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ALDO DE LUCA

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ON SOME PROPERTIES OF THE SYNTACTIC SEMIGROUP OF A VERY PURE SUBSEMIGROUP (*)

by Aldo DE LUCA (1)

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Abstract. — *Some synchronizing properties of "very pure" subsemigroups of a free semigroup are shown by means of an analysis of the properties of their syntactic semigroups.*

Résumé. — *On démontre certaines propriétés de synchronisation pour les sous-semigroupes « très purs » d'un semigroupe libre, en analysant les propriétés de leurs semigroupes syntactiques.*

0. INTRODUCTION

In this paper we consider a family of subsemigroups of a given semigroup S which, following A. Restivo [10], we call "very pure". For the case when $S = X^+$ is the free semigroup generated by an alphabet X, the bases of very pure subsemigroups, which are called "very pure codes", have been introduced by M. P. Schützenberger in the factorizations of free monoids [16] and in the construction of the bases of free Lie algebras [14]. A remarkable result of Restivo [10] shows that the class of finitely generated free subsemigroups of X^+ coincides with the class of finitely generated free subsemigroups having a "bounded synchronization delay". Moreover, very pure subsemigroups, considered as languages, are "strictly locally testable" in the sense of McNaughton and Papert [7].

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(1) Laboratorio di Cibernetica del C.N.R., Arco Felice, Napoli, Italie.
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A recent theorem of Restivo [11] shows that in the more general case in which the subsemigroups are "recognizable" the equivalence between "very pure" and "bounded synchronization delay" holds only if one makes the auxiliary hypothesis, called condition $F(p)$, that any word of the base does not contain, as a factor, a product of a number of code-words greater than a suitable integer $p \geqq 2$.

In this paper we start by considering "very pure" subsemigroups A of an arbitrary semigroup S. The property of "very pure" characterizes the homomorphic image $A\sigma$ of any very pure subsemigroup in its syntactic semigroup $S(A)$. Some general propositions on the structure of the syntactic semigroup of a very pure semigroup A are shown under the further hypothesis that A is "synchronizing". By means of these propositions we obtain when S is a free semigroup, the following two main results:

1. *A recognizable very pure subsemigroup of S is synchronizing* (*cf.* proposition 4.3). 2. *Let A be a free subsemigroup of S such that all the elements of $A\sigma$ but a finite number, are idempotents. Then the following propositions are equivalent:* (i) *A has a bounded synchronization delay.* (ii) *A is very pure and satisfies the condition $F(p)$ for a suitable natural number p.* (iii) *For all the idempotents $e \in A\sigma$, $eS(A)e \subseteqq \{e, 0\}$* (*cf.* proposition 4.5).

This last result and another equivalent proposition concerning the structure of the 0-minimal ideal of the syntactic semigroups $S(A)$, give a characterization of free subsemigroups, of a free semigroup, having a bounded synchronization delay which is more general than that of the theorem of Restivo [11], and also that of a recent result obtained by the author, D. Perrin, A. Restivo and S. Termini [3] under the hypothesis that A is finitely generated.

1. PRELIMINARIES

For the notations and definitions which are not reported in the paper the reader is referred to [2 and 5].

Let S be a semigroup. We call *product* the associative binary operation defined in S and for all $a, b \in S$ we denote by ab their product. For all $A, B \subseteqq S$ we set $AB = \{ab \in S \mid a \in A, b \in B\}$. Any stable subset A of S, i.e. $A^2 \subseteqq A$, is a subsemigroup of S. For any $A \subseteqq S$, A^+ will denote the smallest subsemigroup containing A (i. e. the subsemigroup generated by A). We recall that a semigroup S is *free* if there exists a subset $X \subseteqq S$ such that $X^+ = S$ and any map φ: $X \rightarrow T$, T being any semigroup, can be extended to a unique morphism of S in T. X is called the *base* of S.

A relative notion of freedom is given by the following definition due to Schützenberger [13].

DEFINITION 1.1: A subsemigroup A of S is free *in* S if for all a, $b \in A$ and $s \in S$, as, $sb \in A$ imply $s \in A$.

A classical theorem of Schützenberger [12] shows that *a subsemigroup A of a free semigroup S is itself free if and only if A is free in S.*

For any $A \subseteq S$ the *syntactic congruence* $\sigma_A \sigma_A^{-1}$ of A is the maximal congruence $\varphi\varphi^{-1}$ (where φ denotes any morphism of S in any semigroup) such that φ is *compatible* with A, i.e. $A\varphi\varphi^{-1} = A$; $S(A) = S/\sigma_A \sigma_A^{-1}$ is usually called the *syntactic semigroup* of A and σ_A, that we shall also denote simply by σ, is the *canonical epimorphism* $\sigma: S \to S(A)$. One can easily see that for all $s_1, s_2 \in S$:

$$s_1 \sigma\sigma^{-1} s_2 \leftrightarrow \quad \text{for all } s, t \in S^1 \; (ss_1 t \in A \leftrightarrow ss_2 t \in A).$$

A subset A of S is called *recognizable* if the syntactic semigroup $S(A)$ is *finite*. If A is a subsemigroup of S one has that A is free in S if and only if A is free in the syntactic semigroup $S(A)$.

Let S be a semigroup. We say that S has a *zero element* 0 if the cardinality $|S|$ of S is greater than 1 and for all $s \in S$, $s0 = 0s = 0$. A zero element is obviously unique.

A subsemigroup A of a given semigroup S is called *dense* if for all $s \in S$, $S^1 s S^1 \cap A \neq \emptyset$, i. e. A meets all the two-sided ideals of S.

Let A be a nondense subsemigroup of S and set $Q = \{ s \in S \mid S^1 s S^1 \cap A = \emptyset \}$. One has that $Q\sigma = 0 \in S(A) \setminus A\sigma$. Moreover by using the Schützenberger theorem one can easily derive that if A is a subsemigroup of S free in S then A is dense if and only if $S(A)$ does not contain a zero element. Thus if A is a subsemigroup of S free in S, $A\sigma$ never contains a zero of $S(A)$.

Let us denote, for any $A \subseteq S$, by $\sqrt{A}$ the subset of S defined as:

$$\sqrt{A} = \{ s \in S \mid s^+ \cap A \neq \emptyset \}.$$

Clearly $A \subseteq \sqrt{A}$; a subsemigroup A of S is called "pure" if $A = \sqrt{A}$. This definition entails that when A is pure for all $s \in S$ and $p \geq 1$, $s^p \in A \to s \in A$.

