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PARTIALLY ABELIAN SQUAREFREE WORDS (*)

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Abstract. – *The notions of square-freeness and abelian squarefreeness of words are generalized by introducing the definition of* θ*-square free words for a commutation* θ *in the free monoid. Properties involving finiteness or infiniteness of the set of* θ*-square free words are obtained for alphabets of three and four letters.*

Résumé. – *On généralise la notion de mots sans carré et de mots sans carré abélien en introduisant celle de mot sans carré partiellement abélien pour une relation de commutation* θ. *Des résultats concernant le caractère fini ou infini de l'ensemble des mots sans carré partiellement abélien sont obtenus dans le cas des alphabets de trois ou quatre lettres.*

The determination of avoidable properties of words is one of the main chapters in the combinatorial theory of the free monoid [2, 10]. Among these properties, the one of containing a square has been investigated by many authors (*see* the survey of Berstel [3]). Since the work of Thue [15] it is known that there exist infinitely many square-free words in a three letter alphabet. Another avoidable property is the abelian square-freeness, an abelian square being a word fg such that f and g possess the same number of occurrences of each letter of the alphabet; Pleasants [12] has shown that the set of words which do not contain an abelian square over an alphabet of five letters is infinite. The same question for a 4-letter alphabet is still open.

The recent interest for free partially commutative monoids (introduced by Cartier and Foata [7]) motivated by the modelization of concurrency [1, 11], suggests the definition of a new notion of a square. It is that of a square

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with respect to a commutation relation $\sim_\theta$, called a θ-square in this article. It is a word fg such that $f \sim_\theta g$. If θ is empty then the ordinary squares are obtained and if θ is the whole set $A \times A$ then the θ-squares are the abelian squares. A different definition is given by A. Carpi and A. De Luca [6]. As a consequence of the result of Pleasants, for any alphabet A containing at least five letters and for any relation θ the set of θ-square-free words is infinite. We thus restrict our invesgitation to the infiniteness of the set of θ-square-free words in the case of three or four letter alphabets.

For a three letter alphabet, we prove that if two or three pairs of letters commute then the set of θ-square-free words is finite. If only one pair of letters commute then it is infinite and we give a characterisation of those θ-square-free words in terms of excluded factors.

For a four letter alphabet infiniteness is proved in the case that strictly less than five pairs of letters commute; the case of five and six commutations remains an open problem.

1. PRELIMINARIES

The definitions and notation follow M. Lothaire [10] (*see* chapters 1 and 2).

A is a finite alphabet, A^* is the *free monoid* generated by A, whose elements are called *words*, **1** is the empty word. The *length* of a word w is denoted by $|w|$ and the number of occurrences of the letter a in w by $|w|_a$. The word u is a *factor* of w if $w = w_1 u w_2$. A *morphism* φ between two free monoïds A^* and B^* is a mapping φ such that:

$$\forall u, v \in A^*, \qquad \varphi(uv) = \varphi(u) . \varphi(v).$$

Square-free words

A *square* is a word $w = uu$ with $u \neq 1$, and a *square-free* word is such that none of its factors is a square. If A is a 2-letter alphabet there are only six square-free words namely a, b, ab, ba, aba, bab. If the alphabet has cardinality greater than 2, Thue [15] has shown that there are infinitely many square free words; for instance the sequence $u_1 = abc$, $u_{i+1} = \varphi(u_i)$ where φ is the morphism:

$$\varphi(a) = abc, \qquad \varphi(b) = ac, \qquad \varphi(c) = b$$

consists of square-free words. An *infinite word* w is a mapping from the set N of natural integers into A; such *a* word w is square-free if $w = w_1 uuw'$ (where w, u are finite and w' infinite) implies $u = \mathbf{1}$. Clearly the existence of infinite square free words is equivalent to the infiniteness of the set of square-free finite words.

Commutation relation

A symmetrical subset θ of $A \times A$ generates a relation denoted by $\sim_\theta$ on A^* as the least congruence for with $ab \sim_\theta ba$, for all $(a, b) \in \theta$. In other words, two elements f, g of A^* are equivalent under $\sim_\theta$ if there exist $h_1, h_2, \ldots, h_k$ such that:

$$h_1 = f,\ h_k = g, \qquad \text{and} \quad \forall\, i\,(1 \leqq i < k)\, h_i = h_i' a_i b_i h_i'',$$
$$h_{i+1} = h_i' b_i a_i h_i''\ (a_i, b_i) \in \theta.$$

Note that it is generally assumed that $(a, a) \notin \theta$ for all a but this assumption has no importance here.

