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DECIDABILITY OF PERIODICITY FOR INFINITE WORDS (*)

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Abstract. — *We show that it is decidable whether an infinite word generated by iterated morphism is ultimately periodic or not.*

Résumé. — *Nous montrons qu'on peut décider si un mot infini engendré par morphisme itéré est ultimement périodique.*

1. INTRODUCTION

Let X be a finite alphabet and g a morphism of the free monoïd X^*, prolongable in $u_0 \in X^+$, i. e. such that $g(u_0)=u_0 u$, $u \in X^+$. Then:

$$g^i(u_0)=g^{i-1}(u_0)\,g^{i-1}(u)$$

and g defines a unique word, in general infinite, denoted by:

$$g^{\omega}(u_0)=u_0\,u g(u)\ldots g^i(u)\ldots$$

An infinite word $\mathscr{M}$ is *(ultimately) periodic* if $\mathscr{M}=vw^{\omega}=vwww\ldots$ for finite words v and w. The question of deciding whether $g^{\omega}(u_0)$ is periodic or not has been raised recently, in connection with the ω-sequence equivalence problem for DOL systems [1, 2], the adherence equivalence problem for DOL languages [3], and with the subword complexity of infinite words [4].

We give a simple proof of decidability for this question, using the notion of elementary morphism (*see* [5]). After some preliminaries, we give an algorithm for elementary morphisms in section 2 and for arbitrary morphisms in section 3.

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A subword u of an infinite word $\mathcal{M}$ is *biprolongable* if and only if there exist distinct letters x and y such that ux and uy are subwords of $\mathcal{M}$. Let $c(n)$ be the number of distinct subwords of $\mathcal{M}$ of length n. Then $c(n) \leqq c(n+1)$ and $\mathcal{M}$ is periodic if and only if $c(n)$ is bounded. But if u is a biprolongable subword of $\mathcal{M}$, $|u|=n$, then $c(n+1) \geqq c(n)+1$. Hence the following property.

LEMMA 1: *An infinite word $\mathcal{M}$ is ultimately periodic if and only if the length of its biprolongable subwords is bounded.*

Let $g: X^* \to X^*$ be a morphism. It is *simplifiable* if there exist an alphabet Y, $|Y| < |X|$ and two morphisms $f: X^* \to Y^*$, $h: Y^* \to X^*$ such that $g = h \circ f$. A morphism g is *elementary* if and only if it is not simplifiable. In this case g is injective and the set $\{g(x), x \in X\}$ is a code with bounded delay from left to right ([5], p. 131). In particular if $g(xu)$ is a prefix of $g(yv)$, $x \neq y$ then $g(xu)$ has a bounded length.

Finally a letter $x \in X$ is *growing* (for g) if $|g^n(x)|$, $n \geqq 0$ is unbounded. We denote $C \subset X$ the set of growing letters and $B = X \setminus C$ the set of *bounded* letters.

2. THE CASE OF ELEMENTARY MORPHISMS

LEMMA 2: *The infinite word $\mathcal{M} = g^{\omega}(u_0)$, with g elementary, is ultimately periodic if and only if $\mathcal{M}$ has no biprolongable subword of the form xu, $x \in C$, $u \in B^*$.*

Proof: Assume xu_1 is biprolongable. There exist infinite suffixes $xu_1 y_1 v_1$ and $xu_1 z_1 w_1$ of $\mathcal{M}$ for distinct letters y_1 and z_1. But since $\mathcal{M} = g(\mathcal{M})$, $g(y_1 v_1)$ and $g(z_1 w_1)$ are also suffixes of $\mathcal{M}$. Because g is elementary, their greatest common prefix, u_2, is finite, and $g(xu_1)u_2$ is biprolongable. Similarly there exists u_3 such that $g(g(xu_1)u_2)u_3$ is biprolongable, and so on. Thus we can construct an infinite sequence of biprolongable words, with unbounded length since x is growing. Hence $\mathcal{M}$ is not periodic by lemma 1.