For each semigroup S we denote by $E(S)$ the set of all its *idempotent* elements, i. e. if $e \in E(S)$ then $e^2 = e$. When S is finite one has that $S = \sqrt{E}(S)$. A semigroup S is called a *band* if $S = E(S)$. A *semilattice* is a commutative band. A semigroup S is said *quasi-idempotent-commutative* if for all $e \in E(S)$ the subsemigroup eSe is a semilattice. A finite semigroup S is called *aperiodic* (*cf.* Eilenberg [5]) if it has only trivial subgroups or, equivalently, a positive integer p exists such that for all $s \in S$

$s^p = s^{p+1}$. It is easy to verify that if S is a finite semigroup such that for all $e \in E(S)$, eSe is a band then S is aperiodic.

2. VERY PURE SUBSEMIGROUPS

Let us now introduce a class of subsemigroups of a given semigroup S that, following A. Restivo [10] we call "very pure".

DEFINITION 2.1: A subsemigroup A of S is called *very pure* if for all $s, t \in S$, $st, ts \in A$ imply $s, t \in A$.

Any very pure subsemigroup A of S is also pure whereas the converse is not generally true.

PROPOSITION 2.1: *If A is a very pure subsemigroup of S then A is free in S.*

Proof: Let $a, b \in A$ and $s \in S$ be such that $as, sb \in A$. One has then $bas, sba \in A$. Since A is very pure it follows that $s \in A$. □

We state, without proof, the following proposition which is a straightforward consequence of definition 2.1.

PROPOSITION 2.2: *If B is a very pure subsemigroup of S and A a very pure subsemigroup of B then A is a very pure subsemigroup of S.*

PROPOSITION 2.3: *Let φ: $S \to T$ be a morphism of the semigroup S in the semigroup T. One has that:*

(i) *If B is a very pure subsemigroup of T then $B\varphi^{-1}$ is a very pure subsemigroup of S.*

(ii) *If A is a very pure subsemigroup of S and φ is compatible with A, then $A\varphi$ is very pure in $S\varphi$. Moreover if $S\varphi$ is very pure in T then $A\varphi$ is very pure in T.*

Proof: Let $s, t \in S$ such that $st, ts \in B\varphi^{-1}$. One has then $(st)\varphi = s\varphi\, t\varphi \in B$ and $(ts)\varphi = t\varphi\, s\varphi \in B$. Since B is very pure in T then $s\varphi$, $t\varphi \in B$ that implies $s, t \in B\varphi^{-1}$; $B\varphi^{-1}$ is therefore a very pure subsemigroup of S that concludes the proof of proposition (i).

Suppose now that $s', t' \in S\varphi$ and $s't', t's' \in A\varphi$. Let $s, t \in S$ be such that $s\varphi = s'$, $t\varphi = t'$. One has then $(st)\varphi = s\varphi\, t\varphi = s't' \in A\varphi$, $(ts)\varphi = t\varphi\, s\varphi = t's' \in A\varphi$ and $st, ts \in A\varphi\varphi^{-1} = A$. Since A is very pure $s, t \in A$ and $s\varphi = s' \in A\varphi$, $t\varphi = t' \in A\varphi$. Thus $A\varphi$ is a very pure subsemigroup of S. Finally if $S\varphi$ is very pure in T from proposition 2.2 it follows that $S\varphi$ is very pure in T. □

Let A be a subsemigroup of a given semigroup S. We give the following definition of "synchronizing pair" for A (*cf.* Schützenberger [15]).

DEFINITION 2.2: A pair $(s, t) \in S \times S$ is a *synchronizing pair* for A if $S^1 st S^1 \cap A \neq \emptyset$ and for all $u, v \in S^1$, $ustv \in A \to us, tv \in A$. The subsemigroup A is called *synchronizing* if it has at least a synchronizing pair.

It follows from definition 2.2 that A is synchronizing if and only if there exists a synchronizing pair for A in $A \times A$.

DEFINITION 2.3: A subsemigroup A of S has a *bounded synchronization delay* if a positive integer k exists such that all the pairs in $A^k \times A^k$ are synchronizing. The least integer k for which the previous condition is verified is called the *synchronization delay* of A.

We note that from the previous definition delays of synchronization greater than 1 can be achieved only if $A^2 \subset A$. This last condition is verified, for instance, when A is a *locally finite semigroup* (*cf.* Eilenberg [5]).

One can easily verify that A is synchronizing if and only if $A\sigma$ is so. Moreover A has a synchronization delay equal to s if and only if $A\sigma$ has a synchronisation delay equal to s.

PROPOSITION 2.4: *If A is a subsemigroup of S free in S having a bounded synchronization delay then A is very pure.*

Proof: Let $s, t \in S$ be such that $st, ts \in A$ and $s, t \notin A$. Let us show that for all positive integers n the pairs $((ts)^n, (ts)^n) \in A^n \times A^n$ are not synchronizing. In fact if we suppose that a positive integer k exists such that $((ts)^k, (ts)^k)$ is a synchronizing pair for A one would have $(st)^{2k+1} = s\,(ts)^k\,(ts)^k\,t \in A$, so that $s\,(ts)^k, (ts)^k\,t \in A$. Since A is free in S, from the equalities:

$$s\,(ts)^k = (st)^k\,s \in A,\ (ts)^k\,t = t\,(st)^k = A,$$

it would follow $s, t \in A$ which is a contradiction. □

3. THE SYNTACTIC SEMIGROUP OF A VERY PURE SUBSEMIGROUP

During all this section A will denote a subsemigroup of a given semigroup S, A' the homomorphic image $A\sigma$ of A in the syntactic semigroup $S(A)$ and $S^1(A)$ the semigroup $[S(A)]^1$.

PROPOSITION 3.1: *A is a very pure subsemigroup of S if and only if A' is a very pure subsemigroup of $S(A)$.*

Proof: Let A be a very pure subsemigroup of S. Since $A\sigma\sigma^{-1} = A$, where $\sigma: S \to S(A)$ is the canonical epimorphism of S in the syntactic semigroup $S(A)$, from property (ii) of proposition 2.3 it follows that A' is a very pure subsemigroup of $S\sigma = S(A)$. Vice versa if A' is very pure in $S(A)$ from property (i) of proposition 2.3 one has that $A\sigma\sigma^{-1}=A$ is a very pure subsemigroup of S. □

PROPOSITION 3.2: *Let A be a subsemigroup of S free in S such that $A\subseteqq\sqrt{E}(S)$. If for all $e\in A\cap E(S)$, eSe is a semilattice then A is a very pure subsemigroup of S.*

