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ALEX WEISS
DENIS THERIEN

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VARIETIES OF FINITE CATEGORIES (*)

by Alex WEISS [1] and Denis THERIEN [2]

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Abstract. – *Many new results in the algebraic theory of finite-state machines are based on the idea of using finite categories as the mathematical model for automata. In this article, we study varieties of finite categories. Our main goal is to point out the similarities and distinctions between C-varieties and varieties of finite monoids that underlie the more traditional approach to the theory.*

Résumé. – *Plusieurs résultats nouveaux en théorie algébrique des machines à états finis découlent de l'utilisation de catégories finies comme modèle mathématique des automates. Dans cet article, nous étudions les variétés de catégories finies. Notre but est d'indiquer les similitudes et les différences entre les C-variétés et les variétés de monoïdes finis de l'approche traditionnelle.*

0. INTRODUCTION

The classical point of view in algebraic automata theory uses monoids (or semi-groups) as models for finite-state machines. Underlying this choice of formalization is the assumption that any sequence of symbols, drawn from a finite input alphabet, can be fed to the machine. Denoting the input alphabet by A, the universe of possible inputs is then the free monoid A^* and a finite-state machine can be thought of as a quotient of A^* by a finite-index congruence β.

In some interesting situations the assumption above is not realistic: for example, when two machines are connected in series, the input sequence processed by the "tail" machine is essentially the output sequence produced

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(1) Department of Mathematics, Concordia University, 7141 Sherbrooke Street W., Hingston Hall, Montréal, Québec, Canada H4B 1R6.
(2) School of Computer Science, McGill University, Montréal, Québec, Canada H3A 2K6.

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by the "front" machine. Because of the preprocessing done, the input universe to the "tail" machine is no longer a free monoid.

A more convenient formalization is to view the input alphabet as edges of a graph. The possible input sequences are then paths in this graph, and a finite-state machine becomes a finite category. This generalizes the former point of view since a free monoid can be viewed as the set of paths in a one-vertex graph.

The categorical approach is *not* just rhetorical sophistication. It has already produced results which were not obtainable within the old framework. The new approach was implicitely used in [2], [4], [5] and [12] to solve decidability problems about the wreath product. Recent work [10, 8, 6] is fully exploiting the power of the categorical model. In this paper, we will present some basic ideas and techniques that are relevant to this area.

1. DEFINITIONS

A *category* C is given by a non-empty set of objects Ob_C and, for each i, $j \in \mathrm{Ob}_C$, families of arrows $H_C(i, j)$. We write H_C for the union of all $H_C(i, j)$ and drop the subscript C whenever the context is clear. For all $i, j, k \in \mathrm{Ob}$, a binary operation is given from $H(i, j) \times H(j, k)$ to $H(i, k)$ subject to the following axioms:

(i) for any $x \in H(i, j)$, $y \in H(j, k)$, $z \in H(k, l)$ $(xy)z = x(yz)$;

(ii) for each $j \in \mathrm{Ob}$, there exists an arrow $1_j \in \mathrm{H}(j, j)$ such that $x 1_j = x$ for all x in $H(i, j)$ and $1_j y = y$ for all y in $H(j, k)$.

We always assume that Ob is a finite set.

Given a directed multigraph G with vertex set V and edge set A, the *free category* G^* is defined by $\mathrm{Ob}_{G^*} = V$ and $H_{G^*}(i, j)$ being the set of all paths of finite length from vertex i to vertex j. Concatenation of consecutive paths is the operation. Note that we include for each vertex i a trivial path 1_i, which acts as the identity arrow.

A *congruence* β on a category C is a family of equivalence relations, one for each set $H(i, j)$ such that for any $x_1, y_1 \in H(i, j)$, $x_2, y_2 \in H(j, k)$ we have $x_1 \beta y_1$ and $x_2 \beta y_2$ imply $x_1 x_2 \beta y_1 y_2$. Note that two arrows can be congruent only if they are coterminal.