Definition 1.1: *A square with respect to the relation* θ, *or a* θ-*square, is a word* w *such that* $w = uv$ *and* $u \sim_\theta v$. *A word* w *is* θ-square-free *if none of its factors is a* θ-*square. The set of* θ-*square-free words is denoted by* $L_2(\theta)$.

Note that if θ and ρ are such that $\theta \subset \rho$, then each θ-square is also a ρ-square and then $L_2(\theta)$ contains $L_2(\rho)$. If θ is empty then θ-squares are the usual squares and if θ contains all pairs (a, b) for $a \neq b$ then θ-squares are the abelian squares.

A. Carpi and A. Deluca [6] have introduced another notion of square-freeness in the quotient monoïd $A^*/\sim_\theta$. A word is square-free in $A^*/\sim_\theta$ if all words of its $\sim_\theta$ class are square-free. It is easy to verify that if a word is square-free in $A^*/\sim_\theta$ then it is also θ-square-free, but the converse is not true. For instance in $\{a, b\}^*$ with $ba \sim_\theta ab$, the word *aba* is θ-square-free but not square-free in $A^*/\sim_\theta$ (it is equivalent to *aab*).

We end this section with a characterisation of θ-squares.

Let a, b the two letters of A and let $\pi_{a,b}$ be the morphism of A^* onto $\{a, b\}^*$ defined by:

$$\pi_{a,b}(a) = a, \qquad \pi_{a,b}(b) = b, \qquad \pi_{a,b}(c) = \mathbf{1}, \quad \forall\, c \notin \{a, b\}.$$

The following proposition is a reformulation of Proposition 1.1 of [8].

PROPOSITION 1.2: *The word u.v is a θ-square if and only if conditions (i) and (ii) are satisfied:*

(i) $|u|_a = |v|_a$, $\forall a \in A$.

(ii) $\pi_{a,b}(u) = \pi_{a,b}(v)$, $\forall (a, b) \notin \theta$.

2. PARTIALLY ABELIAN SQUARE FREE WORDS IN $\{a, b, c\}^*$

In this section A is the alphabet consisting of the three letters $\{a, b, c\}$ and θ_1 is the relation consisting of the two pairs $\{(a, c), (c, a)\}$, θ_2 consists of $\{(b, c), (c, b)\}$ and θ_3 of $\{(a, b), (b, a)\}$. We will prove that there are only finitely many $(\theta_1 \cup \theta_2)$ square-free words. We first give some necessary conditions for a word to be θ_1-square-free. Further investigation along these lines would probably lead one to a generalization to θ_1-square-free words of the results obtained by Shelton and Soni [14] on square-free words in $\{a, b, c\}^*$.

PROPOSITION 2.1: *Let f be a θ_1-square-free word such that $f = f_1\, bacbf_2$ or $f = f_1 bcabf_2$. Then at least one of the two words f_1 or f_2 is of length strictly less than* 2.

Proof: Because of the symmetric role played by $\underline{a}$ and $\underline{c}$, we can restrict ourselves to $f = f_1\, bacbf_2$. Suppose that f_1 has length at least 2; then $f_1 = f'_1\, bc$, as well as any other end for f_1, gives a square (this is the case for ab, cb, ba, ac) or a θ_1-square (this is the case for ca). This gives :

$$f = f'_1\, b\,c\,b\,a\,c\,b\, f_2.$$

If f_2 begins with an $\underline{a}$ then $cba\,cba$ is a square; thus f_2 begins with a $\underline{c}$ and this occurrence of c can be followed neither by a $\underline{b}$ (square $cbcb$) nor by an $\underline{a}$ (θ_1-square $bac\,bca$) thus f_2 is of length at most 1 giving the result. ■

Let us introduce the following subsets of $\{a, b, c\}^*$:

$$Y = \{ba, baca\}, \qquad Z = \{bc, bcac\}, \qquad X = Y \cup Z$$

$$U = \{\mathbf{1}, a, c, ac, ca, aca, cac, bac, bca, abac, abca, cbac, cbca\}$$

$$V = \{\mathbf{1}, b, bac, bca, bcab, bacb, bcaba, bcabc, bacba, bacbc\}.$$

PROPOSITION 2.2: *The set $L_2(\theta_1)$ of θ_1-square-free words is a subset of UX^*V. Moreover, if w is a θ_1-square-free word such that*

