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AN ALGORITHM FOR THE WORD PROBLEM IN *HNN* EXTENSIONS AND THE DEPENDENCE OF ITS COMPLEXITY ON THE GROUP REPRESENTATION (*)

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ABSTRACT. — *A well known method for solving the word problem of a finite presented group given as HNN extension* $G^* = \langle G, a; \bar{a}ua = \varphi(u), u \in U \rangle$ *is to reduce it to the word problem of G by means of an a-reduction function* [1, 3, 4]. *It is shown that the complexity of this method depends very strongly on the representation of* G^*. *For any Grzegorczyck class* $\mathcal{E}_n$, *the complexity of the method applied to* G^* *as above may be not less than in* $\mathcal{E}_n$, *while the method applied to an other presentation of* G^*, $G^* = \langle H, s; \bar{s}vs = \psi(v), v \in V \rangle$ *may be of complexity in* $\mathcal{E}_4$. *For a recursively presented group* G^* *the method applied to one presentation may not work at all, while applied to an other presentation is very easy.*

Résumé. — *Une méthode bien connue pour résoudre le problème des mots dans un groupe donné comme extension de HNN,* $G^* = \langle G, a; \bar{a}ua = \varphi(u), u \in U \rangle$ *est de le réduire au problème des mots dans G au moyen d'une fonction de a-réduction. On montre que la complexité de cette méthode dépend très fortement de la représentation de* G^*. *Pour toute classe de Grzegorczyck* $\mathcal{E}_n$, *la complexité de la méthode, appliquée à* G^* *donné comme ci-dessus peut être dans* $\mathcal{E}_n$, *alors que la même méthode, appliquée à une autre présentation de* G^*, $G^* = \langle H, s; \bar{s}vs = \psi(v), v \in V \rangle$, *peut être de complexité dans* $\mathcal{E}_4$. *Pour un groupe* G^* *récursivement présenté, la méthode, appliquée à une certaine présentation, peut ne pas aboutir du tout, alors qu'elle est très facile pour une autre présentation.*

1. INTRODUCTION

In order to do effective computations in a group G, G is normally given by a set S of generators and a set L of defining relators. Any group element can then be represented by a word w in the generators $s \in S$ and their formal inverses $\bar{s}$, $s \in S$. The word problem is to decide for any word w whether it represents the unit element of the group. Its solvability is basic for any effective computation in G.

Let S be a set, $\bar{S} = \{ \bar{s} \mid s \in S \}$ and S^* the set of words over $S \cup \bar{S}$. We denote by e the empty word, by $\equiv$ the identity of S^* and by $|w|$ the length of the word w. Let $L \subseteq S^*$, then the set of congruence classes $[x]$ of the congruence generated by $w = e$ for $w \in L \cup \{ s\bar{s}, \bar{s}s \mid s \in S \}$ forms a group G under multiplication $[x] . [y] = [xy]$ and unit element $[e]$. We write $G = \langle S; L \rangle$ for this group.

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Any group H is isomorphic to such a group $\langle S; L\rangle$ and we call $\langle S; L\rangle$ a presentation of H. We will consider only groups given by presentations with finite S, this means finitely generated groups. $\langle S; L\rangle$ is finitely presented (f. p) resp. recursively presented (r. p.), if L is finite resp. recursively enumerable.

If we represent a group element $[x]$ by the word x, then the problem whether $[x]=[y]$ — we write $x=y$ in G in this case-reduces to the word problem $xy^{-1}=e$ in G. It is well known that the word problem is unsolvable in general even for f. p. groups, but there are large classes of groups for which it is solvable.

One method for solving the word problem for a class of groups is to use group theoretical structure properties of the groups in the class. This means to use how the groups are built up from smaller groups for reducing the word problem to the word problem of the smaller groups. This is done for instance for solving the word problem in the class of groups with only one defining relator.

The composition of a group out of smaller ones may not be unique, so one may ask whether one decomposition is better than an other one with respect to the complexity of the algorithm for solving the word problem induced by the decomposition.

We will describe algorithms in G by appropriate word functions $f: S^* \to S^*$ and identify the complexity of the algorithm with the complexity of f. In this paper the well known Grzegorczyk classes $\mathscr{E}_n$ are used to measure the complexity of the algorithms. $\mathscr{E}_n$ is the smallest class of functions containing the initial functions :

$$\varepsilon: \quad S^{*0} \to S^* \text{ constant } e,$$

$$R_a: \quad S^* \to S^*, \qquad w \to wa \quad \text{for } a \in S \cup \bar{S},$$

$$\pi_i^k: \quad S^{*k} \to S^*, \qquad (w_1, \ldots, w_k) \to w_i, \quad 1 \leqq i \leqq k,$$

together with the n-th Ackermann-function and which is closed under substitution and bounded recursion.

Here, f is derived from the functions g, h_a and b by bounded recursion means that:

$$f(x, e) \equiv g(x), \qquad f(x, ya) \equiv h_a(x, y, f(x, y))$$

and :

$$|f(x, y)| \leqq |b(x, y)|,$$

for all $x \in S^{*k}$, $y \in S^*$, $a \in S \cup \bar{S}$.

The Ackermann-functions are given by:

$$A_1(x, y) \equiv s_1^{|x|+|y|} (s_1 \in S \text{ fixed}),$$

$$A_{n+1}(x, e) \equiv s_1, \qquad A_{n+1}(x, ya) \equiv A_n(x, A_{n+1}(x, y)).$$

It can be shown that an unbounded recursion with $\mathcal{E}_n$-functions may lead to an $\mathcal{E}_{n+1}$ function. We have $\mathcal{E}_n \subseteq \mathcal{E}_{n+1}$ and the union of all the classes $\mathcal{E}_n$ coincides with the class of primitive recursive word functions.

One of the most important tools for the construction of groups with specified properties and for the analysis of algorithmic problems in groups – specially the word problem – is the concept of HNN-extension [1, 2, 4, 6]: If G is a group with isomorphic subgroups U, V and $\varphi : U \to V$ an isomorphism then the group:

$$G^* = \langle G,\ a;\ \bar{a}ua = \varphi(u),\ u \in U \rangle,$$

is called an HNN-Extension of G with stable letter a.

G is a subgroup of G^* [6], so its word problem cannot be harder than that of G^*. We want to reduce the word problem of G^* to that of G. If $G = \langle S; L \rangle$, then G^* is generated by $S_a := S \cup \{a\}$ and has as defining relators L together with the relations given above. If $w \in S_a^*$ has the form $w \equiv x\bar{a}uay$ with $u \in U$, then $w = x\,\varphi(u)\,y$ in G^*. A word $w \in S_a^*$ is a-reduced, if it does not contain a "pinch" $\bar{a}ua$, $u \in U$ or $av\bar{a}$, $v \in V$, as a segment.

A function $r : S_a^* \to S_a^*$ is an a-reduction function, if $r(w) = w$ in G^* and $r(w)$ is a-reduced.

The systematic elimination of pinches leads to an a-reduction function. More formally we can define an a-reduction function $r : S_a^* \to S_a^*$ by:

$$r(e) \equiv e,$$

$$r(wc) \equiv r(w)\,c, \qquad c \in S \cup \bar{S};$$

$$r(wa) \equiv \begin{cases} x\,\varphi(u), & w \equiv x\bar{a}u, \quad u \in S^*, \quad u \in U, \\ r(w)\,a, & \text{else;} \end{cases}$$

$$r(w\bar{a}) \equiv \begin{cases} x\,\varphi^{-1}(v), & w \equiv xav, \quad v \in S^*, \quad v \in V, \\ r(w)\,\bar{a}, & \text{else.} \end{cases}$$

By means of an a-reduction function the word problem of G^* is reducible to that of G, by [6] we get:

$$w = e \text{ in } G^* \text{ iff } r(w) \text{ is } a\text{-free and } r(w) = e \text{ in } G.$$