Proof: Let s, $t\in S$ be such that st, $ts\in A$. From the hypothesis $A\subseteqq\sqrt{E}(S)$ it follows that a positive integer p exists for which $(st)^p, (ts)^p\in A\cap E(S)$. Since for all $e\in A\cap E(S)$, eSe is a semilattice the following relations will hold:

$$\left.\begin{aligned} (st)^p s(st)^p t(st)^p &= (st)^p t(st)^p s(st)^p,\\ (st)^p t(st)^p t(st)^p &= (st)^p t(st)^p,\\ (ts)^p s(ts)^p t(ts)^p &= (ts)^p t(ts)^p s(ts)^p,\\ (ts)^p t(ts)^p t(ts)^p &= (ts)^p t(ts)^p. \end{aligned}\right\}\tag{3.1}$$

From equations $(3.1)_1$ and $(3.1)_3$ one gets:

$$\left.\begin{aligned} s(ts)^p(st)^p(ts)^p t &= (st)^p(ts)^{p+1}(st)^p\in A,\\ t(st)^p(ts)^p(st)^p s &= (ts)^p(st)^{p+1}(ts)^p\in A \end{aligned}\right\}\tag{3.2}$$

and from equations $(3.1)_2$ and $(3.1)_4$:

$$\left.\begin{aligned} (st)^p(ts)^p tt(st)^p &= (st)^p(ts)^p t,\\ (ts)^p tt(st)^p(ts)^p &= t(st)^p(ts)^p. \end{aligned}\right\}\tag{3.3}$$

Multiplying both the sides of equations $(3.3)_1$ and $(3.3)_2$ respectively on the right by s and on the left by s one has:

$$\left.\begin{aligned} (st)^p(ts)^p t(ts)^{p+1} &= (st)^p(ts)^{p+1}\in A,\\ (st)^{p+1} t(st)^p(ts)^p &= (st)^{p+1}(ts)^p\in A. \end{aligned}\right\}\tag{3.4}$$

Similarly multiplying both the sides of equations $(3.2)_1$ and $(3.2)_2$ respectively on the right by $(ts)^{p+1}$ and on the left by $(st)^{p+1}$ one obtains:

$$s(ts)^p(st)^p(ts)^p t(ts)^{p+1}\in A,$$
$$(st)^{p+1} t(st)^p(ts)^p(st)^p s\in A,$$

with

$$(ts)^p(st)^p(ts)^p t(ts)^{p+1}\in A,$$
$$(st)^{p+1} t(st)^p(ts)^p(st)^p\in A,$$

because of equations $(3.4)_1$ and $(3.4)_2$ and the fact that $(ts)^p, (st)^p\in A$.

Since A is free in S it follows that $s \in A$ and, by the hypothesis that st, $ts \in A$, that $t \in A$. Hence A is a very pure subsemigroup of S. $\square$

A consequence of propositions 3.1 and 3.2 is the following corollary the proof of which is straightforward.

COROLLARY 3.1: *Let A be a recognizable subsemigroup of S free in S. If for all $e \in A' \cap E(S(A))$, $e\,S(A)\,e$ is a semilattice then A is a very pure subsemigroup of S.*

Let us now introduce the two subsets $F_A(S)$ and $G_A(S)$ of S defined as:

$$\left.\begin{aligned} F_A(S) &= \{ s \in S \mid t, t' \in S^1,\ tst' \in A \to \exists s_1, s_2 \in S,\\ &\qquad s_1 s_2 = s;\ ts_1, s_2 t' \in A \},\\ G_A(S) &= \{ s \in S \mid t, t' \in S^1,\ tst' \in A \to \exists s_1, s_2 \in S,\\ &\qquad s_1 s_2 = s;\ ts_1, s_2 t', s_2 s_1 \in A \}. \end{aligned}\right\} \tag{3.5}$$

Obviously $G_A(S) \subseteq F_A(S)$. Moreover, from the definition $F_A(S)$, if nonempty, is a two-sided ideal of S. We observe that if $(s, t) \in S \times S$ (resp. $A \times A$) is a synchronizing pair for A then $st \in F_A(S)$ [resp. $G_A(S)$]. Let us set

$$D_A(S) = F_A(S) \cap E(S), \qquad E_A(S) = G_A(S) \cap E(S). \tag{3.6}$$

In the following when S is the syntactic semigroup $S(A)$ of A we shall denote $F_{A\sigma}(S(A))$, $G_{A\sigma}(S(A))$, $E_{A\sigma}(S(A))$ and $D_{A\sigma}(S(A))$ simply by F_A, G_A, E_A and D_A. One can easily derive that:

$$F_A(S)\sigma \subseteq F_A, \qquad G_A(S)\sigma \subseteq G_A,$$

and moreover

$$\left.\begin{aligned} F_A(S)\sigma \cap E(S(A)) &= \sqrt{F}_A(S)\sigma \cap E(S(A)),\\ G_A(S)\sigma \cap E(S(A)) &= \sqrt{G}_A(S)\sigma \cap E(S(A)). \end{aligned}\right\} \tag{3.7}$$

It holds the following proposition the proof of which is reported in the Appendix.

PROPOSITION 3.3: *If A is a recognizable subsemigroup of S then $E_A(S) = D_A(S)$ and $E_A = D_A$.*

LEMMA 3.1: *If $e \in E_A \setminus 0$ then there exists an idempotent $e' \in (E_A \setminus 0) \cap A'$ which is in the same $\underline{D}$-class as e.*

Proof: Since $e \neq 0$ one has $S^1(A)\,e\,S^1(A) \cap A' \neq \emptyset$ (in fact, otherwise, e will be equal to 0) so that there exist $s_1, s_2 \in S^1(A)$ for which $s_1 e s_2 \in A'$. Moreover since $e \in E_A$ there will exist $e_1, e_2 \in S(A)$ such that $e_1 e_2 = e$ and $s_1 e_1, e_2 s_2, e_2 e_1 \in A'$. Let us set $s = e_1 e_2 e_1$, $s' = e_2 e_1 e_2$. One has then $s = ss's$ and $s' = s'ss'$, that is s

and s' are mutually inverse. From a classical result of Miller and Clifford (*cf.* Clifford and Preston [2]) s, s', $ss'=e$ and $s's=(e_2 e_1)^2 \in A'$ belong to the same $\underline{D}$-class. This implies $e' \neq 0$. Let us now prove that the idempotent $e'=(e_2 e_1)^2$ belongs to E_A. Let t, $t' \in S^1(A)$ be such that $tet' \in A'$. We can write $te't'=te_2 ee_1 t' \in A'$ so that since $e \in E_A$ there will exist e'_1, $e'_2 \in S(A)$ such that

$$te_2 e'_1,\ e'_2 e_1 t',\ e'_2 e'_1 \in A' \qquad \text{and} \qquad e'_1 e'_2 = e = e_1 e_2.$$