The category $D = C/\beta$ is then defined by $\mathrm{Ob}_D = \mathrm{Ob}_C$ and $H_D(i, j) = \{ [x]_\beta \mid x \in H_C(i, j) \}$ with the operation being $[x]_\beta [y]_\beta = [xy]_\beta$.

Every category C can be obtained as the quotient of a free category by a congruence. Let G be a graph with vertex set Ob_C and edge set any generating set for H_C. On G^* define $x \beta y$ iff $x=y$ in C: then C can be identified with G^*/β in an obvious way.

A *relational morphism* $\langle \varphi, \psi \rangle : C \to D$ between two categories consists of an object function φ: $\mathrm{Ob}_C \to \mathrm{Ob}_D$ and a morphism relation ψ: $H_C(i, j) \to H_D(i\varphi, j\varphi)$ such that

(i) $x\psi \neq \varnothing$ for any x in H_C;

(ii) $1_{i\varphi} \in 1_i \psi$;

(iii) $(x\psi)(y\psi) \subseteq (xy)\psi$.

C is a *subcategory* of D if φ and ψ are injective functions. C is a *morphic image* of D if φ is a bijection and ψ^{-1} is a surjective function. We say that *Cdivides* D, written $C \prec D$, if, for all $x, y \in H_C(i, j)$, $x\psi \cap y\psi \neq \varnothing$ implies $x=y$. Note that if C and D are monoids, i. e. one-object categories, this definition of division is the same as the one given in [3]. C and D are *equivalent*, denoted by $C \simeq D$, if $C \prec D$ and $D \prec C$. We will write $C \prec_{1-1} D$ if $C \prec D$ with the object function φ being injective.

LEMME 1.1: *$C \prec D$ iff C is a morphic image of a subcategory of E where $E \simeq D$.*

— Sufficiency of the condition follows from transitivity of $\prec$. As for necessity, let $\langle \varphi, \psi \rangle$: $C \to D$ be the division. Define E by $\mathrm{Ob}_E = \mathrm{Ob}_C \times \mathrm{Ob}_D$ and

$$H_E((i, j), (i', j')) = \{ x \mid x \in H_D(j, j') \}.$$

We observe that $E \simeq D$ via $\langle \varphi_1, \psi_1 \rangle$: $E \to D$, defined by $(i, j)\varphi_1 = j$ and $x\psi_1 = x$, and $\langle \varphi_2, \psi_2 \rangle$: $D \to E$, defined by $j\psi_2 = (i_0, j)$ for some fixed i_0 and $x\psi_2 = x$. Next consider the subcategory F of E given by $\mathrm{Ob}_F = \{ (i, i\varphi) \mid i \in \mathrm{Ob}_C \}$ and

$$H_F((i, i\varphi), (j, j\varphi)) = \{ y \mid y \in x\psi \text{ for some } x \in H_C(i, j) \}.$$

Let $\langle \varphi_3, \psi_3 \rangle$: $C \to F$ be defined by

$$i\varphi_3 = (i, i\varphi), \qquad x\psi_3 = \{ y \mid y \in H_F((i, i\varphi), (j, j\varphi)), y \in x\psi \}.$$

It is then checked that C is a morphic image of F. $\square$

A category C is *trivial* iff $|H_C(i, j)| \leqq 1$ for all objects i, j. The *direct product* of two categories C and D is given by $\mathrm{Ob}_{C \times D} = \mathrm{Ob}_C \times \mathrm{Ob}_D$ and

$$H_{C \times D}((i, j), (i', j')) = \{ (x, y) \mid x \in H_C(i, i'), y \in H_D(j, j') \}.$$

LEMME 1.2: *$C \prec D$ iff $C \prec_{1-1} D \times E$ where E is a trivial category and $E \prec_{1-1} C$.*

