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CHARACTERIZATION AND LOWER BOUNDS FOR ADDITIVE CHARGES FOR HETEROGENEOUS QUESTIONNAIRES

by Bhu Dev SHARMA [1] and Asha GARG [2]

Abstract. — *Information Theory has found interesting applications in Questionnaire Theory. Picard and Campbell have shown connections of noiseless Coding Theorems with average charge of a valid homogeneous questionnaires. Duncan has considered heterogeneous questionnaires and has shown that if log d is considered as the charge for a question of resolution d then the expected charge for the questionnaire is lower bounded by Shannon's Entropy. In this paper we consider heterogeneous questionnaire, and a generalized average charge and characterize the two forms, one classical and the other of order t by considering the additivity property. It has then been proved that for a heterogeneous questionnaire average charge of order t is lower bounded by Rényi's Entropy. It has also been shown that a valid questionnaire will exist for which the average charge of order t per state can be made as close to the Rényi's Entropy as desired.*

Résumé. — *La théorie de l'information a trouvé des applications intéressantes dans la théorie des questionnaires. Picard et Campbell ont mis en évidence des liaisons entre des théorèmes de codage sans bruit et le coût moyen d'un questionnaire homogène. Duncan a considéré des questionnaires hétérogènes et montré que, si log d est considéré comme le coût d'une question de base d, le coût moyen du questionnaire est borné inférieurement par l'information de Shannon. Dans ce papier, on considère les questionnaires hétérogènes et un coût moyen généralisé; on caractérise deux formes, l'une classique, l'autre d'ordre t, en considérant la propriété d'additivité. On prouve ensuite que, pour un questionnaire hétérogène, le coût moyen d'ordre t est borné inférieurement par l'information de Rényi. On montre aussi qu'il existe un questionnaire valide pour lequel le coût moyen d'ordre t par état peut être rendu aussi proche que possible de l'information de Rényi.*

I. INTRODUCTION

Information Theory has found an interesting application in theory of questionnaires (Picard [8]). Picard [9] and Campbell [4] have shown that a charging scheme based on the resolution of questions gives a relationship between questionnaire theory and noiseless coding theory. Duncan [5] has generalized the "only if" part of Kraft's inequality for an arbitrary heterogeneous questionnaire. A charge equal to $\log d$, for each question of resolution d, considered by Duncan [5] follows from an equity principle. Using

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this charging scheme an extended noiseless coding theorem shows that the average charge for a heterogeneous questionnaire is bounded below by the Shannon's Entropy [5].

While discussing the noiseless coding theorem for an arbitrary heterogeneous questionnaire, Duncan [5] has chosen the random charge of a questionnaire which minimizes the average charge subjected to the condition that the questionnaire is valid. He has confined to the case when the average charge is expectation of the random charge. But this may not be the case always. In the present paper we introduce a general measure of average charge for an arbitrary heterogeneous questionnaire. Laying down what may be called the additivity of average charges, the general expression has been characterized, as is done in the case of homogeneous questionnaires [8] and the linear functions turn out to be the only possible case suiting the purpose. The lower bound on the average charge when the function is linear has been obtained by Duncan [5], for heterogeneous questionnaires.

In this paper lower bound has been obtained on the average charge obtained by considering the exponential form of the function. A theorem analogous to the ordinary noiseless coding theorem has been proved which shows that the exponential measure of given order t is arbitrary close to the Rényi's Entropy [7].
true state is p_i $(i=1, 2, \ldots, m)$ and

II

First, we give some notations which are very near to those used by Duncan [5].