$w = ux_1 x_2 \ldots x_k v x_i \in X$, $u \in U$, $v \in V$, *then:*

$$i<k, \quad x_i \in Y, \quad |x_{i+2} \ldots x_k v| \neq 0 \Rightarrow x_{i+1} \in Z$$
$$i<k, \ x_i \in Z, \quad |x_{i+2} \ldots x_k v| \neq 0 \Rightarrow x_{i+1} \in Y.$$

Proof: Let w be a θ_1-square free word. If w contains one or no occurrences of b then the result is easy to obtain by inspection. If w contains more than two occurrences of b, as w is square free the words between two consecutive occurrences of b are square free over $\{a, c\}$ hence one of a, c, ac, ca, aca, cac. We rule out the possibility that they are ac or ca by Proposition 2.1. We can thus obtain:

$$w = \alpha_1 b \alpha_2 b \ldots b \alpha_k b \alpha_{k+1} \quad \text{with} \quad k \geqq 2.$$

If $k \leqq 3$ the result is again obtained by inspection; assume that $k>3$. Since $|\alpha_1 b \alpha_2| \geqq 2$ and $|\alpha_k b \alpha_{k+1}| \geqq 2$. It follows by Proposition 2.1, that $\alpha_i \in \{a, c, aca, cac\}$ for $2<i<k$ and:

$$w = \alpha_1 b \alpha_2 w' b \alpha_k b \alpha_{k+1}$$

with $w' \in X^*$.

If $b\alpha_2$ is an element of X then $\alpha_1 \in \{1, a, c, ac, aca, cac\}$ which is included in U; similarly if $b\alpha_k$ is an element of X then $b\alpha_{k+1}$ belongs to X or to $\{bac, bca\}$ giving the result.

We can thus suppose $b\alpha_2$, $b\alpha_k \notin X$; then $\alpha_2, \alpha_k \in \{ac, ca\}$; and an easy inspection shows in this case $\alpha_1 b \alpha_2 \in U$ and $b \alpha_k b \alpha_{k+1} \in V$ as these words do not contain θ_1-squares.

Let us now consider a decomposition of a θ_1-square free word w in:

$$w = ux_1 \ldots x_k v, \quad u \in U, \quad v \in V, \quad x_i \in X$$

then as *babaca* contains a square, we obtain:

$$i<k; \quad x_i = ba \Rightarrow x_{i+1} \in \{bc, bcac\}.$$

If $x_i = baca$ and $x_{i+1} = ba$ then if $x_{i+2} \ldots x_k v$ begins with the letter $\underline{b}$; this gives the square *abab*, so that $x_{i+2} \ldots x_k v$ is empty. ■

PROPOSITION 2.3 : *The length of a* $(\theta_1 \cup \theta_2)$*-square free word is at most* 15.

Proof: Let w be a $(\theta_1 \cup \theta_2)$-square free word; w being θ_1-square free it can be written as

$$w = u x_1 \ldots x_k v.$$

From Proposition 2.1 applied to θ_2-square-free words we deduce that none of the x_i for $i=1\ldots k-2$ is *bcac* since in that case x_{i+1} would be from the set $\{ba, baca\}$ giving the factor *acba* for *w*.

The longest θ_1-square-free word belonging to $\{ba, bc, baca\}^*$ are:

babcba, babcbacabcbabc, bacabcbabc,

bacabcbacaba, bcbabc, bcbacabcbabc

This gives the two $(\theta_1 \cup \theta_2)$ square free words of length 15:

cabaca bc bacabac

cbabcbacabcbabc. ■

Remark 2.4: Recall that $L_2(\theta)$ is the set of θ-square-free words. In next section we will prove that $L_2(\theta_1)$ (and symmetrically $L_2(\theta_2)$, and $L_2(\theta_3)$) is infinite. By easy but tedious considerations (or by using a computer) it is possible to verify that:

$$L_2(\theta_1 \cup \theta_2) = L_2(\theta_1) \cap L_2(\theta_2)$$
$$L_2(\theta_1 \cup \theta_2 \cup \theta_3) = L_2(\theta_1) \cap L_2(\theta_2) \cap L_2(\theta_3).$$

Note that these equalities do not hold for any θ, θ' since if we consider the four letter alphabet $\{a, b, c, d\}$ and the two relations $\theta_1 = \{(a, b), (b, a)\}$ and $\theta_2 = \{(c, d), (d, c)\}$ then *abcdbadc* belongs to $L_2(\theta_1) \cap L_2(\theta_2)$ but not to $L_2(\theta_1 \cup \theta_2)$.