One then has

$$e_2 e'_1 e'_2 e_1 = e_2 e_1 e_2 e_1 = e'$$

and

$$e'_2 e_1 e_2 e'_1 = e'_2 e'_1 e'_2 e'_1 = (e'_2 e'_1)^2 \in A'. \quad \square$$

LEMMA 3.2: *Let A be a very pure subsemigroup of S. If $e \in A' \cap E_A$ then $eA'e=\{e\}$ and $eS(A)e \subseteqq \{e, 0\}$.*

Proof: Let us first prove that if $e \in A' \cap E_A$ then $eA'e=\{e\}$. Let h be an element of A' and suppose that $sehes' \in A'$ for some suitable s, $s' \in S^1(A)$. Since $e \in E_A$ there exist $e_1, e_2, e'_1, e'_2 \in S(A)$ such that $e=e_1 e_2=e'_1 e'_2$ and se_1, $e_2 he'_1$, $e'_2 s'$, $e_2 e_1$, $e'_2 e'_1 \in A'$. As $e \in A'$ and from proposition 3.1 A' is a very pure subsemigroup of $S(A)$, it follows that $e_1, e_2, e'_1, e'_2 \in A'$. Moreover $ses'=se^2 s'=se_1 e_2 e'_1 e'_2 s' \in A'$. Thus one has that $sehes' \in A' \rightarrow ses' \in A'$. Let us now suppose, on the contrary, that $ses' \in A'$ for suitable s, $s' \in S^1(A)$. Since $e \in E_A$ there exist $e_1, e_2 \in S(A)$ such that $e_1 e_2 = e \in A'$ and se_1, $e_2 s'$, $e_2 e_1 \in A'$. We then have $e_1, e_2 \in A'$ and $sehes'=se_1 e_2 he_1 e_2 s' \in A'$. This shows that $ses' \in A' \rightarrow sehes' \in A'$. Hence, as the syntactic congruence of A' is the identity, $ehe=e$ and $eA'e=\{e\}$.

Let us now prove that if $e \in E_A \cap A'$ then $eS(A)e \subseteqq \{e, 0\}$. Let g be an arbitrary element of $S(A)$. One has that either $ege=0$ (this can occur only if A is not dense) or there exist s, $s' \in S^1(A)$ for which $seges' \in A'$. Since $e \in E_A \cap A'$ there will exist elements $e_1, e_2, e'_1, e'_2 \in S(A)$ such that:

$$e_1 e_2 = e'_1 e'_2 = e \in A', \qquad e_2 e_1,\ e'_2 e'_1,\ se_1,\ e_2 ge'_1,\ e'_2 s' \in A'.$$

Since A' is very pure in $S(A)$ it follows that $e_1, e_2, e'_1, e'_2 \in A'$ and $ege=e_1(e_2 ge'_1)e'_2 \in A'$. This proves that in any case $eS(A)e \subseteqq \{e, 0\}$. $\square$

PROPOSITION 3.4: *Let A be a very pure subsemigroup of S. If $e \in E_A$ then $eS(A)e \subseteqq \{e, 0\}$.*

Proof: The result is trivially true if $S(A)$ has a zero element 0 and $e=0$. Let us then suppose $E_A \setminus 0 \neq \emptyset$ and take $e \in E_A \setminus 0$. From lemma 3.1 an idempotent

$e' \in E_A \cap A'$ exists such that $e' \underline{D} e$. As e' and e are in the same $\underline{D}$-class, elements t, $s \in S(A)$ exist for which $e = te's$. Thus one has $eS(A)\,e = te'sS(A)\,te's$. From lemma 3.2 it follows $e'sS(A)\,te' \subseteqq \{e', 0\}$ so that $eS(A)\,e \subseteqq \{e, 0\}$. □

Let us observe that under the hypothesis of proposition 3.4 for all $e \in E_A$, $eS(A)e$ is a trivial group with a zero element if $S(A)$ contains a 0. This implies that all subgroups of $S(A)$ whose identity elements are in E_A are *trivial* or, that is the same, all regular $\underline{D}$-classes of $S(A)$ having at least an idempotent in E_A are $\underline{H}$-trivial.

PROPOSITION 3.5: *Let A be a very pure subsemigroup of S. A is synchronizing if and only if $E_A \setminus 0 \neq \emptyset$.*

Proof: Let $E_A \setminus 0 \neq \emptyset$ and $e \in E_A \setminus 0$. From lemma 3.1 there exists an element $e' \in E_A \cap A'$ in the same $\underline{D}$-class as e. Let $g\sigma = e'$ with $g \in A$. We prove that the pair $(g, g) \in A \times A$ is a synchronizing pair for A. Let $s, s' \in S^1$ be such that $sggs' \in A$. This implies $te't' \in A'$ with $t = s\sigma$ and $t' = s'\sigma$. Since $e' \in E_A \cap A'$ there exist $e_1, e_2 \in S(A)$ such that $e' = e_1 e_2$ and $e_2 e_1$, te_1, $e_2 t' \in A'$; A' being very pure in $S(A)$, $e_1, e_2 \in A'$ so that te', $e't' \in A'$ and sg, $gs' \in A$. Vice versa let us suppose that A is synchronizing, i. e. there exists a synchronizing pair $(u, v) \in A \times A$ for A. One has that $uv \in G_A(S) \cap A$ and $(uv)\sigma \in G_A \cap A'$. Moreover $(uv)\sigma \in E(S(A))$. In fact for all $s, s' \in S^1$, $suvs' \in A$ if and only if $suvuvs' \in A$ so that $(uv)\sigma = (uv)^2\sigma = ((uv)\sigma)^2$. Thus $(uv)\sigma \in E_A \cap A'$ and $E_A \setminus 0 \neq \emptyset$. □

By means of an argument similar to that of the proof of the previous proposition one can derive that *if A is a very pure subsemigroup of S, A has a bounded synchronization delay if and only if there exists a positive integer k for which $A^k\sigma \subseteqq E_A$.*

Let S be a semigroup having a 0-minimal ideal J which is completely 0-simple. We recall that $J \setminus 0$ is a $\underline{D}$-class and that the $\underline{H}$-classes in $J \setminus 0$ containing an idempotent are isomorphic to a same abstract group that is called the (structure) *group of J*.