The complexity of this method is the least upper bound of the complexities of r and of G's word problem. In order to compute r as defined above one has to decide for $u \in S^*$ whether $u \in U\,(u \in V)$ and in the positive case to compute $\varphi(u)\,(\varphi^{-1}(u))$. If this can be done by an $\mathcal{E}_n$-process, then $r \in \mathcal{E}_{n+1}$, since r is defined by recursion with $\mathcal{E}_n$-functions in this case. Let G have a $\mathcal{E}_n$-decidable word problem, then G^* has an $\mathcal{E}_{n+1}$ decidable word problem. If in addition r can be bounded by an $\mathcal{E}_n$-function, it is itself an $\mathcal{E}_n$-function, hence G^* has $\mathcal{E}_n$-decidable word problem, too. So two questions arise:

(1) *Question:* Is this apparent jump of the complexity from $\mathscr{E}_n$ to $\mathscr{E}_{n+1}$ really possible for all n, that is: Given any $n \in \mathbb{N}$, does there exist a f. p. HNN-extension $G^* = \langle G, a; \bar{a}ua = \varphi(u), u \in U \rangle$ such that the decisions for U, V and the computations of φ, φ^{-1} are in $\mathscr{E}_n$ but there is no a-reduction function in $\mathscr{E}_n$ for G^*

(2) *Question:* How does the complexity of the described method for solving the word problem of G^* change if one chooses an other HNN-decomposition $G^* = \langle H, b; \bar{b}vb = \psi(v), v \in V \rangle$ for the group.

The first question came up to us in studying the complexity of algorithmic problems in one relator groups in [2]: Every one relator group $G = \langle S; p \rangle$ with $|p| < 2n$ can be embedded in a HNN-extension G^* of a one relator group $H = \langle S_0; p_0 \rangle$ with $|p_0| < 2n - 2$. An induction on n gives that G has $\mathscr{E}_n$-decidable word problem. If it could be proved that the jump from $\mathscr{E}_n$ to $\mathscr{E}_{n+1}$ in the reduction for G^* does not occur for some n, one would have a bound for the complexity of the word problem for one relator groups independent of the length $|p|$ of the relator. Such bound is not known [2, 5], it is conjectured however that every one relator group has $\mathscr{E}_4$-decidable word problem.

Our goal is to give for $n \geqq 4$ examples of groups G^* with a finite presentation $G^* \equiv \langle G, a; \bar{a}ua = \varphi(u), u \in U \rangle$ such that the method gives an $\mathscr{E}_4$-solution of the word problem for G^* and with an other finite presentation $G^* = \langle H, b; \bar{b}vb = \psi(v), v \in V \rangle$ such that the subgroups are $\mathscr{E}_n$-decidable and the isomorphisms are $\mathscr{E}_n$-computable but for which no b-reduction in $\mathscr{E}_n$-exists. These examples were given in [2], without proof. This paper can be viewed as an continuation of [2], but is of independent interest and the knowledge of [2] is not indispensable for this paper.

The examples answer both questions given above. So the method of solving the word problem for a group G^* given as HNN-extension depends very strongly on the HNN-representation chosen for G^*. For "bad" decompositions it may be extremely inefficient. If one allows recursively presented HNN-extensions G^* we show that for one representation there may be no recursive reduction function while for an other one there is an $\mathscr{E}_2$-reduction function and the word problem for G^* is $\mathscr{E}_2$-decidable.

2. MAIN RESULTS

In this section we first give an example of r. p. HNN-extension with $\mathscr{E}_2$-decidable word problem, such that no recursive reduction function exists. Then we turn to the more complicated case of finitely presented groups and state the main results. As usual in the determination of the complexity of algorithmic problems the main technical difficulties are in proving that some naturally given lower bound for the complexity of the problem is sharp. This will occupy most of section 3 of this paper were the proofs are given.

We need some more terminology and concepts from group theory. If $A \subseteqq S^*$, then A^{-1} is the set $A^{-1} = \{x^{-1} \mid x \in A\}$. Now A^* is the set of words in the $x \in A \cup A^{-1}$, while $\langle A \rangle$ is the set of words representing elements of the subgroup generated by A: For $G = \langle S;\ L \rangle$ and $A \subseteqq S^*$ we have $w \in \langle A \rangle$ iff there is an $u \in A^*$ such that $w = u$ in G. In the computation of a reduction function for an HNN-extension of G with a subgroup generated by a set A, we must decide wheter a word lies in $\langle A \rangle$ and in the positive case apply the isomorphism to the word of A^*. For subgroups of the free group this can be done easily if A is a set of Nielsen reduced words [6]. Let ρ be the free reduction function, then A is called Nielsen reduced if the following three conditions hold:

(1) If $x \in A$ then $x^{-1} \notin A$ and $\rho(x) \equiv x$.

(2) If $x, y \in A \cup A^{-1}$, $|\rho(xy)| > 0$ then $|\rho(xy)| \geqq |x|$.

(3) If $x, y, z \in A \cup A^{-1}$, $|\rho(xy)| > 0$, $|\rho(yz)| > 0$ then:

$$|\rho(xyz)| > |x| - |y| + |z|.$$

A Nielsen reduced set A has two advantages: Its elements freely generate the subgroup $\langle A \rangle$ and the problem $w \in \langle A \rangle$ is reducible to the problem $x \in A$.

Now we give our first example. Let $f : \mathbb{N} \to \mathbb{N}$ be an injective $\mathcal{E}_2$-function which enumerates a recursively enumerable but not decidable subset of $\mathbb{N}$. The group:

$$G^* = \langle a,\ b,\ c,\ d;\ \bar{d}\bar{b}^{f(n)} c b^{f(n)} d = \bar{a}^n \bar{b}^{f(n)} c b^{f(n)} a^n \ (n \in \mathbb{N}) \rangle,$$

is a HNN-Extension of the free group $G = \langle a,\ b,\ c;\ \emptyset \rangle$ with stable letter d. There is no computable d-reduction function r for G^*, because of $m \in f(\mathbb{N})$ iff $r(\bar{d}\bar{b}^m c b^m d)$ is d-free.

On the other hand another representation for G^* as a HNN-extension of a free group is:

$$G^* = \langle a,\ b,\ d,\ c;\ \bar{c} b^{f(n)} a^n \bar{d}\bar{b}^{f(n)} c = b^{f(n)} a^n \bar{d}\bar{b}^{f(n)} \ (n \in \mathbb{N}) \rangle.$$

Here the stable letter is c, $U = V$ and the identity is the isomorphism. The set $\{ b^{f(n)} a^n \bar{d}\bar{b}^{f(n)} \mid n \in \mathbb{N} \}$ is Nielsen reduced and $\mathcal{E}_2$-decidable in $\langle a,\ b,\ d;\ \emptyset \rangle$. So $U$and V are $\mathcal{E}_2$-decidable and the definition of paragraph 1 gives a c-reduction function for G^* in $\mathcal{E}_2$. This means that the word problem for G^* is $\mathcal{E}_2$ decidable, though there is no d-reduction function for the first representation of G^*. Of course similar examples can be constructed for any complexity degree, but notice that the group in this example is not finitely presented.

We turn now to the class of finitely presented groups. We define two sequences of groups G_n and H_n which are built up as HNN-extensions. To simplify notations we set $s_0 \equiv b^2$.

$$G_0 = \langle a,\, b;\, \emptyset \rangle,$$
$$G_n = \langle G_{n-1},\, s_n;\, \bar{s}_n a s_n = \bar{b} a b,\, \bar{s}_n b s_n = s_{n-1} \rangle, \qquad n > 0,$$
$$H_0 = \langle b;\, \emptyset \rangle,$$
$$H_n = \langle H_{n-1},\, s_n;\, \bar{s}_n b s_n = s_{n-1} \rangle, \qquad n > 0,$$

where $S = \{ s_n \mid n > 0 \}$ is an infinite set of letters different from a, b. It is easy to see that the groups H_n are all one relator groups.

2.1. Theorem: For any $n \geqq 1$:

(*a*) *G_n is a HNN-Extension of G_{n-1} with stable letter s_n. There exists a s_n-reduction function $r_n \in \mathcal{E}_{n+2}$, but no s_n-reduction function lies in $\mathcal{E}_{n+1}$.*

(*b*) *H_n is a HNN extension of H_{n-1} with stable letter s_n. There exists a s_n-reduction function $\rho_n \in \mathcal{E}_4$ and for $n \geqq 2$ no s_n-reduction function lies in $\mathcal{E}_3$.*

(*c*) *G_n is a HNN-extension of H_n with stable letter a and there exists an a-reduction function in $\mathcal{E}_4$.*

(*d*) *Both, G_n and H_n, have $\mathcal{E}_4$-decidable word problems.*

We will prove this theorem in paragraph 3.