– Let $\langle \varphi, \psi \rangle: C \to D$ be a division. Let E be defined by $Ob_E = Ob_C$ and $H_E(i, j) = \{(i, j)\}$ if $H_C(i, j) \neq \varnothing$, $H_E(i, j) = \varnothing$ otherwise: the product in E is given by $(i, j)(j, k) = (i, k)$. It is trivial that $E \prec_{1-1} C$. Define next $\langle \varphi_1, \psi_1 \rangle$: $C \to D \times E$ by $i\varphi_1 = (i\varphi, i)$ and $x\psi_1 = (x\psi, (i, j))$ for $x \in H_C(i, j)$: this establishes that $C \prec_{1-1} D \times E$. The converse follows from the fact that $D \times E \prec D$ whenever E is trivial. □

A *C-variety* **V** is a collection of finite categories such that $D_1, D_2 \in \mathbf{V}$ and $C \prec D_1$ imply $C \in \mathbf{V}$ and $D_1 \times D_2 \in \mathbf{V}$. This generalizes the notion of M-varieties where only monoids (i. e. one- object categories) are considered. Similarly to the monoid case dealt with in [3] and [9] one can naturally define notions of varieties of congruences on free categories [13] and varieties of rational languages over free categories [11], such that 1-1 correspondance can be set up between all three types of varieties.

2. RESTRICTED C-VARIETIES

Since any non-empty C-variety admits categories on more than one object as elements, M-varieties are not C-varieties. One way of recapturing M-varieties as special cases is to allow 1-1 division only. A *restricted C-variety* is defined to be a class of finite categories closed under 1-1division and direct product. As will be seen below, restricted C-varieties are essentially obtained by restricting the type of free categories under consideration.

Let $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$ be directed multigraphs: the direct product $G_1 \times G_2$ is defined by

$$(V_1 \times V_2, (A_1 \times A_2) \cup (A_1 \times \{1_i \mid i \in V_2\}) \cup (\{1_i \mid i \in V_1\} \times A_2)).$$

Observe that $H_{G_1^*} \times H_{G_2^*} \neq \varnothing$ iff $H_{G_1^*} \neq \varnothing$ and $H_{G_2^*} \neq \varnothing$. Also, we will say that G_1 is *covered* by G_2 if there exists a 1-1 function φ from V_1 to V_2 such that whenever there is a path from i to j in G_1 there is also a path from $i\varphi$ to $j\varphi$ in G_2. A family F of free categories will be said to be *admissible* if whenever it contains two free categories induced by the multigraphs G_1 and G_2 it also contains the free category induced by $G_1 \times G_2$ and any free category induced by a graph G that is covered by G_1. For any (unrestricted) C-variety **V**, define

$$\mathbf{V}_F = \{C \mid C \in \mathbf{V},\ C = G^*/\beta \text{ for some } G^* \in F\}.$$

THEOREM 2.1: **W** *is a restricted C-variety iff* $\mathbf{W} = \mathbf{V}_F$ *for some unrestricted C-variety* **V** *and some admissible family F of free categories.*

— Let $\mathbf{W}$ be a restricted C-variety and let $\mathbf{V}$ be the smallest C-variety containing $\mathbf{W}$. Let $F = \{ G^* \mid \text{there exists } C = G^*/\beta \in \mathbf{W} \}$. By definition $\mathbf{W} \subseteq \mathbf{V}_F$. Let $C = G^*/\beta \in \mathbf{V}$ with $G^* \in F$: thus $C \prec D$ with $D \in \mathbf{W}$ and there exists some $B = G^*/\gamma \in \mathbf{W}$. By lemma 1.2, $C \prec_{1-1} D \times E$ where E is trivial and $E \prec_{1-1} C$. Then $E \prec_{1-1} B$ and $C \in \mathbf{W}$: hence $\mathbf{W} = \mathbf{V}_F$. Suppose G_1^* and G_2^* are in F: there thus exists γ_1 and γ_2 such that $C_1 = G_1^*/\gamma_1$ and $C_2 = G_2^*/\gamma_2$ are in $\mathbf{W}$. The congruence $\gamma_1 \times \gamma_2$ on $(G_1 \times G_2)^*$, defined by $(x_1, x_2)\gamma_1 \times \gamma_2\ (y_1, y_2)$ iff $x_1 \gamma_1 y_1$ and $x_2 \gamma_2 y_2$ is such that $(G_1 \times G_2)^*/\gamma_1 \times \gamma_2$ is isomorphic to $C_1 \times C_2$; hence $(G_1 \times G_2)^*$ belongs to F. Now suppose that G_2^* is in F and that the graph G_1 is covered by the graph G_2 via the 1-1 function φ: define on G_1^* the congruence $\varphi\gamma_2$ by $x\, \varphi\gamma_2\, y$ iff $x\varphi\, \gamma_2\, y\varphi$; then $G_1/\varphi\gamma_2 \prec_{1-1} G_2^*/\gamma_2$ so that G_1^* is in F. Conversely let $C_1 = G_1^*/\gamma_1$, $C_2 = G_2^*/\gamma_2$. If they are both in $\mathbf{V}_F$ then $C_1 \times C_2 \in \mathbf{V}$ and it can be obtained as the quotient of the free category $(G_1 \times G_2)^*$ by the congruence $\gamma_1 \times \gamma_2$ defined above; hence $C_1 \times C_2 \in \mathbf{V}_F$. Also if $C_2 \in \mathbf{V}_F$ and $C_1 \prec_{1-1} C_2$, it must be that G_1 is covered by G_2. Hence $C_1 \in \mathbf{V}_F$. This proves that $\mathbf{V}_F$ is a restricted C-variety. □