Remark 2.5: The number of words of length k for $(1 \leq k \leq 15)$ of $L_2(\theta_1)$, $L_2(\theta_1 \cup \theta_2)$, $L_2(\theta_1 \cup \theta_2 \cup \theta_3)$ is given by the following table:

k	$L_2(\theta_1)$	$L_2(\theta_1 \cup \theta_2)$	$L_2(\theta_1 \cup \theta_2 \cup \theta_3)$
1.	3	3	3
2.	6	6	6
3.	12	12	12
4.	18	18	18
5.	30	30	30
6.	38	34	30
7.	46	32	18
8.	48	22	0
9.	60	24	0
10.	68	24	–
11.	88	30	–
12.	96	28	–
13.	98	18	–
14.	100	6	–
15.	100	2	–

3. SUFFICIENT CONDITIONS FOR θ_1-SQUARE-FREENESS

In this section we give conditions for a word w which imply that w is θ_1-square-free and we prove that these conditions are satisfied by the sequence of Thue-Morse. We also give some conditions which have to be satisfied by a morphism in order that the image of a square-free word is a θ_1-square-free word.

DEFINITION 3.1: A word f satisfies condition (F) if neither $bacb$ nor $bcab$ is a factor of f.

PROPOSITION 3.2: *Let f be a finite square-free word satisfying* (F), *and containing a θ_1-square as a factor, then f admits one of the following decompositions:* (α) $f=f_1\,ac\,u\,aca\,u\,af_2$; ($\beta$) $f=f_1\,ca\,u\,cac\,u\,cf_2$; ($\gamma$) $f=f_1\,a\,u\,aca\,u\,caf_2$; ($\delta$) $f=f_1\,c\,u\,cac\,u\,ac\,f_2$.

Moreover in such a decomposition one of f_1 or f_2 is of length at most 1.

Proof: Let f be such a word. Then:

$$f=f_1\,gh\,f_2 \qquad \text{and} \qquad g\sim_{\theta_1}h.$$

As f is square-free and satisfies condition (F) the only possible words between two occurrences of b are from the set $B=\{a, c, aca, cac\}$. Note that two different words in this set are not equivalent under $\sim_{\theta_1}$. Let g and h be decomposed in the following way:

$$g=g_1\,b\,g_2\ldots bg_p, \qquad \forall i=1, p: g_i\in\{a, c\}^*$$
$$h=h_1\,b\,h_2\ldots bh_q, \qquad \forall i=1, q: h_i\in\{a, c\}.$$

From Proposition 1.1 we get $p=q$ and $g_i\sim h_i$ for $i=1, \ldots, p$.

From $g_i\in B$ for $i=2, \ldots, p-1$, we get $g_i=h_i$ for $i=2, \ldots, p-1$. As f is square-free $g_1\neq h_1$ or $g_p\neq h_p$, by our previous remark $g_p\,h_1$ is an element of B and $g_p\,h_1=aca$ or $g_p\,h_1=cac$. As $\underline{a}$ and $\underline{c}$ play symmetric roles we can suppose $g_p\,h_1=aca$, this gives:

$$g_p=a \quad \text{and} \quad h_1=ca \qquad \text{or} \qquad g_p=ac \quad \text{and} \quad h_1=a;$$

in the first case $h_p=a$ and $g_1=ca$ giving decomposition (α); in the second case $h_p=ca$ and $g_1=a$ giving decomposition (γ).