The groups G_n, H_n can be pictured in the following way.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
G_0 \arrow[r, "s_1", "{\substack{a \to \bar{b}ab \\ b \to b^2}}"'] & G_1 \arrow[r, "s_2", "{\substack{a \to \bar{b}ab \\ b \to s_1}}"'] & G_2 \cdots\cdots & G_{n-1} \arrow[r, "s_n", "{\substack{a \to \bar{b}ab \\ b \to s_{n-1}}}"'] & G_n \cdots \\
H_0 \arrow[u, "a"] \arrow[r, "s_1", "b \to b^2"'] \arrow[ur, phantom, "e \to e"] & H_1 \arrow[u, "a"] \arrow[r, "s_2", "b \to s_1"'] \arrow[ur, phantom, "b\bar{s}_1 \to b\bar{s}_1"] & H_2 \cdots\cdots \arrow[u, "a", "{\substack{b\bar{s}_1 \to b\bar{s}_1 \\ b\bar{s}_2 \to b\bar{s}_2}}"'] & H_{n-1} \arrow[u, "a"] \arrow[r, "s_n", "b \to s_{n-1}"'] \arrow[ur, phantom, "{\substack{b\bar{s}_1 \to b\bar{s}_1 \\ \vdots \\ b\bar{s}_{n-1} \to b\bar{s}_{n-1}}}"] & H_n \cdots \arrow[u, "a", "{\substack{b\bar{s}_1 \to b\bar{s}_1 \\ \vdots \\ b\bar{s}_n \to b\bar{s}_n}}"']
\end{tikzcd}$$

We draw some consequences of the theorem.

2.2. Corollary: *For $n \geqq 3$ there is a finitely presented HNN-extension $G^* = \langle G,\, a;\, \bar{a} u a = \varphi(u),\, u \in U \rangle$ with:*

(i) *U and $\varphi(U)$ are $\mathcal{E}_n$-decidable subgroups of G.*

(ii) *The isomorphisms φ and φ^{-1} are $\mathcal{E}_n$-computable.*

(iii) *There is no a-reduction function for G^* in $\mathcal{E}_n$.*

2.3. Corollary: *For $n \geqq 3$ there is a finitely presented HNN-extension:*

$$G^* = \langle G, s; \bar{s}us = \varphi(u), u \in U \rangle = \langle H, a; \bar{a}va = \varphi(v), v \in V \rangle,$$

with:

(i) *G and H both have $\mathscr{E}_4$-decidable word problem.*

(ii) *There exists an a-reduction function for G^* in $\mathscr{E}_4$.*

(iii) *There is no s-reduction function for G^* in $\mathscr{E}_n$.*

(iv) *G^* has $\mathscr{E}_4$-decidable word problem.*

These corollaries follow from the proof of theorem 2.1 in paragraph 3. For 2.2 take G_{n-1} as HNN-extension of G_{n-2} and for 2.3 take G_{n-1} as HNN-extension of G_{n-2} and of H_{n-1}.

3. PROOF OF THEOREM 2.1

3.1. Lemma: *For $n \geqq 1$ we have:*

(*a*) *$H_n = \langle H_{n-1}, s_n; \bar{s}_n b s_n = s_{n-1} \rangle$ is a HNN-extension of H_{n-1}.*

(*b*) *$G_n = \langle H_n, a; \bar{a} b \bar{s}_j a = b \bar{s}_j, j = 1, \ldots, n \rangle$ is a HNN-extension of H_n.*

(*c*) *$G_n = \langle G_{n-1}, s_n; \bar{s}_n a s_n = \bar{b} a b, \bar{s}_n b s_n = s_{n-1} \rangle$ is a HNN-extension of G_{n-1}.*

Proof: We have to prove the isomorphism condition for HNN-extensions.

(*a*) $\varphi : b \to s_{n-1}$ is an isomorphism, since $\langle b \rangle$ and $\langle s_{n-1} \rangle$ are free subgroups of H_{n-1} (notice that $s_0 \equiv b^2$).

(*b*) The isomorphism is the identity.

(*c*) Since $\langle a, b \rangle$ is a free subgroup of rank two of G_{n-1}, it suffices to prove that $\langle \bar{b} a b, s_{n-1} \rangle$ is a free subgroup of rank two also. Let $U_n = \{ s_1 \bar{b}, \ldots, s_n \bar{b} \}$ and $\omega \in U_n^*$ be freely reduced in the $(s_j \bar{b})^{\pm 1}$. An induction on n shows:

(i) ω is fully reduced, i.e. contains no s_i-pinches $(1 \leqq i \leqq n)$;

(ii) $\bar{b} \omega b = s_n^k$ in $H_n \Leftrightarrow k = 0$ and $\omega \equiv e$;

(iii) $\omega = b^k$ in $H_n \Leftrightarrow k = 0$ and $\omega \equiv e$.

By (i) the subgroup $\langle U_n \rangle$ of H_n – and hence of G_n too – is freely generated by the $s_j \bar{b}$. By (ii) any $\omega \in \{ \bar{b}ab, s_n \}^*$ freely reduced in $\bar{b} a^{\pm 1} b, s_n^{\pm 1}$ is a-reduced in G_n. Hence $\langle \bar{b} ab, s_n \rangle$ is a free subgroup of rank two of G_n. △

We want to prove that there is no s_n-reduction function for G_n in $\mathscr{E}_{n+1}$ but there is an s_n-reduction function for H_n in $\mathscr{E}_4$. For this we show that the words $\bar{s}_n^i \bar{b} a b s_n^i$ become very long after s_n-reduction. In order to measure the increase of the length we introduce functions $f : \mathbb{N} \to \mathbb{N}$ and $f_n : \mathbb{N} \to \mathbb{N}\, (n \geqq 0)$ by the following equations.

For $i, n \in \mathbb{N}$ let :

(a) $f(0)=1, f(i+1)=2^{f(i)}$;

(b) $f_0(i)=2i+1, f_{n+1}(i)=f_n^{(i)}(1)$ (f_n iterated i-times).

Then we get :

3.2. LEMMA: *(a) The function f is a generating function of the arithmetic class $\mathscr{E}_4$ and the f_n are generating functions of the classes $\mathscr{E}_{n+2}(n \geqq 1)$:*

(b) Let $x_0 \equiv b$ and $x_{i+1} \equiv \bar{s}_2 x_i^{-1} s_2 b \bar{s}_2 x_i s_2 (i \geqq 1)$, then $x_i = b^{f(i)}$ in H_2.

(c) For $i, n \in \Pi$, $\bar{s}_n^i \bar{b} a b s_n^i = \bar{b}^{f_n(i)} a b^{f_n(i)}$ in G_n.

Proof: For (a) *see* f. i. [7], $f \in \mathscr{E}_4 - \mathscr{E}_3$ and $f_n \in \mathscr{E}_{n+2} - \mathscr{E}_{n+1} (n \geqq 1)$.