The family of all free categories is certainly admissible. It turns out that there are only three other such non-empty families. Define

$$M = \{ G^* \mid |\mathrm{Ob}_{G^*}| = 1 \},$$
$$Q = \{ G^* \mid H(i, j) = \varnothing \text{ if } i \neq j \},$$
$$P = \{ G^* \mid H(i, j) \neq \varnothing \text{ implies } H(j, i) = \varnothing \text{ or } i = j \}.$$

THEOREM 2.2: *M, Q, P and the set of all free categories are the only admissible non-empty families of free categories.*

— That M, Q and P are indeed admissible is straightforward. Conversely if F is non-empty, it must contain the one-object, one arrow category: note that the underlying free category is generated by the empty set. The underlying graph covers any one-object graph: hence $M \subseteq F$. If $M \subsetneq F$ there must be in F a k-object category G^*, with $k \geqq 2$: any graph underlying a free category in Q can be covered by a direct product of copies of the graph G. Thus $Q \subseteq F$. If $Q \subsetneq F$, a k-object category G^* can be found in F with $k \geqq 2$ and $H(i, j) \neq \varnothing$ for some $i \neq j$. Any graph underlying a free category in P can be covered by a direct product of copies of the graph G, so that $P \subseteq F$. Finally if $P \subsetneq F$ then F contains some G^* with objects $i_0, i_1, \ldots, i_k$ all different such that $H(i_0, i_1), H(i_1, i_2), \ldots, H(i_k, i_0)$ are all non-empty. Any graph can be covered by a direct product of copies of G so that F must then include all free categories. □

The M-varieties are seen to correspond exactly to the restricted C-varieties of the form V_M.

3. INDUCING C-VARIETIES FROM M-VARIETIES

Let $\mathbf{W}$ be a M-variety: we can view $\mathbf{W}$ as the restriction of some C-variety $\mathbf{V}$ to one-object categories, i. e. $\mathbf{W}=\mathbf{V}_M$. $\mathbf{V}$ certainly determines $\mathbf{W}$ uniquely: we will see that the converse does not always hold.

$$\text{Define } \mathbf{gW}=\{\, C \mid C \prec M \text{ for some } M \in \mathbf{W} \,\}$$

and

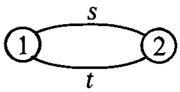
$$\mathbf{lW}=\{\, C \mid M \prec C \text{ and } |\mathrm{Ob}_M|=1 \text{ imply } M \in \mathbf{W} \,\}.$$

The two families form C-varieties and we have $\mathbf{gW}_M=\mathbf{lW}_M=\mathbf{W}$. We say that $\mathbf{gW}$ is the C-variety *globally induced* by $\mathbf{W}$, and $\mathbf{lW}$ is *locally induced* by $\mathbf{W}$.