Let us consider now the decomposition:

$$f=f_1\,ac\,u\,aca\,u\,a\,f_2$$

and let us show that at least one of f_1 or f_2 is of length at most 1; a symmetric proof will give the other ones. In such a decomposition u begins and ends with the letter $\underline{b}$. If u is of length more than 1, then u has one of the following decompositions:

$$u = babu', \qquad u = bacabu', \qquad u = bcbu', \qquad u = bcacbu'.$$

The first one gives a square $abab$, the second one $bacabaca$ (with the $\underline{b}$ at the end of the first occurence of u). The third one $cbcb$, as to the fourth we have

$$f = f_1\, acbcacbu'\, aca\, u\, a\, f_2.$$

Since $bacb$ is not a factor of f, f_1 doesn't end with $\underline{b}$; it doesn't end with $\underline{c}$ or $\underline{a}$ either, since f is square-free; thus f_1 is empty. If u is of length 1, then:

$$f = f_1\, ac\, b\, aca\, b\, a\, f_2.$$

And f_2 doesn't begin with $\underline{a}$ (square aa) nor with $\underline{b}$ (square $abab$); the first letter of f_2 is thus $\underline{c}$ and one can easily prove that this $\underline{c}$ is not followed by any other letter so that f_2 has length 1. ■

COROLLARY 3.3: *Any infinite square free word of* $\{a, b, c\}^*$ *begining with a letter b and satisfying (F) is* θ_1*-square-free.*

Proof: Let w be such a word and assume it has a θ_1-square then $w = w_1\, gh\, w_2\, w'$ with $|w_2| \geqq 2$. Since $w_1\, gh\, w_2$ satisfies the hypothesis of Proposition 3.1 this gives $|w_1| \leqq 1$, since w begins a letter b, we get $w_1 = b$ and among the decompositions of $w_1\, gh\, w_2$ only the following remain because of condition (*F*):

$$ba\, u\, aca\, u\, ca\, w_2, \qquad bc\, u\, cac\, u\, ac\, w_2.$$

Then u is of length greater than one and ends with $cacb$ or $acab$. This implies $|w_2| \leqq 1$, a contradiction. ■

COROLLARY 3.4: *The infinite sequence of words obtained from the Thue Morse sequence by deleting the first letter consists of* θ_1*-square-free words. Thus* $L_2(\theta_1)$ *is infinite.*

Proof: Set $u_0 = abc$, and $u_i = \varphi(u_{i-1})$ where φ is defined by $\varphi(a) = abc$, $\varphi(b) = ca$, $\varphi(c) = b$. Remark first that cbc is not a factor of u_i since $\{abc, ac, b\}^* \cap A^*\, cbc\, A^*$ is empty. We observe that, if $\varphi(u_i)$ has $bcab$ as a factor, then u_i contains aa and is not square-free. If $\varphi(u_i)$ contains the factor *bcab* then u_i contains necessarily *cbc* with is a contradiction by the previous remark. We thus obtain the result as a consequence of Corollary 3.2. ■

Note that each u_i is also θ_1-square-free but the technical proof of this fact is of poor interest and is omitted here.

4. PARTIALLY ABELIAN SQUARE FREE WORDS IN A FOUR LETTERS ALPHABET

In this section, A is the alphabet $\{a, b, c, d\}$. We consider the two relations ρ_1 and ρ_2 which are obtained by symmetrization of:

$$\rho_1' = \{(a, c), (a, d), (b, d), (c, d)\}$$
$$\rho_2' = \{(a, c), (a, d), (b, c), (b, d)\}.$$

We will show that there exist an infinite number of ρ_1-square-free words and of ρ_2-square-free words. By the symmetric role of a, b, c, d and using the fact that if $\theta \subset \rho$, any ρ-square-free word is also θ-square-free, it is easy to verify that if ρ is any relation with at most four pairs of commutations then the set of ρ-square free words is infinite. The cases where ρ has five or six pairs of commutations remain an open question, the last one is a reformulation of the problem of the existence of an infinite word without an abelian square, in a four letters alphabet.

To prove these results we use the Thue Morse sequence t defined by the iteration of morphism φ : $\varphi(a) = abc$, $\varphi(b) = ac$, $\varphi(c) = b$, or any infinite sequence with no θ_1-square.

Let ψ be the morphism defined by

$$\psi(a) = a; \qquad \psi(b) = bd; \qquad \psi(c) = c;$$

then we have

THEOREM: *$\psi(t)$ is a ρ_1 and a ρ_2-square free infinite word.*

1. It is not difficult to prove that $\psi(t)$ is ρ_1-square free. Assume $\psi(t)$ contains a ρ_1-square uv, then by Proposition 1.1.:

$$\pi_{a,b}(u) = \pi_{a,b}(v) \qquad \text{and} \qquad \pi_{b,c}(u) = \pi_{b,c}(v).$$

Let u' and v' be obtained from u and v by deleting all the occurences of d. Let:

$$t = t_1 u' v' t_2$$

and

$$\pi_{a,b}(\mathrm{u}') = \pi_{a,b}(v'), \qquad \pi_{b,c}(u') = \pi_{b,c}(v')$$

giving a θ_1-square for t which is in contradiction with Corollary 3.3.