(b) By induction on i :

$$x_{i+1} = \bar{s}_2 \bar{x}_i^1 s_2 b \bar{s}_2 x_i s_2 = \bar{s}_2 \bar{b}^{f(i)} s_2 b \bar{s}_2 b^{f(i)} s_2 = \bar{s}_1^{f(i)} b s_1^{f(i)} = b^{2^{f(i)}} \quad \text{in } H_2.$$

(c) By induction on n, i :

$$\bar{s}_{n+1}^{i+1} \bar{b} a b s_{n+1}^{i+1} = \bar{s}_{n+1} \bar{b}^{f_{n+1}(i)} a b^{f_{n+1}(i)} s_{n+1} = \bar{s}_n^{f_{n+1}(i)} \bar{b} a b s_n^{f_{n+1}(i)}$$

$$= \bar{b}^{f_n(f_{n+1}(i))} a b^{f_n(f_{n+1}(i))} \quad \text{in } G_{n+1}. \quad \triangle$$

In the sequel we will use the following abbreviation:

$$k_{n,i} :\equiv \begin{cases} \bar{b}^{f_n(i)} a b^{f_n(i)}, & i > 0, \\ b^{-(i+1)} a b^{i+1}, & i \leqq 0. \end{cases}$$

By Lemma 3.2 the following equation holds:

$$k_{n,i} = \bar{s}_n^i \bar{b} a b s_n^i \quad \text{in } G_n \quad \text{for } n \geqq 1 \text{ and } i \geqq -n.$$

3.3. LEMMA: *Let $n \geqq 1$ and $K_n = \langle k_{n,j} \mid j \geqq -n \rangle$ in G_0.*

Then for any $\omega \in G_n$ we have $\omega \in \langle \bar{b} a b, s_n \rangle$ in G_n iff there are $l, m \geqq 0$ and $u \in \{a, b\}^*$, $u \in K_n$ such that $\omega = s_n^l u s_n^{-m}$ in G_n and $s_n^l u s_n^{-m}$ is s_n-reduced.

This means that the decision for $w \in \langle \bar{b} a b, s_n \rangle$ in G_n is reduced after s_n-reduction to the decision $u \in K_n$ in the free group $G_0 = \langle a, b; \emptyset \rangle$.

Proof: "$\Rightarrow$" Since $k_{n,j} = \bar{s}_n^j \bar{b} a b s_n^j \in \langle \bar{b} a b, s_n \rangle$, we have $\omega = s_n^l u s_n^{-m} \in \langle \bar{b} a b, s_n \rangle$ if $u \in K_n$.

"$\Rightarrow$" Let $\omega \in \{\bar{b} a b, s_n\}^*$. Because of $\bar{s}_n k_{n,j} s_n = k_{n,j+1}$ in G_n for $j \geqq -n$, we can shift in ω all $\bar{s}_n$ to the right and all s_n to the left. This gives a word $v \in \{k_{n,i} \mid i \geqq 0\}^*$ with $\omega = s_n^p v s_n^{-q}$ in G_n $(p, q \geqq 0)$.

The word $s_n^p v s_n^{-q}$ may not be s_n-reduced. We show by induction on n that the s_n-reduction of such a word $s_n^p v s_n^{-q}$ results in a word $s_n^l u s_n^{-m}$ with $u \in K_n$, $l, m \geqq 0$. To do this, it suffices to prove that $s_n v \bar{s}_n \in K_n$, if $v \in K_n$ and $s_n v \bar{s}_n$ is a s_n-pinch.

Notice that the $k_{n,j}$ freely generate K_n in the free group $\langle a, b; \emptyset \rangle = G_0$ because they are Nielsen reduced.

For $n=1$, if $s_1 v \bar{s}_1$ is a s_1-pinch we have $v \in \langle k_{1,j} | j \geqq -1 \rangle$ and $v \in \langle \bar{b} ab, b^2 \rangle$ in G_0. So $v \in \langle k_{1,j} | j \geqq 0 \rangle$ and $s_1 v \bar{s}_1 \in \langle k_{1,j} | j \geqq -1 \rangle = K_1$.

For the induction step: If $s_n v \bar{s}_n$ is a s_n-pinch then:

$$v \in \langle k_{n,j} | j \geqq -n \rangle \cap \langle \bar{b} ab, s_{n-1} \rangle .$$

By the induction hypothesis v cannot contain the factor $k_{n-1,-n} = k_{n,-n}$. We get

$v \in \langle k_{n,j} | j > -n \rangle$ and $s_n v \bar{s}_n \in \langle k_{n,j} | j \geqq -n \rangle = K_n$. △

With this Lemma we can now prove the existence of a s_n-reduction function in $\mathscr{E}_{n+2}$ for G_n.

3.4. LEMMA: *There is a s_n-reduction function $r_n \in \mathscr{E}_{n+2}$ for $G_n = \langle G_{n-1}, s_n; \bar{s}_n a s_n = \bar{b} ab, \bar{s}_n b s_n = s_{n-1} \rangle$; but for $n \geqq 1$ there is no s_n-reduction function for G_n in $\mathscr{E}_{n+1}$.*

Proof: The proof is again by induction on n. For $n=0$, G_0 is the free group $\langle a, b; \emptyset \rangle$ and we choose r_0 to be the free reduction on G_0. We will define r_n such that $r_n(\omega)$ is fully reduced, i. e. contains no s_i-pinch as a subword ($1 \leqq i \leqq n$).

Suppose $r_{n-1} \in \mathscr{E}_{n+1}$ has been defined. For the definition of r_n we use the following auxiliary function h_n:

$$\begin{aligned}
&h_n(e) \equiv e, \\
&h_n(\omega x^\varepsilon) \equiv h_n(\omega) x^\varepsilon, \qquad x \in \{a, b, s_1, \ldots, s_{n-1}\} \qquad \varepsilon = \pm 1, \\
&h_n(\omega s_n) \equiv \begin{cases} u \varphi(v), & h_n(\omega) \equiv u \bar{s}_n v, \quad v \in \langle a, b \rangle, \\ h_n(\omega) s_n, & \text{otherwise,} \end{cases} \\
&h_n(\omega \bar{s}_n) \equiv \begin{cases} u \varphi^{-1}(v), & h_n(\omega) \equiv u s_n v, \quad v \in \langle \bar{b} ab, s_{n-1} \rangle, \\ h_n(\omega) \bar{s}_n, & \text{otherwise} \end{cases}
\end{aligned}$$

(in both cases v is s_n-free) and put:

$$r_n(\omega) \equiv r_{n-1}(\omega_0) s_n^{\varepsilon_1} r_{n-1}(\omega_1) \ldots s_n^{\varepsilon_p} r_{n-1}(\omega_p),$$

if:

$$h_n(\omega) \equiv \omega_0 s_n^{\varepsilon_1} \omega_1 \ldots s_n^{\varepsilon_p} \omega_p \qquad \text{with} \quad \omega_i \quad s_n\text{-free.}$$

We will have $r_n \in \mathscr{E}_{n+2}$, if we can prove $h_n \in \mathscr{E}_{n+2}$. We show that the subgroups $\langle a, b \rangle$ and $\langle \bar{b}ab, s_{n-1} \rangle$ of G_{n-1} are $\mathscr{E}_{n+1}$-decidable and the isomorphisms φ and φ^{-1} are $\mathscr{E}_{n+1}$-computable. This immediately gives $h_n \in \mathscr{E}_{n+2}$.

Let $A_i = \{ a, b, s_1, \ldots, s_i \}$. For $v \in A_{n-1}^*$ we have $v \in \langle a, b \rangle$ in G_{n-1} iff $r_{n-1}(v) \in \{ a, b \}^*$, since $r_{n-1}(v)$ is fully reduced.

We get $\varphi(v)$ if we replace a by $\bar{b}ab$ and b by s_{n-1} in $r_{n-1}(v)$. So $\langle a, b \rangle$ is $\mathscr{E}_{n+1}$-decidable and φ is $\mathscr{E}_{n+1}$-computable, since $r_{n-1} \in \mathscr{E}_{n+1}$.