THEOREM 3.1: *Let* $\mathbf{V}$ *a C-variety and* $\mathbf{W}=\mathbf{V}_M$. *Then* $\mathbf{gW} \subseteq \mathbf{V} \subseteq \mathbf{lW}$.

– If $C \in \mathbf{gW}$ then $C \prec M$ for some $M \in \mathbf{W}$. Since $\mathbf{W} \subseteq \mathbf{V}$, we have $M \in \mathbf{V}$ and $C \in \mathbf{V}$ as well. Let now $C \in \mathbf{V}$: since $\mathbf{V}_M=\mathbf{W}$ any monoid dividing C is in $\mathbf{W}$. Hence $C \in \mathbf{lW}$. □

Thus $\mathbf{W}=\mathbf{V}_M$ uniquely determines $\mathbf{V}$ iff $\mathbf{gW}=\mathbf{lW}$. This equality holds in a number of interesting cases: for example, whenever $\mathbf{W}$ is a non-trivial variety of groups [12], and when $\mathbf{W}$ is the variety of nilpotent monoids [14]. In the next section we will prove that $\mathbf{gA}_1=\mathbf{lA}_1$ where $\mathbf{A}_1$ is the M-variety of idempotents monoids. On the other hand examples are known where the equality does not hold ([12, 4]). The simplest such case is when $\mathbf{W}=\mathbf{1}$ is the M-variety consisting of the one-element monoid only. It is clear that $\mathbf{g1}$ is the C-variety of all trivial categories. The category E_2 described as



$s \neq t$

is in $\mathbf{l1}$ but is not trivial. Hence $\mathbf{g1} \neq \mathbf{l1}$. The C-variety $\mathbf{l1}$, despite its apparent simplicity, seems to be playing an important role when decomposing machines (*see* [8] and [6] for example). We indicate below some interesting properties of this variety.

Let G^* be a free category with H_{G^*} generated by A. Define a preorder on Ob_{G^*} by $i \leqq j$ iff $H(i, j) \neq \varnothing$. Let $i \equiv j$ iff $i \leqq j$ and $j \leqq i$. The preorder naturally induces a partial order on the $\equiv$classes. Let $A_F \subseteq A$ be defined by $a \in A_F$ iff $a \in H(i, j)$ with $i \not\equiv j$. Next define on $H_{G^*}(i, j)$, $x \,\beta_F\, y$ iff for all $a \in A_F$ $x=x_0 a x_1$ iff $y=y_0 a y_1$. Thus x and y are β_F-equivalent iff they traverse the same set of edges, where only edges between distinct $\equiv$-classes are considered. It is easy to check that G^*/β_F is a well-defined category.

THEOREM 3.2: $C=G^*/\beta \in \mathbf{l1}$ *iff* $\beta \supseteq \beta_F$.

– If $\beta \supseteq \beta_F$ then $|H_C(i, i)| = 1$ for all $i \in \mathrm{Ob}_C$. Hence $C \in \mathbf{l1}$. Conversely let $C \in \mathbf{l1}$, $x, y \in H_{G^*}(i, j)$, $x \beta_F y$. Then $x = x_0 a_1 x_1 \ldots a_n x_n$, $y = y_0 a_1 y_1 \ldots a_n y_n$ with $a_1, \ldots a_n \in A_F$ and $x_0, \ldots, x_n$, $y_0, \ldots, y_n \in (A - A_F)^*$. Suppose $x_i, y_i \in H_{G^*}(k, l)$: there exists z_i such that $x_i z_i \in H_{G^*}(k, k)$ and $z_i y_i \in H_{G^*}(l, l)$. Thus $x_i z_i \beta 1_k$ and $z_i y_i \beta 1_l$. This gives $x_i \beta x_i z_i y_i \beta y_i$ and $\beta \supseteq \beta_F$. □

An important property of categories in **l1** is that $C \prec M$ for any monoid M that is sufficiently large. This is equivalent to the following theorem of Tilson [10].