2. Suppose that $w = \psi(t)$ contains a ρ_2-square uv, let t_1 (resp. $t_1 x$) be the longest factor of t such that $\psi(t_1)$ is a left factor of w_1 (resp. $\psi(t_1 x)$ is a left factor of $w_1 u$), and let $t_1 xy$ be the smallest such that $\psi(t_1 xy)$ has $w_1 uv$ as a left factor.

Then we have:

$$t = t_1 xy t_2, \qquad w = w_1 u v w_2$$

and one of the following pair (i), (j)′ of conditions holds:

(1) $u = \psi(x)$	(1′) $v = \psi(y)$
(2) $u = \psi(x) b$	(2′) $bv = \psi(y)$
(3) $u = d\psi(x)$	(3′) $vd = \psi(y)$
(4) $u = d\psi(x) b$	(4′) $bvd = \psi(y)$.

Note that as uv is a ρ_2-square we have

$$|u|_b = |v|_b \quad \text{and} \quad |u|_d = |v|_d.$$

This gives that the only possible combinations are:

- (1) or (4) with (1)′ or (4)′,
- (2) with (3)′,
- (3) with (2)′.

As x and y are to be consecutive in t and u and v are in w then (1) with (4′), (4) with (1)′, (2) with (3)′ and (3) with (2)′ are to be discarded:

- (1) with (4′) gives $ubvd = \psi(x)\psi(y)$,
- (4) with (1)′ gives $uv = d\psi(x) b\psi(y)$,
- (2) with (3′) gives $uvd = \psi(x) b\psi(y)$,
- (3) with (2′) gives $ubv = d\psi(x)\psi(y)$.

We have only to consider (1), (1)′ and (4), (4)′.

If (1) and (1)′ hold then:

$$uv = \psi(x)\psi(y);$$

uv being a ρ_2-square this gives:

$$\pi_{a,b}(\mathrm{u}) = \pi_{a,b}(\mathrm{v}) \quad \text{and} \quad \pi_{c,d}(u) = \pi_{c,d}(v).$$

But $\pi_{a,b}(u)=\pi_{a,b}(x)$ and $\pi_{c,d}(u)$ is obtained from $\pi_{b,c}(x)$ replacing the occurences of b by d. Thus:

$$\pi_{a,b}(\mathrm{x})=\pi_{a,b}(\mathrm{y}) \qquad \text{and} \qquad \pi_{b,c}(\mathrm{x})=\pi_{b,c}(\mathrm{y})$$

and again by Proposition 1.1, xy is a ρ_1-square in t, a contradiction.

If (4) and (4)′ hold then:

$$uvd=d\psi(x)\psi(y)$$

and as uv is a ρ_2-square, $\pi_{a,b}(u)=\pi_{a,b}(v)$ and $\pi_{c,d}(u)=\pi_{c,d}(v)$. We thus get

$$\pi_{a,b}(\mathrm{bud})=\pi_{a,b}(bvd), \qquad \pi_{c,d}(bud)=\pi_{c,d}(bvd).$$

From (4), and (4′) we obtain:

$$\pi_{a,b}(b\psi(x)b)=\pi_{a,b}(\psi(y)), \qquad \pi_{c,d}(d\psi(x)d)=\pi_{c,d}(\psi(y))$$

$\pi_{c,d}(\psi(x))$ is obtained from $\pi_{b,c}(x)$ by replacing the occurences of b by d; we obtain

$$\pi_{a,b}(bxb)=\pi_{a,b}(y) \qquad \text{and} \qquad \pi_{b,c}(bxb)=\pi_{b,c}(y).$$

Thus bxb and y are equivalent under $\sim_{\theta_1}$, giving $y=by'b$ and $x\sim_{\theta_1}y'$ (b commutes with no letter under θ_1) since t contains the factor xy, we have $xy=xby'b$ which is a θ_1-square, and we also obtain a contradiction. ■

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