Conversely, assume that there is no biprolongable factor of the form xu. We consider two cases.

First case: $\mathcal{M}$ contains only a finite number of occurrences of growing letters. Then there is only one such occurrence, and $g(u_0) = u_0 u$ with $u \in B^+$. Moreover $|g^i(u)|$, $i \geqq 0$, is bounded, and there is a smallest n such that $g^{n+1}(u) = g^i(u)$, $i \leqq n$. But then:

$$\mathcal{M} = u_0 u g(u) \dots g^{i-1}(u) [g^i(u) \dots g^n(u)]^{\omega}$$

is ultimately periodic.

Second case: $\mathcal{M}$ contains an infinite number of occurrences of growing letters, $\mathcal{M}=\alpha_0 x_1 \alpha_1 x_2 \alpha_2 \ldots$, $x_i \in C$, $\alpha_i \in B^*$. Let n be the smallest integer such that $x_{n+1}=x_i$, $i \leqq n$. Since there is no biprolongable word of the form xu, we have $\mathcal{M}=\alpha_0 x_1 \alpha_1 \ldots x_{i-1} \alpha_{i-1}\ [x_i \alpha_i \ldots x_n \alpha_n]^\omega$ and $\mathcal{M}$ is ultimately periodic. ■

COROLLARY 3: *If* $\mathcal{M}=g^\omega(u_0)$, *with* g *elementary, we can decide if* $\mathcal{M}$ *is ultimately periodic.*

Proof: Consider the following procedure:

Compute the subset of growing letters, C.

If $g(u_0)$ contains only one occurrence of letter from C then $\mathcal{M}$ is ultimately periodic.

If $g(u_0)$ contains several occurrences of letters from C then:

– compute the shortest prefix p of $\mathcal{M}$ containing two occurrences of the same growing letter x_i:

$$p=\alpha_0 x_1 \alpha_1 \ldots x_i \alpha_i x_{i+1} \ldots x_n \alpha_n x_i;$$

– for all xu prefix of some $x_j \alpha_j$, $i \leqq j \leqq n$, check if xu is biprolongable;

– $\mathcal{M}$ is ultimately periodic if and only if no xu is biprolongable.

This procedure gives the right answer by lemma 2. Moreover each step is effectively computable: one can determine if a letter is growing, and one can determine if a given word xu is biprolongable (this comes from the fact that for a given n one can compute all subwords of length n of $\mathcal{M}$, *see* [5], p. 210-212). ■

3. THE CASE OF ARBITRARY MORPHISMS

THEOREM 4: *It is decidable whether* $\mathcal{M}=g^\omega(u_0)$ *is ultimately periodic or not for an arbitrary morphism* g.

Proof: By induction on the size of the alphabet, $|X|$.

If $|X|=1$ then $\mathcal{M}$ is always periodic.

Assume the theorem is true for alphabets of size $<|X|$ and let $g: X^* \to X^*$ be an arbitrary morphism. If g is elementary then we decide if $\mathcal{M}$ is periodic by corollary 3. If g is not elementary we compute Y, $f: X^* \to Y^*$ and $h: Y^* \to X^*$ such that $g=h \circ f$ and $|Y|<|X|$. Let $g'=f \circ h$, and $u'_0=f(u_0)$. Then:

$$g'(u'_0)=g'(f(u_0))=f(g(u_0))=f(u_0 u)=u'_0\, f(u),$$

where $g(u_0)=u_0 u$. So $g'(u'_0)$ starts with u'_0 and defines an infinite word $\mathcal{M}'=g'^{\omega}(u'_0)$. Moreover $\mathcal{M}=h(\mathcal{M}')$ and $\mathcal{M}'=f(\mathcal{M})$, and $\mathcal{M}$ is ultimately periodic if and only if $\mathcal{M}'$ is. Therefore, by induction hypothesis we can decide if $\mathcal{M}'$ is periodic or not. Since the construction of g' from g is effective (*see* [5], p. 17), we can decide whether $\mathcal{M}$ is periodic or not. This proves the inductive step. ■

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