Let $\Theta=\{\theta_1, \theta_2, \ldots, \theta_m\}$ be a finite state space and $P=(p_1, p_2, \ldots, p_m)$ be the probability vector over the state space such that probability of θ_i being the true state is p_i $(i=1, 2, \ldots, m)$ and

$$\sum_{i=1}^{m} p_i=1, \qquad p_i \geqq 0 \quad (i=1, 2, \ldots, m). \tag{1}$$

Let Q be a questionnaire defined on Θ and n_{id} represent the number of questions of resolution d required to reach the state θ_i. Now, if a heterogeneous questionnaire Q is valid and uses precisely n_{id} questions of resolution d $(d=1, 2, \ldots)$ to determine θ_i $(i=1, 2, \ldots, m)$, then (*cf.* Duncan [5]):

$$\sum_{i=1}^{m} \prod_{d=1}^{\infty} d^{-n_{id}} \leqq 1. \tag{2}$$

Also, if Q is a valid heterogeneous questionnaire and $C(Q)$ is the random charge when $\log d$ is the charge for each question of resolution d, then expected charge for Q is given by

$$E_p C(Q) = \sum_{i=1}^{m} \sum_{d=1}^{\infty} p_i n_{id} \log_2 d = \sum_{i=1}^{m} p_i \log_2 \prod_{d=1}^{\infty} d^{n_{id}} \tag{3}$$

which is ordinary average of the random charge

$$C(Q) = \sum_{d=1}^{\infty} n_{id} \log_2 d \tag{3}$$

In general, the random charge may be a function of this quantity. So that if we take a continuous, strictly increasing function viz. $\varphi: [1, \infty[\to R$, the random charge for Q may be given by

$$C(Q) = \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{id}}\right).$$

Consequently, the generalized average charge for Q may be taken as

$$E_P^{\varphi} C(Q) = \varphi^{-1}\left[\sum_{i=1}^{m} p_i \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{id}}\right)\right], \tag{4}$$

φ being a continuous, strictly increasing function, φ^{-1} exists.

It is interesting to see that (4) reduces to the classical average charge $E_P C(Q)$ in two different situations. The first case arises when a questionnaire Q uses same number of questions of each resolution d to determine every θ_i $(i = 1, 2, \ldots, m)$ i. e. when $n_{1d} = n_{2d} = \ldots = n_{md} = n_d$ (say), so that we have for any φ,

$$E_P^{\varphi} C(Q) = \sum_{d=1}^{\infty} n_d \log_2 d = E_P C(Q).$$

Next, we consider that φ is a linear function i. e. if

$$\varphi(x) = \varphi_0(x) = ax + b; \qquad a \neq 0, \quad x \in [1, \infty[$$

then, also

$$E_P^{\varphi_0} C(Q) = \sum_{i=1}^{m} \sum_{d=1}^{\infty} p_i n_{id} \log_2 d = E_P C(Q).$$

For reasons that will become clear in the next section, another useful function φ is the one given by

$$\varphi(x) = \varphi_t(x) = 2^{tx} \qquad (x \in [1, \infty[), \quad t \neq 0.$$

For the function φ_t,

$$E_P^{\varphi_t} C(Q) = \frac{1}{t} \log_2\left(\sum_{i=1}^{m} p_i \prod_{d=1}^{\infty} d^{t n_{id}}\right). \tag{5}$$

We call this average charge to be the exponential average charge of order t for Q.

It can be seen easily that $\lim_{t \to 0} E_P^{\varphi_t} C(Q) = E_P^{\varphi_0} C(Q) = E_P C(Q)$.

III

In this section, we jointly characterize the average charges given in (3) and (5).

Consider two independent state spaces

$$\Theta = \{ \theta_1, \theta_2, \ldots, \theta_J \} \qquad \text{and} \qquad \Theta^* = \{ \theta_1^*, \theta_2^*, \ldots, \theta_K^* \}$$

with associated probability distributions $P=(p_1, p_2, \ldots, p_J)$ and $U=(u_1, u_2, \ldots, u_K)$ such that $p_j \geqq 0$, $\sum_{j=1}^{J} p_j = 1$, $(j=1, 2, \ldots, J)$ and $u_k \geqq 0$, $\sum_{k=1}^{K} u_k = 1$ $(k=1, 2, \ldots, K)$. Since Θ and Θ^* are independent, the probability of the pair (θ_j, θ_k^*) is $p_j u_k$ $(j=1, 2, \ldots, J;\ k=1, 2, \ldots, K)$.