If $v \in \langle \bar{b}ab, s_{n-1} \rangle$ in G_{n-1} then $v = s_{n-1}^l u s_{n-1}^{-m}$ in G_{n-1}, where l, $m \geqq 0$, $u \in K_{n-1}$ and $s_{n-1}^l u s_{n-1}^{-m}$ is s_{n-1}-reduced by lemma 3.3. Because of $r_{n-1}(v) = v = s_{n-1}^l u s_{n-1}^{-m}$ in G_{n-1}, $r_{n-1}(v)$ and $s_{n-1}^l u s_{n-1}^{-m}$ must be s_{n-1}-parallel. We get $v \in \langle \bar{b}ab, s_{n-1} \rangle$ in G_{n-1} $\Leftrightarrow$

$$r_{n-1}(v) \equiv v_0 s_{n-1} v_1 \ldots s_{n-1} v_l \bar{s}_{n-1} v_{l+1} \ldots \bar{s}_{n-1} v_{l+m}$$
$$\wedge\, r_{n-1}(\bar{s}_{n-1}^l r_{n-1}(v) s_{n-1}^m) \in K_{n-1}.$$

Because $\{ k_{n-1,\, j} \mid j \geqq -n+1 \}$ is a Nielsen reduced system of generators for K_{n-1} in the free group G_0, it is $\mathscr{E}_{n+1}$-decidable whether a word $x \in A_{n-1}^*$ is in K_{n-1}, and to rewrite it as a word in the $k_{n-1,\, j}$. The $k_{n-1,\, j}$ can in turn be written as words in $\bar{b}ab$, s_{n-1}. For the computation of φ^{-1} replace $\bar{b}ab$ by a and s_{n-1} by b in this word. So the subgroup $\langle \bar{b}ab, s_{n-1} \rangle$ is $\mathscr{E}_{n+1}$-decidable in G_{n-1} and φ^{-1} is $\mathscr{E}_{n+1}$-computable.

We have now proved $r_n \in \mathscr{E}_{n+2}$. Assume $r'_n \in \mathscr{E}_{n+1}$ is a s_n-reduction function for G_n. Since $r_{n-1} \in \mathscr{E}_{n+1}$ and $r'_n(s_n^{-i} \bar{b}abs_n^i)$ is s_n-free, we can define the words $\omega_i \equiv r_{n-1}(r'_n(s_n^{-i} \bar{b}abs_n^i))$ $(i \geqq 0)$ and $|\omega_i|$ must be bounded by an $\mathscr{E}_{n+1}$-function. But of course:

$$\omega_i = \bar{b}^{f_n(i)} ab^{f_n(i)} \quad \text{in } G_0,$$

and $f_n(i)$ grows faster than any $\mathscr{E}_{n+1}$-function. The right side of the last equation is freely reduced and hence of minimal length among all words equal to ω_i in G_0. So the $|\omega_i|$ are not bounded by an $\mathscr{E}_{n+1}$-function and no s_n-reduction function in $\mathscr{E}_{n+1}$ exists for G_n. $\triangle$

Our next lemma will be concerned with reduction functions for the groups H_n.

3.5. Lemma: *There is a s_n-reduction function $\rho_n \in \mathscr{E}_4$ for $H_n = \langle H_{n-1}, s_n;\ \bar{s}_n b s_n = s_{n-1} \rangle$. For $n \geqq 2$ there is no s_n-reduction function for H_n in $\mathscr{E}_3$.*

Proof: We define the functions ρ_n by induction on n and we use again some auxiliary functions h_n as in the proof of 3.4. They are defined in a different way in order to ensure some special properties needed later on. (It costs some effort to show $\rho_n \in \mathscr{E}_4$.)

Let again ρ_0 be the free reduction on $H_0 = \langle b : \emptyset \rangle$. The functions h_n are defined as follows under the assumption that $\rho_{n-1} \in \mathscr{E}_4$ has been found:

$$h_n(e) \equiv e,$$
$$h_n(\omega x^{\varepsilon}) \equiv h_n(\omega) x^{\varepsilon}, \quad x \in \{ b, s_1, \ldots, s_{n-1} \}, \quad \varepsilon = \pm 1,$$

$$h_n(\omega s_n) \equiv \begin{cases} u s_{n-1}^i, & h_n(\omega) \equiv u \bar{s}_n v, \quad v = b^i, \\ u \bar{s}_n \rho_{n-1}(v) s_n, & h_n(\omega) \equiv u \bar{s}_n v, \quad v \notin \langle b \rangle, \\ u s_n \rho_{n-1}(v) s_n, & h_n(\omega) \equiv u s_n v, \\ \rho_{n-1}(v) s_n, & h_n(\omega) \equiv v, \end{cases}$$

$$h_n(\omega \bar{s}_n) \equiv \begin{cases} u b^i, & h_n(\omega) \equiv u s_n v, \quad v = s_{n-1}^i, \\ u s_n \rho_{n-1}(v) \bar{s}_n, & h_n(\omega) \equiv u s_n v, \quad v \notin \langle s_{n-1} \rangle, \\ u \bar{s}_n \rho_{n-1}(v) \bar{s}_n, & h_n(\omega) \equiv u \bar{s}_n v, \\ \rho_{n-1}(v) \bar{s}_n, & h_n(\omega) \equiv v. \end{cases}$$

Here we assume v to be s_n-free.

If $\omega \in H_n$ is given, we define ω', $\rho_n(\omega)$ as follows. If:

$$\omega \equiv \omega_0 s_n^{\varepsilon_1} \omega_1 \ldots s_n^{\varepsilon_p} \omega_p \qquad (\omega_i \cdot s_n\text{-free}),$$

then:

$$\omega' \equiv \rho_{n-1}(\omega_0) s_n^{\varepsilon_1} \rho_{n-1}(\omega_1) \ldots s_n^{\varepsilon_p} \rho_{n-1}(\omega_p) \qquad \text{and} \qquad \rho_n(\omega) \equiv \hat{h}_n(\omega'),$$

where $\hat{h}_n(\omega')$ comes from $h_n(\omega')$ by replacing the last s_n-free syllabe u in $h_n(\omega')$ by $\rho_{n-1}(u)$.

The following three facts are easily proved:

(i) $\rho_n(\omega)$ is full reduced and, if ω is full reduced, $\rho_n(\omega) \equiv \omega$.

(ii) The order of pinch resolutions in the computation of $\hat{h}_n(\omega')$ is left most – inner most. In particular if $\omega' \equiv \omega_1 s_n^{\varepsilon} \omega_2 \bar{s}_n^{\varepsilon} \omega_3$, where $s_n^{\varepsilon} \omega_2 \bar{s}_n^{\varepsilon}$ is the last pinch resolved, then $\hat{h}_n(\omega') \equiv \hat{h}_n(\hat{h}_n(\omega_1) s_n^{\varepsilon} \hat{h}_n(\omega_2) \bar{s}_n^{\varepsilon} \omega_3)$.

(iii) If $\rho_n(\omega) \equiv u_0 s_n^{\varepsilon_1} u_1 \ldots s_n^{\varepsilon_p} u_p$, then there is a decomposition: $\omega \equiv \omega_0 s_n^{\varepsilon_1} \omega_1 \ldots s_n^{\varepsilon_p} \omega_p$ with $u_i \equiv \rho_n(\omega_i)$.

(For the last two properties we need the special definition of h_n.)

To prove $\rho_n \in \mathscr{E}_4$ we need another fact: If u, v are fully reduced and $i \in \mathbb{Z}$ then we have:

$$(\star) \qquad |\rho_n(u b^i v)| \leqq \hat{\imath}\, 2^{|uv|} \qquad \text{and} \qquad |\rho_n(u s_n^i v)| \leqq \hat{\imath} s^{|uv|},$$

where $\hat{\imath}$ is max $\{1, |i|\}$.

For the function $\hat{h}_n$ we have $|\hat{h}_n(\omega')| \leqq \hat{f}(|\omega'|)$ with ω' as above and $\hat{f}$ as in definition 3.2. This is proved by induction on the number of s_n-pinches in the computation of $\hat{h}_n(\omega')$ using (ii) and ($\star$).

A further induction on n gives then the final result:

$$|\omega'| \leqq \hat{f}^{n-1}(|\omega|) \qquad \text{and} \qquad |\rho_n(\omega)| \leqq \hat{f}^n(|\omega|).$$

This means the reduction functions ρ_n is bounded by an $\mathscr{E}_4$-function and hence is itself an $\mathscr{E}_4$-function.

To prove ($\star$) we use an induction on n to prove the following fact:

If $u, v \in H_n$ are fully reduced and $i, j, k, l \in \mathbb{Z}$ then:

$$(a) \qquad \begin{array}{l} ub^l s_n^i = b^j \ \text{ in } H_n \ \Rightarrow \ |j| \leqq \hat{l}.2^{|u|}, \\ ub^l s_n^i = s_{n-1}^j \ \text{ in } H_n \ \Rightarrow \ |j| \leqq \hat{l}.2^{|u|}; \end{array}$$

$$(b) \qquad \begin{array}{l} |\rho_n(ub^i s_n^k v)| \leqq \hat{i}\hat{k}\, 2^{|uv|}, \\ |\rho_n(us_n^k b^i v)| \leqq \hat{i}\hat{k}\, 2^{|uv|}. \end{array}$$

We will show only the induction step from $n-1$ to n, for the induction basis for $n=1$ is proved in a similar way.