THEOREM 3.3: $\mathbf{l1} \subseteq \mathbf{gW}$ *for any non-trivial M-variety* $\mathbf{W}$.

– By theorem 3.2 it suffices to show that $C = G^*/\beta_F$ divides some monoid in $\mathbf{W}$. Suppose $A_F = \{a_1, \ldots, a_n\}$. Let $M \in \mathbf{W}$ with $|M| > 1$ and choose any m in M different from the identity. Define $\langle \varphi, \psi \rangle$: $C \to M \times \ldots \times M$ (n times) by letting $i\varphi$ be the unique object of $M \times \ldots \times M$ for all $i \in \mathrm{Ob}_C$ and, for $a_i \in A_F$, $a_i \psi = (1, \ldots, i, \ldots, 1)$ where the unique m in the vector $a_i \psi$ occurs in the i-th position: if $a \in A - A_F$ then $a\psi = (1, \ldots, 1)$: ψ is extended in a unique way to H_C. Since a path x in G^* can traverse an edge in A_F at most once we get $x\psi = (u_1, \ldots, u_n)$ where $u_i = m$ if $x = x_0 a_i x_1$ and $u_i = 1$ otherwise. This yields that $x\psi$ characterizes $[x]_{\beta_F}$, i.e. $\langle \varphi, \psi \rangle$ is a division. □

In general, it is not known at present if **gW** and **lW** are the only possible C-varieties $\mathbf{V}$ such that $\mathbf{V}_M = \mathbf{W}$. This is probably not so but no examples are known. At least the case $\mathbf{W} = \mathbf{1}$ is settled.

THEOREM 3.4: *If* $\mathbf{V}_M = \mathbf{1}$ *then* $\mathbf{V} = \mathbf{g1}$ *or* $\mathbf{V} = \mathbf{l1}$.

– Suppose that $\mathbf{g1} \subsetneq \mathbf{V}$: there exists $C \in \mathbf{V}$ and $i, j \in \mathrm{Ob}_C$ such that $|H(i, j)| \geqq 2$. It cannot be that $i = j$ otherwise C would not be in **l1**. Hence $E_2 \prec C$. We claim that G^*/β_F divides a direct product of copies of E_2 for any free category G^*. The theorem follows from this claim.

Let $A_F = \{a_1, \ldots, a_n\}$: then $\beta_F = \beta_1 \cap \ldots \cap \beta_n$ where $x \beta_i y$ iff $x, y \in H(j, k)$ and $x = x_0 a_i x_1$ iff $y = y_0 a_i y_1$. It thus suffices to show that $G^*/\beta_i \prec E_2$. Suppose $a_i \in H(u_i, v_i)$. Partition the objects of G^* in three sets: $V_1 = \{v \mid H(v, u_i) \neq \varnothing\}$, $V_2 = \{v \mid H(v_i, v) \neq \varnothing\}$ and V_3 consists of the remaining vertices. Define $\langle \varphi, \psi \rangle$: $G^*/\beta_i \to E_2$ by $v\varphi = 2$ if $v \in V_2$ and $v\varphi = 1$ otherwise, and $a\psi = s$ if $a = a_i$, $a\psi = 1_1$ if $a \in H(u, v)$ with $v \in V_1 \cup V_3$, $a\psi = 1_2$ if $a \in H(u, v)$ with $u \in V_2$ and $a\psi = t$ otherwise. The reader will check that $\langle \varphi, \psi \rangle$ is a well defined relational morphism: moreover $x\psi = s$ iff $x = x_0 a_i x_1$ so that $\langle \varphi, \psi \rangle$ is indeed a division. □

It is clear that $\mathbf{gW}_M = \mathbf{lW}_M$ and that $\mathbf{gW}_Q = \mathbf{lW}_Q$. The category E_2 exemplifies that $\mathbf{g1}_p \neq \mathbf{l1}_p$. On the other hand we have the following.

THEOREM 3.5: *For any M-variety* $\mathbf{W} \neq \mathbf{1}$ $\mathbf{gW}_P = \mathbf{lW}_P$.

