mathcal{H}_r\mathcal{H}\subseteq\mathcal{H}$.

If we do not require that the morphism classes in the conjecture are closed under renaming then $\mathscr{F}_*$ can be dressed into a compositional form because every class of simple transductions has an "artificial" representation as morphic compositions. To see this let $\mathscr{K}$ be a class of simple transductions, say $\mathscr{K}=\{\tau_i \mid i=1, 2, \ldots\}$. Each τ_i has a representation as a morphic composition, $\tau_i=\alpha_{i1}^{-1}\alpha_{i2}\alpha_{i3}^{-1}\alpha_{i4}$. Moreover, these morphisms can be chosen in such a way that the domain alphabets of α_{ik} and α_{jk} are disjoint for all different i, j $(k=1, 2, 3, 4)$. Hence the morphisms connected to different transductions do not mix, and, indeed, $\mathscr{F}_*=\mathscr{H}_1^{-1}\mathscr{H}_2\mathscr{H}_3^{-1}\mathscr{H}_4$, where

$$\mathscr{H}_k=\{\alpha_{ik} \mid i=1, 2, \ldots\} \qquad \text{for} \quad k=1, 2, 3, 4.$$

However, if we require closure under renamings, then we can prove that $\mathscr{F}_*$ does not match with our compositional representations of length three.

THEOREM 7.1: *There are no classes $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ of morphisms such that $\mathscr{F}_*=\mathscr{H}_1^{\varepsilon_1}\mathscr{H}_2^{\varepsilon_2}\mathscr{H}_3^{\varepsilon_3}$ and $\mathscr{H}_3$ is closed under renamings.*

Proof: First of all $\mathscr{F}_*\neq\mathscr{H}_1^{-1}\mathscr{H}_2\mathscr{H}_3^{-1}$ by Lemma 6.2.

Let us assume then that $\mathscr{F}_*=\mathscr{H}_1\mathscr{H}_2^{-1}\mathscr{H}_3$ and that $\mathscr{H}_3$ is closed under renamings. There is a morphic composition $\tau=\alpha_1\alpha_2^{-1}\alpha_3$ with a nonfree domain and thus by Lemma 5.1 this composition is ambiguous. Hence there is a word u such that $\mathbf{card}(u\alpha_1\alpha_2^{-1})\geqq 2$. We need only to prove that for all integers n there is a morphism α in $\mathscr{H}_3$ such that α is injective on Σ^*, where $\mathbf{card}\ (\Sigma)=n$. This is because $u\alpha_1\alpha_2^{-1}(\rho\alpha)\notin\mathscr{F}_*$ for a suitable renaming ρ and injective $\alpha\in\mathscr{H}_3$.

Let then Σ be an alphabet and consider the identity function $\iota:\Sigma^*\to\Sigma^*$. Clearly, $\iota\in\mathscr{F}_*$, and hence by assumption there are morphisms $\beta_i\in\mathscr{H}_i$, $i=1, 2, 3$, such that $\iota=\beta_1\beta_2^{-1}\beta_3$. Suppose $\beta_3:\Sigma_3^*\to\Sigma^*$. For each $a\in\Sigma$ there is a letter $a'\in\Sigma_3$ such that $a'\beta_3=a$. But now β_3 is injective on $\{a' \mid a\in\Sigma\}^*$ as required. □

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