For (a) we make a further induction on $p=|i|=|u|_{s_n}$.

Let $p=0$: $ub^l=b^j$ in H_n implies $u\equiv b^k$, since u is fully reduced, and we get $|j| \leqq |k|+|l| \leqq \hat{l}\cdot 2^{|u|}$.

If $ub^l=s_{n-1}^j$ then $|j|=|u|_{s_{n-1}} \leqq |u| \leqq 2^{|u|}$.

Induction step $p-1 \to p$: Let $u\equiv u_0 s_n^\varepsilon u_1$ where u_0 is s_n-free and $\varepsilon=-sgn(i)$. We may assume $\varepsilon=-1$, the case $\varepsilon=1$ is analogous.

Both, $ub^l s_n^i=b^j$ and $ub^l s_n^i=s_{n-1}^j$ in H_n imply $ub^l s_n^i \in H_{n-1}$ and

$$ub^l s_n^i \equiv u_0 \bar{s}_n u_1 b^l s_n^{i-1} s_n = u_0 \bar{s}_n b^k s_n = u_0 s_{n-1}^k \quad \text{in } H_n,$$

where $u_1 b^l s_n^{i-1}=b^k$ in H_n and so $|k| \leqq \hat{l}\cdot 2^{|u_1|}$.

For $ub^l s_n^i=b^j$ in H_n we get $b^j=u_0 s_{n-1}^k$ in H_{n-1} which leads to:

$$|j| \leqq 2^{|u_0|} \leqq 2^{|u|}.$$

For $ub^l s_n^i=s_{n-1}^j$ in H_n we get $s_{n-1}^j=u_0 s_{n-1}^k$ in H_{n-1} which leads to:

$$|j| \leqq |u_0|_{s_{n-1}} + |k| \leqq |u_0| + \hat{l}\cdot 2^{|u_1|} \leqq \hat{l}\cdot 2^{|u|}.$$

So (a) is proved.

To prove (b) we first assume $k=0$. Then we may assume $\rho_n(ub^i v)$ to be s_n-free. For otherwise $u\equiv u_0 u_1$ and $v\equiv v_1 v_0$ where u_0 and v_0^{-1} are empty or end with s_n^ε and $\rho_n(ub^i v)\equiv u_0 \rho_n (u_1 b^i v_1) v_0$ with $\rho_n (u_1 b^i v_1)$ s_n-free. Now, of course, $|\rho_n(u_1 b^i v_1)| \leqq \hat{i}\, 2^{|u_1 v_1|}$ implies $|\rho_n (ub^i v)| \leqq \hat{i}\, 2^{|uv|}$.

Since $\rho_n(ub^i v)$ is s_n-free we must have $|u|_{s_n}=|v|_{s_n}$. We make now an induction on $p=|u|_{s_n}$. If $p=0$ then $\rho_n(ub^i v)\equiv\rho_{n-1}(ub^i v)$ and by induction hypothesis (for $n-1$) we get $|\rho_n(ub^i v)|\leqq\hat{\imath}2^{|uv|}$. Let $u\equiv u_0 s_n^{-\varepsilon}u_1$ and $v\equiv v_1 s_n^{\varepsilon}v_0$ where u_0 and v_0 are s_n-free. Since $ub^i v\in H_{n-1}$, $s_n^{-\varepsilon}u_1 b^i v_1 s_n^{\varepsilon}$ must be a pinch, which means $u_1 b^i v_1=b^j$ or $=s_{n-1}^j$ depending on whether $\varepsilon=-1$ or $\varepsilon=1$ and in both cases we have, by the induction hypothesis (for $p-1$), $|j|\leqq|\rho_n(u_1 b^i v_1)|\leqq\hat{\imath}2^{|u_1 v_1|}$. We get:

$$\rho_n(ub^i v)\equiv\begin{cases}\rho_{n-1}(u_0 b^j v_0), & \varepsilon=-1,\\ \rho_{n-1}(u_0 s_{n-1}^j v_0), & \varepsilon=1.\end{cases}$$

The induction hypothesis (for $n-1$) gives $|\rho_n(ub^i v)|\leqq\hat{\jmath}.2^{|u_0v_0|}\leqq\hat{\imath}.2^{|uv|}$.

Let now $k\neq 0$. Because of the symmetry we only consider $\rho_n(us_n^k b^i v)$ and $k>0$. Let $k=j+m+l$ such that the first j s_n in s_n^k react with u, the last l s_n in s_n^k react with $b^i v$ and m s_n remain unpinched. This means that we may decompose $u\equiv u_1 u_0$, $v\equiv v_0 v_1$ with $u_0 s_n^j=s_{n-1}^{j_1}$ and $s_n^l b^i v_0=b^{l_1}$ in H_n, where u_0 and v_0^{-1} are empty or begin with $\bar{s}_n$ and $|j_1|\leqq 2^{|u_0|}$, $|l_1|\leqq\hat{\imath}2^{|v_0|}$ by (a). If $m>0$ then:

$$\rho_n(us_n^k b^i v)\equiv\rho_n(u_1 s_{n-1}^{j_1})s_n^m\rho_n(b^{l_1}v_1)$$

and:

$$|\rho_n(us_n^k b^i v)|\leqq\hat{\jmath}_1.2^{|u_1|}+m+|l_1|+|v_1|\leqq 2^{|u_0u_1|}+m+\hat{\imath}2^{|v_0|}+|v_1|\leqq\hat{\imath}\hat{k}2^{|uv|}.$$

If $m=0$ then s_n-pinches may occur between u_1 and v_1. We assume that u_1 and v_1 are not s_n-free and let $u_1\equiv u_3 s_n^{\varepsilon}u_4$ and $v_1\equiv v_4 s_n^{\zeta}v_3$ where u_4 and v_4 are s_n-free.

If a s_n-pinch occurs then $\mathscr{E}=-\zeta$ and $u_4 s_{n-1}^{j_1}b^{l_1}v_4$ is equal to b^q or s_{n-1}^q in H_{n-1}, depending on $\varepsilon=+1$ or $\varepsilon=-1$.

By induction hypothesis (for $n-1$) we have $|q|\leqq\hat{l}_1.\hat{\jmath}_1.\hat{\jmath}_2 2^{|u_4v_4|}$ and we get:

$$\rho_n(us_n^k b^i v)\equiv\begin{cases}\rho_n(u_3\bar{s}_n b^q s_n v_3) & \text{if } \varepsilon=-1,\\ \rho_n(u_3 b^q v_3) & \text{if } \varepsilon=1,\end{cases}$$

and:

$$|\rho_n(us_n^k b^i v)|\leqq\hat{q}\cdot 2^{|u_3v_3|+2}\leqq\hat{l}_1\cdot\hat{\jmath}_1\cdot 2^{|u_3u_4v_3v_4|+2}\leqq\hat{\imath}\cdot 2^{|v_0|}.2^{|u_0|}.2^{|u_3u_4v_3v_4|+2}\leqq\hat{\imath}\cdot\hat{k}\cdot 2^{|uv|}.$$

If no further s_n-pinch occurs then:

$$\rho_n(us_n^k b^i v)\equiv u_3 s_n^{\varepsilon}\rho_{n-1}(u_4 s_{n-1}^{j_1}b^{l_1}v_4)s_n^{\zeta}v_3$$

and:

$$|\rho_n(us_n^k b^i v)|\leqq|u_3|+1+\hat{l}_1\cdot\hat{\jmath}_1\cdot 2^{|u_4v_4|}+1+|v_3|\leqq\hat{\imath}\cdot\hat{k}\cdot 2^{|uv|}.$$

Notice that for H_2 there is no s_2-reduction function in $\mathscr{E}_3$: Let $x_0 \equiv b$ and $x_{i+1} \equiv \bar{s}_2 x_i^{-1} s_2 b\bar{s}_2 x_i s_2$. Then $|x_i| \leqq 2^{i+1}$ but $x_i = b^{f(i)}$ in H_2, so $|x_i|$ is $\mathscr{E}_3$-bounded and $b^{f(i)}$ is not. Of course then there cannot exist a s_n-reduction function in $\mathscr{E}_3$ for H_n ($n \geqq 2$). △

Next we want to prove that there is an a-reduction function in $\mathscr{E}_4$ for $G_n = \langle H_n, a; \bar{a}b\bar{s}_i a = b\bar{s}_i, i=1, \ldots, n\rangle$. Since G_n is a HNN-extension of H_n with the identity as isomorphism we just have to show that the problem $\omega \in \langle b\bar{s}_1, \ldots, b\bar{s}_n\rangle$ in H_n is $\mathscr{E}_4$-decidable. To do this we first need an auxiliary lemma.