– Since $\mathbf{gW} \subseteq \mathbf{lW}$ we have $\mathbf{gW}_P \subseteq \mathbf{lW}_P$. Let $C = G^*/\beta \in \mathbf{lW}_P$, where $\mathrm{Ob}_C = \{1, \ldots, n\}$. By hypothesis $H(i, i) \prec M_i$ for some $M_i \in \mathbf{W}$. Define β_i on G^* by $x \beta_i y$ iff $x, y \in H(j, k)$ and either x, y have no prefix in $H(j, i)$ or $x = x_0 u x_1$, $y = y_0 v y_1$ with $u \beta v$ where $u(v)$ is the maximal length segment of $x(y)$ that is in $H(i, i)$. Then β_i is a congruence on G^* and $G^*/\beta_i \prec M_i$. Moreover $\beta = \beta_1 \cap \ldots \cap \beta_n \cap \beta_F$ so

$$G^*/\beta \prec G^*/\beta_1 \times \ldots \times G^*/\beta_n \times G^*/\beta_F.$$

Since $G^*/\beta_i \prec M_i$ and $G^*/\beta_F \prec M$ for some $M \in W$ by theorem 3.3, we deduce $C \in \mathbf{gW}_P$. □

This last result has consequences for decidability problem concerning the wreath product. Given monoids S, T and an M-variety $\mathbf{W}$ we want to determine if there exists $X \in \mathbf{W}$ such that $S \prec X \circ T$. It can be shown [12, 10] that this problem reduces to deciding if a specific (constructible) category belongs to $\mathbf{gW}$. This is decidable whenever $\mathbf{gW} = \mathbf{lW}$ and membership in $\mathbf{W}$ is decidable. If T is R-trivial, the category in question is of the form G^*/β for some G^* in P. By theorem 3.5, the problem above can thus be solved whenever $\mathbf{W}$ has a decidable membership problem.

4. $\mathbf{gA}_1 = \mathbf{lA}_1$

Let $\mathbf{A}_1$ be the M-variety of idempotent monoids i.e. $M \in \mathbf{A}_1$ iff $m = m^2$ for all $m \in M$. We will show that $\mathbf{gA}_1 = \mathbf{lA}_1$. The proof given here is typical of similar results.

We first need a general fact.

LEMME 4.1: *Let* $\mathbf{W}$ *be an M-variety. Let* $C = G^*/\beta$ *where* $\mathrm{Ob}_{G^*} = \mathrm{Ob}_C$ *and* H_{G^*} *is generated by A. Then* $C \in \mathbf{gW}$ *iff there exists a congruence* γ *on the free monoid* A^* *such that* $M = A^*/\gamma \in \mathbf{W}$ *and for any* $x, y \in H_{G^*}(i, j)$ $x \gamma y$ *implies* $x \beta y$.

– Suppose there exists such γ. Define $\langle \varphi, \psi \rangle$: $C \to M$ by $i\varphi = 1$, the unique object of M, for *all* $i \in \mathrm{Ob}_C$ and $[x]_\beta \psi = \{[y]_\gamma \mid x \gamma y\}$. This is indeed a division so that $C \in \mathbf{gW}$. Conversely let $\langle \varphi, \psi \rangle$: $C \to M$ be a division for some $M \in \mathbf{W}$. Let $A = \{a_1, \ldots, a_n\}$: choose for each i an arbitrary element $m_i \in a_i \varphi$. We define a new relational morphism $\langle \varphi, \psi_1 \rangle$: $C \to M$ by $[x]_\beta \psi_1 = \{m_{i_1} \ldots m_{i_k} \mid$ there exists $w \beta x$, where $w = a_{i_1} \ldots a_{i_k}\}$. The image of C by ψ_1 is a submonoid M_1 of M that is generated by A. Also if $[x]_\beta$ and

$[y]_\beta$ are coterminal and $[x]_\beta \psi_1 \cap [y]_\beta \psi_1$ is not empty then $[x]_\beta \psi \cap [y]_\beta \psi$ is not empty either so that $x \beta y$. Hence ψ_1 is a division. □