PROPOSITION 3.6: *Let A be a very pure subsemigroup of S such that $E_A \setminus 0 \neq \emptyset$. The syntactic semigroup $S(A)$ contains a unique 0-minimal ideal J which is completely 0-simple and has a trivial group. Moreover $E_A = J \cap E(S(A))$.*

Proof: Since $E_A \setminus 0 \neq \emptyset$ from lemma 3.1 one has that $E_A \cap A' \neq \emptyset$ so that by using proposition 3.4 for all $e \in E_A \cap A'$, $eS(A)e \subseteqq \{e, 0\}$. Let us now define for a fixed $e \in E_A \cap A'$ the two-sided ideal $J = S(A)eS(A)$. Since $0 \notin A'$ one has $J \neq 0$ and $J^2 \neq 0$; J is a 0-minimal ideal of $S(A)$. In fact let I be any two-sided ideal of $S(A)$ such that $I \neq 0$ and $I \subseteqq J$. One has $I \cap A' \neq \emptyset$ since, otherwise,

$S^1(A)IS^1(A) \cap A' = \emptyset$ and $I = 0$ which is a contradiction. If $s \in I \cap A'$ one obtains $ese = e$ since $eS(A)e \subseteqq \{e, 0\}$ and

$$J = S(A)eS(A) = S^1(A)eS^1(A) \quad = S^1(A)eseS^1(A) \subseteqq S^1(A)sS^1(A) \subseteqq I.$$

Hence $I = J$.

J is the *unique* 0-minimal ideal of $S(A)$. In fact if I in any other 0-minimal ideal one has $I \cap A' \neq \emptyset$, so that if $s \in I \cap A'$ one would have $es \in I \cap J \cap A' = 0 \cap A'$ which is a contradiction.

J is *completely* 0-*simple*. In fact since $J^2 \neq \emptyset$, J is simple and moreover the idempotent e is *primitive* (*cf.* Clifford and Preston [2]). This is shown by the fact that if f is any idempotent such that $ef = fe = f$ one obtains $efe = fe = f$ and, as $eS(A)e \subseteqq \{e, 0\}$, $f = 0$ or $f = e$.

The group of J is trivial. Since J is completely 0-simple, $J \setminus 0$ is a $\underline{D}$-class containing the $\underline{H}$-class $eS(A)e \setminus 0 = \{e\}$ which is a trivial group. Thus all the $\underline{H}$-classes of $J \setminus 0$ contain a unique element.

Let us now prove that $E_A = J \cap E(S(A))$. We first show that $E_A \subseteqq J \cap E(S(A))$. If $e = 0$ the result is trivially true. Let us then suppose $e \in E_A \setminus 0$. From lemma 3.1 there exists an idempotent $f \in E_A \cap A'$ such that $e\underline{D}f$. Since A' is very pure in $S(A)$ it follows from lemma 3.2 that $fA'f = \{f\}$. Taking $s \in A' \cap J$ one obtains $fsf = f \in J$ and, as $e\underline{D}f$, also $e \in J$.

Let us now show that $E_A \supseteqq J \cap E(S(A))$. Let e be an idempotent of J. If $e = 0$ the result is obvious. Let us then suppose $e \in J \setminus 0$. If $s \in J \cap A'$ one has that s is an idempotent too. In fact J is completely 0-simple and $s^2 \neq 0$ so that $s^2 \in H_s$. Being $H_s = \{s\}$ it follows $s^2 = s$. Since $e, s \in J \setminus 0$ one has $e\underline{D}s$. Now $s \in E_A$. In fact if $tst' = tsst' \in A'$ then $ts = tst's \in A'$ and $st' = stst' \in A'$. Since $e\underline{D}s$ it follows $e \in E_A$. $\square$

4. VERY PURE SUBSEMIGROUPS OF A FREE SEMIGROUP.

In this section we suppose that the semigroup S is a free semigroup generated by a finite base X ($S = X^+$). As usual X is called *alphabet*, the elements of X are called *letters* and the elements of X^+ *words*. For any word $f \in X^+$, $|f|$ will denote the *length* of f, i. e. the number of letters in the unique factorization of f in terms of the elements of X.

As we said in the first section a subsemigroup A of X^+ is itself free if and only if A is free in X^+. Since a very pure subsemigroup A of X^+ is free in X^+ it is then free.

For any subsemigroup A of X^+ the set $B=A\setminus A^2$ is the (unique) minimal set of generators of A; B is the *base* of A if A is free. The base of a free subsemigroup A of X^+ is usually called *code* and the base of a very pure subsemigroup A of X^+ *very pure code* (*cf.* [9, 10]).

For a subsemigroup A of X^+ we simply denote by $F_A(X)$ and $G_A(X)$ the sets $F_A(X^+)$ and $G_A(X^+)$ defined in general by equation (3.5).

Let us now introduce the following set:

$$H_A(X)=\{f\in X^+ \mid X^*fX^*\cap B=\emptyset\},$$

where $B=A\setminus A^2$ and $X^*=X^+\cup\{1\}$; $H_A(X)$ if is nonempty is a two-sided ideal of X^+.

PROPOSITION 4.1: *For any subsemigroup A of X^+, $H_A(X)=F_A(X)$.*

Proof: Let us first show that $F_A(X)\subseteq H_A(X)$.

Let $f\in F_A(X)$. The word f has also to belong to $H_A(X)$. In fact, otherwise, there would exist $u,\ v\in X^*$ such that $ufv\in B$. This would imply the existence of f_1, $f_2\in X^+$ such that $f_1f_2=f$ and $uf_1, f_2v\in A$. Thus $B\cap A^2\neq\emptyset$ which is absurd.

We prove now that $H_A(X)\subseteq F_A(X)$. Let $f\in H_A(X)$. If $X^*fX^*\cap A=\emptyset$ obviously $f\in F_A(X)$. Let us then suppose that there exist $u,\ v\in X^*$ such that $w=ufv\in A$; a factorization of w in terms of the elements of B having a parsing-line inside f has then to exist. In fact, otherwise, there would exist words u_1, $v_1\in X^*$, $|u_1|\leq|u|$, $|v_1|\leq|v|$ such that $u_1fv_1\in B$ which is a contradiction. Thus there exist $f_1, f_2\in X^+$ such that $f=f_1f_2$ and $uf_1, f_2v\in A$, i.e. $f\in F_A(X)$. □

LEMMA 4.1: *For any subsemigroup A of X^+, $F_A(X)\sigma\cap E(S(A))\subseteq E_A$.*