3.6. LEMMA: *Let* $U_n = \{s_1\bar{b}, \ldots, s_n\bar{b}\}$, $\omega \in H_n$ *fully reduced and* $l, m \in \mathbb{Z}$. *If* $\omega \notin \langle s_n\rangle$ *in* H_n *and* $s_n^l \omega s_n^m \in \langle U_n\rangle$ *in* H_n *then* $|l|, |m| \leqq |\omega| + n$.

Proof: We will use without special mention the following properties of f_n which are easily proved by induction: Let ω be fully reduced.

(a) If $\omega^{-1} a \omega = \bar{b}^i a b^i$ in G_n with $i \in \mathbb{Z}$, then $|i| \leqq f_n(|\omega|)$.

(b) For $k \geqq 0$, $i \in \mathbb{Z}$, $l \geqq f_n(j)$ and $|\omega| \leqq j$: If $\omega \bar{b}^l a b^l \omega^{-1} = s_n^k \bar{b}^i a b^i \bar{s}_n^k$ in G_n where $s_n \bar{b}^i a b^i \bar{s}_n$ is s_n-reduced, then $i \geqq f_n(j - |\omega|) \geqq 1$ and $k \leqq |\omega|_{s_n}$.

To prove the lemma, let $s_n^l \omega s_n^m \in \langle U_n\rangle$ and $\omega \notin \langle s_n\rangle$. Then in G_n we have $a s_n^l \omega s_n^m \bar{a} = s_n^l \omega s_n^m$ and so $\bar{s}_n^l a s_n^l = \omega s_n^m a \bar{s}_n^m \omega^{-1}$ in G_n.

We consider three cases :

1. $l, m \geqq 0$. This includes by taking inverses the case $l, m \leqq 0$. Reduction of the $s_n a s_n$ blocks on both sides leads to:

$$\bar{b}^{f_n(l-1)} a b^{f_n(l-1)} = \omega s_n^{m-(n-1)} b^{n-1} a \bar{b}^{n-1} \bar{s}_n^{m-(n-1)} \omega^{-1},$$

where:

$$s_n^{m-(n-1)} b^{n-1} a \bar{b}^{n-1} \bar{s}_n^{m-(n-1)},$$

is s_n reduced. Then $m-(n-1) \leqq |\omega|_{s_n}$ or $m \leqq |\omega|_{s_n} + (n-1)$ and:

$$\omega^{-1} \bar{b}^{f_n(l-1)} a b^{f_n(l-1)} \omega = s_n^{m-(n-1)} b^{n-1} a \bar{b}^{n-1} \bar{s}_n^{m-(n-1)}.$$

If $f_n(|\omega|) \leqq f_n(l-1)$ this equality cannot hold which means $l-1 < |\omega|$ and $l \leqq |\omega| + n$.

2. $l \leqq 0$ and $m \geqq 0$. This can be written as $\bar{s}_n^l \omega s_n^m \in \langle U_n\rangle$ with $l, m \geqq 0$. We get $s_n^l a \bar{s}_n^l = \omega s_n^m a s_n^{-m} \omega^{-1}$ in G_n and for $l, m \geqq n-1$ reduction of both $s_n a \bar{s}_n$ blocks leads (as in the proof of lemma 3.3 and 1) to:

$$s_n^{l-(n-1)} b^{n-1} a \bar{b}^{n-1} \bar{s}_n^{l-(n-1)} = \omega s_n^{m-(n-1)} b^{n-1} a \bar{b}^{n-1} \bar{s}_n^{m-(n-1)} \omega^{-1} \quad \text{in } G_n,$$

which in turn may be written as:

$$b^{n-1} a \bar{b}^{n-1} = \bar{s}_n^p \omega s_n^q b^{n-1} a \bar{b}^{n-1} \bar{s}_n^q \omega^{-1} s_n^p \quad \text{in } G_n,$$

with $p=l-(n-1)$ and $q=m-(n-1)$. Here $s_n b^{n-1} a\bar{b}^{n-1}\bar{s}_n$ is s_n-reduced. Suppose $q \geqq |\omega|_{s_n}+2$. For all s_n^q to be cancelled at least two s_n in s_n^q must react with $\bar{s}_n^p$ because all s_n must pinch out in the right side of the equation. But $\omega \notin \langle s_n \rangle$, so this is not possible. We get $q \leqq |\omega|_{s_n}+1 \leqq |\omega|+1$ and by symmetry $p \leqq |\omega|+1$ also. This implies $l, m \leqq |\omega|+n$.

3. $l \geqq 0$ and $m \leqq 0$, which may be written as $s_n^l \omega \bar{s}_n^m \in \langle U_n \rangle$ with $l, m \geqq 0$. We now have $\bar{s}_n^l a s_n^l = \omega \bar{s}_n^m a s_n^m \omega^{-1}$ in G_n from which we get:

$$\bar{b}^{f_n(l-1)} a b^{f_n(l-1)} = \omega \bar{b}^{f_n(m-1)} a b^{f_n(m-1)} \omega^{-1} \quad \text{in } G_n.$$

We show that this equation can only hold if $m, l \leqq |\omega|$.

Let $K_n^+ = \langle \{k_{n,i} \mid i \geqq 0\} \rangle$ in G_n with $k_{n,i} \equiv \bar{b}^{f_n(i)} a b^{f_n(i)}$ as after lemma 3.2. Notice that $K_1^+ \supsetneqq K_2^+ \supsetneqq \ldots \supsetneqq K_n^+ \supsetneqq \ldots$ because of $f_n(m+1) = f_{n-1}(f_n(m))$. By induction on n we prove that, if ω is fully reduced and $\omega \notin \langle s_n \rangle$ with $\omega k_{n,i} \omega^{-1} \in K_n^+$, then $i \leqq |\omega|$. This is of course equivalent to our assertion above.

We only prove the induction step $n-1 \to n$ because the case $n=1$ is handled similarly. Suppose the claim is not true and let ω be a counterexample of minimal length with $\omega k_{n,i} \omega^{-1} \in K_n^+$, $\omega \notin \langle s_n \rangle$ and $0 < |\omega| < i$.

ω is not s_n-free: If $e \neq \omega$ is s_n-free we have $\omega k_{n,i} \omega^{-1} \equiv \omega k_{n-1, f_n(i-1)} \omega^{-1}$ with $f_n(i-1) > i-1 \geqq |\omega|$. So for $\omega \notin \langle s_{n-1} \rangle$ we can use the induction hypothesis and get that $\omega k_{n,i} \omega^{-1} \notin K_{n-1}^+$ which contradicts our assumption.

If $\omega = s_{n-1}^j$ in G_{n-1} then $0 < |j| \leqq |\omega| < i$ and by using 3.3 we get:

$$\omega k_{n-1, f_n(i-1)} \omega^{-1} = k_{n-1, f_n(i-1)-j} \quad \text{in } G_n.$$

But $k_{n-1, f_n(i-1)-j} \notin K_n^+$ because of:

$$0 < |j| < i \quad \text{implies } |j| \leqq f_n(i-1) - f_n(i-2) < f_n(i) - f_n(i-1),$$

and so:

$$f_n(i-1) < f_{n-1}(f_n(i-1) - |j|) < f_n(i) < f_{n-1}(f_n(i-1) + |j|) < f_n(i+1),$$

for $0 < |j| < i$. This again contradicts our assumption.