It is known (*see* [3, Ch. 9]) that $A^*/\gamma \in \mathbf{A}_1$ iff $\gamma \supseteq \alpha$ where α is defined in the following way. For $x \in A^*$ let $A_x = \{ a \mid a \in A,\ x = x_0 a x_1 \}$; if $A_x \neq \emptyset$ let $x\lambda (x\rho)$ be the longest prefix (suffix) of x such that $A_{x\lambda} \neq A_x$ ($A_{x\rho} \neq A_x$). The congruence α is given by $x \alpha y$ iff $A_x = A_y$ and, if $A_x \neq \emptyset$, $x = (x\lambda) au = vb (x, x\rho)$, $y = (y\lambda) aw = zb (y\rho)$ with $(x\lambda) \alpha (y\lambda)$, $(x\rho) \alpha (y\rho)$. Note that in particular x and y have the same initial and terminal letter.

THEOREM 4.2: $\mathbf{gA}_1 = \mathbf{lA}_1$.

— It suffices to show that $\mathbf{lA}_1 \subseteq \mathbf{gA}_1$. On the free category G^* let β be the smallest congruence satisfying $x \beta x^2$ for all $x \in H_{G^*}(i, i)$. Thus $C = G^*/\delta \in \mathbf{lA}_1$ iff $\delta \supseteq \beta$. In view of lemma 4.1 and the canonical property of α defined above, it is sufficient to show that $x \alpha y$ implies $x \delta y$ for any $x, y \in H_{G^*}(i, j)$.

If $|A_x| \leqq 1$ the result is trivial. We proceed by induction on $|A_x|$. We first show that $x \alpha uv$ implies $x \beta uv'$ for some v'. Again if $|A_u| \leqq 1$ the claim follows immediately. Otherwise let u_0 be the longest common prefix of x and u: if $u_0 = u$ we are done. If not, let $u = u_0 a u_1$, $x = u_0 x_0$. If u_0 does not contain the letter a then $x_0 = waz$ and $u_0 \alpha u_0 w$: this follows from the definition of α. Since $|A_{u_0}| < |A_u|$ we deduce $u_0 \beta u_0 w$: thus $x \beta u_0 az$. If u_0 does contain the letter a then $u_0 = waz$ and $x = wazx_0 \beta\ wazazx_0 = u_0 azx_0$. In both cases we have $x \beta x'$ for some x' having a longer common prefix with u. Since $\beta \subseteq \alpha$ we can iterate the argument until this common prefix coincides with u. By symmetry we also have $x \alpha uv$ implies $x \beta u'v$ for some u'. Now going back to the proof of the main result let $x \alpha y$. The statement proved above can be used to deduce $y \beta xz$ for some z: hence we also have $x \alpha xz$. Using the symmetric version of the intermediate result we get $x\ \beta\ wz$ for some w. Then $x \beta wzz \beta xz$, so that $x \beta y$. □

5. CONCLUSION

The theory of varieties of Eilenberg and Schüzenberger, relating algebraic properties of monoids and combinatorial properties of languages, has helped tremendously to organize the body of knowledge that concerns finite-state machines.

What is emerging at this point is simply a refinement of that theory. By relaxing the condition that a machine should have a free monoid as its input space, one is led to introduce categories as the right model for automata.

The notion of variety is easily generalized in a way that the relationship with languages is preserved.

The advantages are two-fold. First, as we have outlined in the introduction, partial multiplication better represents what is happening in a situation where a machine is decomposed into simpler components. Second, there are "more" C-varieties than M-varieties and the generalization from monoids to categories appears to allow enough freedom to express conveniently phenomena that are impossible to describe using exclusively the old framework. For example several results abour wreath product decompositions have been obtained in recent years by using the categorical approach. Also the C-variety **l1** provides a missing link in the theory of maximal proper epimorphisms of Rhodes [7, 8]. We believe that categories could be helpful in studying some important decidability problems like those about the dot-depth hierarchy [1] or the group-complexity hierarchy [3, Ch. 12].

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