Proof: Let $e\in F_A(X)\sigma\cap E(S(A))$. If $S^1(A)\ eS^1\ (A)\cap A'=\emptyset$, i.e. $e=0$ the result is obvious. Let us then suppose that exist $s,\ t\in S^1(A)$ for which $set\in A'$. Let $u,\ v\in X^*$ and $f\in F_A(X)$ be such that $u\sigma=s$, $v\sigma=t$, $f\sigma=e$. One then has that for all positive integers k, $uf^kv\in A$. Let us take $k>|f|$. Since $f\in F_A(X)$ a sequence of pairs $X^+\times X^+$ exists:

$$(f_1, f_2),\ \ldots,\ (f_{2k-1}, f_{2k}),$$

such that $f_{2i-1}f_{2i}=f$ $(i=1,\ \ldots,\ k)$ and for all $i,\ j$ $(i\leq i<j\leq k)$:

$$uf^{i-1}f_{2i-1},\quad f_{2i}f^{j-i-1}f_{2j-1},\quad f_{2j}f^{k-j}v\in A,$$

where when $i=1$ or $k=j, f^0=1$. Since $k>|f|$ two integers i, j exist $1\leq i<j\leq|f|$, for which $f_{2i-1}=f_{2j-1}$ and, therefore, $f_{2i}=f_{2j}$. Setting $f_{2i-1}=f'_1$ and $f_{2i}=f'_2$ we can write

$$uf^{i-1}f'_1,\quad f'_2f^{j-i-1}f'_1,\quad f'_2f^{k-j}v\in A.$$

Let $e_1=f'_1\sigma$ and $e_2=f'_2\sigma$. One then has

$$se^{i-1}e_1,\quad (e_2e_1)^{j-i},\quad e_2e^{k-j}t\in A'. \tag{4.1}$$

Moreover $e^{i-1}e_1e_2e^{k-j}=e$ and $e_2e^{k-j}e^{i-1}e_1=(e_2e_1)^{k-j+i}$. Since $k>|f|$ it follows that $k-(j-i)\geqq 2$ and $(e_2e_1)^{k-j+i}=(e_2e_1)^2$. From equation (4.1) $(e_2e_1)^{j-i}\in A'$ with $j-i\geqq 1$, so that $(e_2e_1)^2\in A'$. This proves that $e\in E_A$. □

REMARK 1: If A is a pure subsemigroup of X^+ the stronger relation $F_A(X)\subseteq\sqrt{G_A}(X)$, which obviously implies $F_A(X)\sigma\cap E(S(A))\subseteq E_A$, can be easily derived by using an argument similar to that of the previous proof.

PROPOSITION 4.2: *If A is a recognizable and free subsemigroup of X^+ then $H_A(X)\cap A\neq\emptyset$ and $E_A\setminus 0\neq\emptyset$.*

Proof: Since A is recognizable so will be $B=A\setminus A^2$. Let us then denote by τ the syntactic epimorphism $\tau: X^+\to S(B)$, where $S(B)=X^+/\tau\tau^{-1}$ is the syntactic semigroup of B which is finite. We show now, by contradiction, that $H_A(X)\cap A\neq\emptyset$. Let us, in fact, suppose that $H_A(X)\cap A=\emptyset$, i.e. for all $f\in A$, $X^*fX^*\cap B\neq\emptyset$. This implies that for all $s\in A\tau$, $S^1(B)sS^1(B)\cap B\tau\neq\emptyset$. As a consequence if $S(B)$ contains a zero element 0 then $0\notin A\tau$. In fact, otherwise, from the previous equation $0\in B\tau$ and for all $s\in S(B)$, $s0=0s=0\in B\tau$. If $f\tau=s$, $b\tau=0$, with $f\in X^+$, $b\in B$, one would obtain $(fb)\tau=(bf)\tau=b\tau$, i.e. fb, $bf\in B$. Since A is free $f\in A$ and $B\cap A^2\neq\emptyset$ that is a contradiction.

$S(B)$ has a unique 0-minimal ideal. In fact if J is a 0-minimal ideal of $S(B)$ then $J\cap B\tau\neq\emptyset$, otherwise, $S^1(B)JS^1(B)\cap B\tau=\emptyset$ that implies $J=0$ which is absurd. If I is any other 0-minimal ideal of $S(B)$ and $I\neq J$ one would have for $s\in I\cap B\tau$ and $t\in J\cap B\tau$ that $st\in I\cap J\cap A\tau=0\cap A\tau=\emptyset$ which is a contradiction.

Let J be the unique 0-minimal ideal of $S(B)$. Since $J\cap B\tau\neq\emptyset$ one has $J^2\neq 0$. In fact if $s\in J\cap B\tau$ then $s^2\in J\cap A\tau$ so that $s^2\neq 0$. Being $S(B)$ finite and $J^2\neq 0$, J is a completely 0-simple semigroup so that from the general theory of semigroups (*cf.* Clifford and Preston [2]) $J\setminus 0$ coincides with a $\underline{D}$-class.

Since $J\cap B\tau\neq\emptyset$ there exists an $\underline{H}$-class H such that $H\cap B\tau\neq\emptyset$. If $s\in H\cap B\tau$ then $s^2\neq 0$ and $s^2\in H$ and H is a group. Denoting by e the identity element of H one has $es=se=s$. Let now $f\tau=e$ and $b\tau=s$, with $f\in X^+$ and $b\in B$; one gets: $(fb)\tau=(bf)\tau=b\tau$, and then fb, $bf\in B$ so that, since A is free, $f\in A$ and $B\cap A^2\neq\emptyset$ which is a contradiction.

Thus $H_A(X)\cap A\neq\emptyset$. From proposition 4.1, $H_A(X)=F_A(X)$ so that if $f\in F_A(X)\cap A$ then for all positive integers k, $f^k\sigma\in F_A(X)\sigma\cap A'$. If p is a suitable positive integer such that $f^p\sigma$ is an idempotent one has from lemma 4.1, $f^p\sigma\in F_A(X)\sigma\cap E(S(A))\subseteq E_A$. Hence $E_A\cap A'\neq\emptyset$ and $E_A\setminus 0\neq\emptyset$. □

PROPOSITION 4.3: *If A is a recognizable and very pure subsemigroup of X^+ then A is synchronizing.*

Proof: From proposition 4.2, $E_A \setminus 0 \neq \emptyset$, so that, by using proposition 3.5 one has that A has a synchronizing pair in $A \times A$. Thus A is a synchronizing subsemigroup of X^+. □

Some examples of (nonrecognizable) very pure subsemigroup of X^+ which are not synchronizing are given in [4]. We say that a subsemigroup A of X^+ satisfies the condition $F(p)$ (*cf.* Restivo [11]) if

$$X^* B^p X^* \cap B = \emptyset, \tag{4.2}$$

where $B = A \setminus A^2$ and p is a suitable natural number ($p \geqq 2$).