We claim that $\omega \equiv u_r \bar{s}_n u_{r-1} \ldots \bar{s}_n u_0$ with u_j s_n-free, r, $|u_j| < |\omega| < i$, $u_0 \not\equiv e$ and $u_r \notin \langle s_{n-1} \rangle$.

Suppose $\omega \equiv \omega' s_n u$, $u s_n$-free. Then $u \notin \langle s_{n-1} \rangle$ in G_{n-1}, for, if $u = s_{n-1}^k$, we have $\omega = \omega' b^k s_n$ in G_n and $\omega' b^k$ would be a shorter counterexample. Now $s_n u k_{n,i} u^{-1} \bar{s}_n$ must be a pinch which means (*see* 3.3) $u k_{n,i} u^{-1} = s_{n-1}^p k_{n-1,l} \bar{s}_n^p$

for some $p \geqq 0$, $l > -(n-1)$ and the right hand side is s_{n-1}-reduced (equal exponents, because u is fully reduced and by symmetry). But we have $|u| < i-1 < f_n(i-1)$ and $k_{n,i} \equiv k_{n-1, f_n(i-1)}$, so we will never reach from $k_{n,i}$ some $k_{n-1,j}$ with $j \leqq 0$ by reducing $uk_{n,i} u^{-1}$. We must have $p=0$ and $uk_{n,i} u^{-1} \in K_{n-1}^{+}$ which contradicts our induction hypothesis.

We have got $\omega \equiv \omega' \bar{s}_n u$ and of course $u \neq e$ (minimality of ω). Now $\bar{s}_n uk_{n,i} u^{-1} s_n$ must be a pinch, which means $uk_{n,i} u^{-1} = \bar{b}^l a b^l$ and so:

$$l \geqq f_{n-1}(f_n(i-1) - |u|) > f_n(i-1) \qquad \text{and} \qquad \bar{s}_n uk_{n,i} u^{-1} s_n = k_{n-1,l}.$$

It is now easy to see that $\omega \equiv u_r \bar{s}_n u_{r-1} \ldots \bar{s}_n u_0$, u_j s_n-free and:

$$\bar{s}_n u_{j-1} \ldots \bar{s}_n u_0 k_{n,i} u_0^{-1} s_n \ldots u_{j-1}^{-1} s_n = k_{n-1, l_j},$$

with $l_j > f_n(i-1)$ for $j = 1, \ldots, r$. The minimality of ω implies that $u_r \notin \langle s_{n-1} \rangle$, so by the induction hypothesis $\omega k_{n,i} \omega^{-1} = u_r k_{n-1,l_r} u_r^{-1} \notin K_{n-1}^{+}$, which finishes our proof. $\triangle$

3.7. LEMMA: *There is an a-reduction function in $\mathscr{E}_4$ for $G_n = \langle H_n, a;\ \bar{a} b \bar{s}_i a = b \bar{s}_i,\ i = 1, \ldots, n \rangle$.*

Proof: It suffices to show that $\langle U_n \rangle = \langle s_1 \bar{b}, \ldots, s_n \bar{b} \rangle$ is a $\mathscr{E}_4$-decidable subgroup of H_n. We prove this by induction on n.

Let $\omega \equiv \omega_0 s_n^{\varepsilon_1} \omega_1 \ldots s_n^{\varepsilon_p} \omega_p$ be fully reduced. If $\omega \in \langle U_n \rangle$ then there is a word $u \in U_n^*$, free reduced in the $s_i \bar{b}$ and hence fully reduced (*see* the proof of lemma 3.1, with $\omega = u$ in G_n. This means $u \equiv u_0 (s_n \bar{b})^{\varepsilon_1} \ldots (s_n \bar{b})^{\varepsilon_p} u_p$ with $u_i \in U_{n-1}^*$. Since $u^{-1} \omega = e$ in H_n, there is a sequence $j_{-1}, j_0, \ldots, j_{p-1}, j_p$ with $j_{-1} = j_p = 0$ and a system of $p+1$ equations of the form:

$$\bar{s}_n u_i^{-1} b s_{n-1}^{j_{i-1}} \omega_i s_n = s_{n-1}^{j_i}, \quad i.\,e.\ u_i^{-1} b s_{n-1}^{j_{i-1}} \omega_i = b^{j_i},$$

or:

$$\bar{s}_n u_i^{-1} b^{j_{i-1}} \omega_i s_n = s_{n-1}^{j_i}, \quad i.\,e.\ u_i^{-1} b^{j_{i-1}} \omega_i = b^{j_i},$$

or:

$$s_n \bar{b} u_i^{-1} b s_{n-1}^{j_{i-1}} \omega_i \bar{s}_n = b^{j_i} \quad i.\,e.\ \bar{b} u_i^{-1} s_{n-1}^{j_{i-1}} \omega_i = s_{n-1}^{j_i},$$

or:

$$s_n \bar{b} u_i^{-1} b^{j_{i-1}} \omega_i \bar{s}_n = \bar{b}^{j_i}, \quad i.\,e.\ b u_i^{-1} b^{j_{i-1}} \omega_i = s_{n-1}^{j_i}.$$

We claim $|j_i| \lneqq |\omega| + n \cdot |\omega|_{s_n}$. Then according to the induction hypothesis one can test by a $\mathscr{E}_4$-process, whether the system of equations is solvable, i. e. whether $u \in \langle U_n \rangle$ with $\omega = u$ in H_n exists. So $\langle U_n \rangle$ is a $\mathscr{E}_4$-decidable subgroup of H_n.

The proof of $|j_i| \leqq |\omega| + n \cdot |\omega|_{s_n}$, $i=1, \ldots, p-1$ is by induction on p. If $\varepsilon_1 = -1$ then $u_0 \omega_0 = s_{n-1}^{j_0}$ in H_{n-1}, hence $|j_0| \leqq |\omega_0| + (n-1)$ by lemma 3.6. (We have $s_{n-1}^{j_0} \omega_0^{-1} \in \langle U_{n-1} \rangle$, and if $\omega_0 \in \langle s_{n-1} \rangle$ then $|j_0| \leqq |\omega_0|$).

So:

$$\omega = u_0^{-1} s_{n-1}^{j_0} \bar{s}_n \omega' = u_0 \bar{s}_n b^{j_0} \omega', \qquad \text{where} \quad \omega' \equiv \omega_1 s_n^{\varepsilon_2} \omega_2 \ldots s_n^{\varepsilon_p} \omega_p.$$

Now we can argue with $b^{j_0} \omega'$ instead of ω (there may be some cancellation of $b's$). By induction hypothesis:

$$|j_i| \leqq |\omega'| + j_0 + (n-1) |\omega'|_{s_n} \leqq |\omega| + n |\omega|_{s_n}.$$

If $\varepsilon_p = +1$ the same argument works with ω^{-1} instead of ω.

Let $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_q = 1$, $\varepsilon_{q+1} = -1$, $q < p$, and $\hat{\omega} \equiv \omega_0 s_n \omega_1 \ldots s_n \omega_q$. The first $q+1$ equations are:

$$u_0^{-1} \omega_0 = b^{j_0},$$
$$u_i b s_{n-1}^{j_{i-1}} \omega_i = b^{j_i}, \qquad i=1, \ldots, q-1,$$
$$\bar{b} u_q b s_{n-1}^{j_{q-1}} \omega_q = s_{n-1}^{j_q}.$$

This implies:

$$|j_{i-1}| \leqq |\omega_i| + |j_i| + 1 + (n-1) \qquad \text{for} \quad i=1, \ldots, q-1$$

and $|j_{q-1}|, |j_q| \leqq |\omega_q| + 1 + (n-1) = |\omega_q| + n$ by lemma 3.6.

This gives:

$$|j_i| \leqq |\hat{\omega}| + n \cdot q \leqq |\omega| + n |\omega|_{s_n} \qquad \text{for} \quad i=0, 1, \ldots, q.$$

Again we get $|j_i| \leqq |\omega| + n |\omega|_{s_n}$ for all i as in the case $\varepsilon_1 = -1$. Δ

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