PROPOSITION 4.4: *Let A be a subsemigroup of X^+. If A satisfies the condition $F(p)$ then $A' \cap E(S(A)) \subseteqq E_A$. Moreover if A is finitely generated then $E_A = E(S(A))$.*

Proof: If A satisfies the condition $F(p)$ for a suitable natural number p, then $A \subseteqq \sqrt{H}_A(X)$. From proposition 4.1, $\sqrt{\mathrm{H}}_{\mathrm{A}}(\mathrm{X}) = \sqrt{\mathrm{F}}_{\mathrm{A}}(\mathrm{X})$ so that by making use of equation (3.7) and lemma 4.1.

$$A' \cap E(S(A)) \subseteqq \sqrt{F}_A(X)\sigma \cap E(S(A)) = F_A(X)\sigma \cap E(S(A)) \subseteqq E_A.$$

If A is finitely generated then $X^+ \subseteqq \sqrt{H}_A(X) = \sqrt{F}_A(X)$, so that

$$X^+ \sigma \cap E(S(A)) = E(S(A)) \subseteqq E_A.$$

Since $E_A \subseteqq E(S(A))$ one derives $E_A = E(S(A))$. □

REMARK 2: The above proposition is true under the only hypotheses that $A \subseteqq \sqrt{H}_A(X)$ and $X^+ \subseteqq \sqrt{H}_A(X)$ which are weaker than the conditions $F(p)$ and to be finitely generated respectively.

PROPOSITION 4.5: *Let A be a free subsemigroup of X^+ such that $A' \setminus E(S(A))$ is a finite subset of $S(A)$. The following propositions are equivalent:*

P_1. *A has a bounded synchronization delay.*

P_2. *A is very pure and satisfies the condition $F(p)$ for a suitable natural number p.*

P_3. *The syntactic semigroup $S(A)$ has a unique 0-minimal ideal J which is completely 0-simple. Moreover $A' \cap E(S(A)) \subseteqq J$ and the group of J is trivial.*

P_4. *For all $e \in A' \cap E(S(A))$, $eS(A)e \subseteqq \{e, 0\}$.*

Proof: We shall prove the equivalence of previous propositions by showing that $P_1 \rightarrow \mathrm{P}_2 \rightarrow \mathrm{P}_3 \rightarrow \mathrm{P}_4 \rightarrow \mathrm{P}_1$.

($P_1 \to P_2$). Let A be a free subsemigroup of X^+ having a bounded synchronization delay. By proposition 2.4 it follows that A is very pure. Moreover the condition $F(p)$ has to be satisfied for a suitable integer $p \geqq 2$. In fact, otherwise, $X^* B^{2s} X^* \cap B \neq \emptyset$, where s is the synchronization delay of A. Thus there would exist $t, t' \in X^*$, $b_1, b_2 \in B^s$ such that $tb_1 b_2 t' \in B$. Since $(b_1, b_2) \in B^s \times B^s$ is a synchronizing pair for A it would follow $tb_1, b_2 t' \in A$ and $B \cap A^2 \neq \emptyset$ which is a contradiction.

($P_2 \to P_3$). Let us now suppose A very pure and such that the condition $F(p)$ is satisfied for a suitable integer $p \geqq 2$. From proposition 4.4 one has $A' \cap E(S(A)) \subseteqq E_A$ and from proposition 3.6 it follows that the syntactic semigroup $S(A)$ has a unique 0-minimal ideal J which is completely 0-simple and has a trivial group. Moreover $E_A = J \cap E(S(A))$ so that $A' \cap E(S(A)) \subseteqq J$.

($P_3 \to P_4$). Since J is completely 0-simple and the group of J is trivial one has that for all the idempotents $e \in J$, $eS(A)e \subseteqq \{e, 0\}$. Being $A' \cap E(S(A)) \subseteqq J$ proposition P_4 follows.

($P_4 \to P_1$). Let us first show that

$$B^{k+1} \sigma \subseteqq E(S(A)) \cap A'$$

where $k = |A' \setminus E(S(A))|$. In fact if $f \in B^{k+1}$ we can express it uniquely as $f = f_1 \ldots f_{k+1}$ with $f_i \in B$ $(1 \leqq i \leqq k+1)$. We construct then the sequence of words: $f_1, f_1 f_2, \ldots, f_1 \ldots f_{k+1}$ and that of their respective images in $S(A)$ $m_1, m_1 m_2, \ldots, m_1 \ldots m_{k+1}$, having set $m_i = f_i \sigma$ $(i = 1, \ldots, k+1)$. Necessarily one element of the previous sequence has to belong to $E(S(A))$. This is obvious if all previous $k+1$ elements are distinct because $k = |A' \setminus E(S(A))|$. Let us then suppose that there exist integers i, j, $1 \leqq i < j \leqq k+1$, such that:

$$m_1 \ldots m_i = m_1 \ldots m_i (m_{i+1} \ldots m_j) = m_1 \ldots m_i (m_{i+1} \ldots m_j)^p,$$

for all positive integers p. We set $u = m_{i+1} \ldots m_j \in A'$ and consider the sequence $u, u^2, \ldots, u^{k+1}$. One of these elements has to be an idempotent. Let r, $1 \leqq r \leqq k+1$, be such that $u^r = e \in E(S(A))$. One has $m_1 \ldots m_i = m_1 \ldots m_i e$ and from the hypothesis $eS(A)e \subseteqq \{e, 0\}$ and the fact that $0 \notin A'$:

$$(m_1 \ldots m_i)^2 = m_1 \ldots m_i e m_1 \ldots m_i e = m_1 \ldots m_i \in E(S(A)).$$

Let $m_1 \ldots m_j \in E(S(A))$ for a suitable integer j, $1 \leqq j \leqq k+1$. One derives that $f\sigma = m_1 \ldots m_j m_{j+1} \ldots m_{k+1}$ and $(f\sigma)^2 = f\sigma \in E(S(A))$. Hence $B^{k+1} \sigma \subseteqq E(S(A)) \cap A'$.

Let us now prove that all the pairs $(f, g) \in B^{k+1} \times B^{k+1}$ are synchronizing. Let $h, h' \in X^*$ be such that $hfgh' \in A$. Setting $h\sigma = s$, $h'\sigma = s'$, $f\sigma = f'$, $g\sigma = g'$ one gets $sf'g's' \in A'$. Since $f', g' \in A' \cap E(S(A))$ and $0 \notin A'$, one has $sf'g's'f' = sf' = (hf)\sigma \in A'$ and, similarly, $g'sf'g's' = g's' = (gh')\sigma \in A'$. Thus $hf, gh' \in A$. □

REMARK 3: The hypothesis that $A' \setminus E(S(A))$ is a finite set is certainly verified if A' is *finite* or when A is a *recognizable subset* of X^+. In this last case the equivalence between propositions P_1 and P_2 has been proved by A. Restivo [11] by means of combinatorial arguments.

An example of a non recognizable code on the alphabet $X = \{x, y\}$ for which we can use proposition 4.5 is given by the set $B = \{x^n y^n \mid n \geqq 1\}$. In this case one can verify that $B^+ \sigma$ is a finite subsemigroup of $S(B^+)$.

In the proof of proposition 4.5 we have also shown that if A has a bounded synchronization delay s then $s \leqq |A' \setminus E(S(A))| + 1$. We note that $|A' \setminus E(S(A))| = |A' \setminus J|$. In fact in this case $E(S(A)) \cap A' \subseteq J$. Moreover (*cf.* proposition 3.6) $J \cap A' \supseteq E(S(A)) \cap A'$ so that $J \cap A' = E(S(A)) \cap A'$.

Let us note, at last, that for the equivalence of P_3 to propositions P_1, P_2, P_4 it is essential, besides the condition that J has a trivial group, that *all* the idempotents of A' are in J. In fact, for instance, if A is a free subsemigroup of X^+ which is a *noncounting regular set* (or a *star-free* event) then $S(A)$ is aperiodic (*cf.* Eilenberg [5]) so that the group of J is trivial. From this one can only derive that A is synchronizing. An example is given, over the alphabet $X = \{x, y, z\}$, by the free subsemigroup $A = \{x, yx^*z\}^+$ of X^+. A is a noncounting regular set. However A does not verify the condition $F(p)$ for any natural number p so that it has not a bounded synchronization delay. From a syntactic point of view $S(A)$ has only trivial subgroups but the idempotent $x\sigma \in A'$ does not belong to the 0-minimal ideal J of $S(A)$.

A subset A of X^+ is called *strictly locally testable* if there exist three finite sets U, V, W such that:

$$A \cap X^l X^* = (UX^* \cap X^* V) \setminus X^* W X^* \tag{4.3}$$

where l is the greatest of the lengths of the words of $U \cup V \cup W$.

A subset A of X^+ is called *locally testable* if it belongs to the Boolean closure of strictly locally testable sets (*cf.* McNaughton and Papert [7]).

Clearly a locally testable set of X^+ is recognizable (via Kleene's theorem). The following characterization of locally testable sets has been given by McNaughton [8], Zalcstein [17], Brzozowski and Simon [1].

PROPOSITION 4.6: *Let A be a recognizable subset of X^+. A is locally testable if and only if for all $e \in E(S(A))$, $eS(A)e$ is a semilattice.*

A subsemigroup $A \subseteqq X^+$ is called locally testable (resp. strictly locally testable) if A is a locally testable (resp. strictly locally testable) subset of X^+.

PROPOSITION 4.7: *If A is a free and locally testable subsemigroup of X^+ then A is very pure.*

Proof: If A is locally testable then from previous proposition for all $e \in E(S(A))$ $eS(A)e$ is a semilattice so that, since A is free, by making use of proposition 3.2 it follows that A is very pure. □

From the definition (4.3) one easily derives that a strictly locally testable subsemigroup of X^+ has a bounded synchronization delay.

Let us now suppose that A is a *finitely generated* subsemigroup of X^+. It holds the following:

PROPOSITION 4.8: *Let A be a free and finitely generated subsemigroup of X^+. The following propositions are equivalent:*

P_1. *A is strictly locally testable.*

P_2. *A is locally testable.*

P_3. *A is very pure.*

P_4. *The 0-minimal ideal J of $S(A)$ has a trivial group and $E(S(A)) \subseteqq J$.*

P_5. *For all $e \in E(S(A))$, $eS(A)e \subseteqq \{e, 0\}$.*

P_6. *A has a bounded synchronization delay.*

The equivalence of propositions P_1, P_3, P_6 has been shown by Restivo [10]. The equivalence of P_1 and P_2 has been proved by Hashiguchi and Honda [6]. An algebraic proof of the equivalence of P_1, P_4, P_5 and P_6 is in De Luca, Perrin, Restivo and Termini [3].

Let us remark that the equivalence of propositions $P_3 - P_6$ is a corollary of proposition 4.5. One has only to observe that if A is finitely generated then the condition $F(p)$ is certainly verified for a suitable integer p, and that if A is very pure by propositions 4.4 and 3.6 it follows $E_A = E(S(A)) \cap J$. One can also prove the equivalence of P_2 and the previous propositions ($P_3 - P_6$) by using proposition 4.7 and the characterization of locally testable languages given by proposition 4.6.

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APPENDIX

PROOF OF PROPOSITION 3.3

From definitions (3.5) and (3.6) one has always that $E_A(S) \subseteqq D_A(S)$. Let us now suppose that A is a recognizable subsemigroup of S; we have to show that $D_A(S) \subseteqq E_A(S)$. Let $e \in D_A(S)$ and $s, t \in S^1$ be such that $set \in A$. Since e is an idempotent for any integer $n \geqq 1$, $e^n = e$ and $se^n t \in A$. From definition (3.5) of $D_A(S)$ a sequence of pairs of elements of S:

$$(e_1, e_2), (e_3, e_4) \ldots (e_{2n-1}, e_{2n})$$

exists such that $e_{2h-1} e_{2h} = e$ $(h = 1, \ldots, n)$ and for all $h, k, 1 \leqq h < k \leqq n$:

$$se^{h-1} e_{2h-1}, e_{2h} e^{k-h-1} e_{2k-1}, \qquad e_{2k} e^{n-k} t \in A, \tag{A1}$$

where $e^0 = 1$. Since A is recognizable the syntactic congruence $\equiv$ of A is of finite index. Hence if $n > |S(A)|$ a pair (i, j) of positive integers exists such that $1 \leqq i < j \leqq n$ and $e_{2i-1} \equiv e_{2j-1}$. By equation (A1) one has

$$e_{2i} e^{j-i-1} e_{2j-1} \equiv e_{2i} e^{j-i-1} e_{2i-1} = (e_{2i} e_{2i-1})^{j-i}.$$

Thus since $e_{2i} e^{j-i-1} e_{2j-1} \in A$ then $(e_{2i} e_{2i-1})^{j-i} \in A$. This implies $(e_{2i} e_{2i-1})^2 \in A$. Setting $f_1 = e^{i-1} e_{2i-1}$, $f_2 = e_{2i} e^{n-i} = e_{2i} e$ one has:

$$f_1 f_2 = e,$$

$$f_2 f_1 = e_{2i} e e^{i-1} e_{2i-1} = e_{2i} e e_{2i-1} = (e_{2i} e_{2i-1})^2 \in A,$$

and $sf_1, f_2 t \in A$. This shows that $e \in E_A(S)$ and then $E_A(S) = D_A(S)$. The equality $E_A = D_A$ is a trivial consequence of this result in the case $S = S(A)$ and $A = A\sigma$. □

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