Let us denote by PU the probability distribution

$$\{ p_1 u_1, p_1 u_2, \ldots, p_1 u_K, p_2 u_1, \ldots, p_2 u_K, \ldots, p_J u_1, p_J u_2, \ldots, p_J u_K \}$$

and let valid heterogeneous questionnaires Q_1 and Q_2 exist on Θ and Θ^*, which use precisely m_{jd} $(j=1, 2, \ldots, J)$ and n_{kd} $(k=1, 2, \ldots, K)$ questions of resolution d respectively to determine θ_j and θ_k^*. A questionnaire say, Q, may now be developed from the above two questionnaires on Θ and Θ^* in which $m_{jd}+n_{kd}$ $(j=1, 2, \ldots, J; k=1, 2, \ldots, K)$ questions of resolution d are required to determine the pair (θ_j, θ_k^*).

Now, because a questionnaire for (θ_j, θ_k^*) exists with $\{ m_{jd}+n_{kd} \}$ questions of resolution d $(d=1, 2, \ldots, \infty)$, we have the inequality

$$\sum_{j=1}^{J} \sum_{k=1}^{K} \prod_{d=1}^{\infty} d^{-(m_{jd}+n_{kd})} \leqq 1. \tag{6}$$

which also follows from the inequalities

$$\sum_{j=1}^{J} \prod_{d=1}^{\infty} d^{-m_{jd}} \leqq 1 \qquad \text{and} \qquad \sum_{k=1}^{K} \prod_{d=1}^{\infty} d^{-n_{kd}} \leqq 1$$

ensured from the existence of questionnaires Q_1 and Q_2.

Further, it is natural to expect that if $E_{PU}^{\varphi} C(Q)$ is a measure of average charge for Q, then it is the sum of the average charges for Q_1 and Q_2 separately i. e.:

$$E_{PU}^{\varphi} C(Q) = E_P^{\varphi} C(Q_1) + E_U^{\varphi} C(Q_2) \tag{7}$$

i. e.:

$$\varphi^{-1}\left[\sum_{j=1}^{J}\sum_{k=1}^{K} p_j u_k \varphi\left(\sum_{d=1}^{\infty}\{m_{jd}+n_{kd}\}\log_2 d\right)\right]$$
$$=\varphi^{-1}\left[\sum_{j=1}^{J} p_j \varphi\left(\sum_{d=1}^{\infty} m_{jd}\log_2 d\right)\right]+\varphi^{-1}\left[\sum_{k=1}^{K} u_k \varphi\left(\sum_{d=1}^{\infty} n_{kd}\log_2 d\right)\right]$$

i. e.:

$$\varphi^{-1}\left[\sum_{j=1}^{J}\sum_{k=1}^{K} p_j u_k \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{m_{jd}+n_{kd}}\right)\right]$$
$$=\varphi^{-1}\left[\sum_{j=1}^{J} p_j \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{m_{jd}}\right)\right]+\varphi^{-1}\left[\sum_{k=1}^{K} u_k \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{kd}}\right)\right]. \quad (8)$$

We call the property (7) as *additivity* of the charge.

Now, we will find all additive, quasiarithmatic average charges which amounts to determining all possible values of φ which satisfy (8) under the condition (6). We restrict ourselves to the case $J=K=2$ and proceed on the lines of Aczél [2].

THEOREM 1: *For a questionnaire Q, the only quasiarithmetic charges* (4) *which are additive* (7) *with $J=K=2$ are the arithmetic and exponential average charges given in* (3) *and* (5).

Proof: For $J=K=2$, (8) can be written as

$$\varphi^{-1}\left[p_1 u_1 \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{m_{1d}+n_{1d}}\right)+p_1 u_2 \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{m_{1d}+n_{2d}}\right)\right.$$
$$\left.+p_2 u_1 \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{m_{2d}+n_{1d}}\right)+p_2 u_2 \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{m_{2d}+n_{2d}}\right)\right]$$
$$=\varphi^{-1}\left[p_1 \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{m_{1d}}\right)+p_2 \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{m_{2d}}\right)\right]$$
$$+\varphi^{-1}\left[u_1 \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{1d}}\right)+u_2 \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{2d}}\right)\right]. \quad (9)$$

where

$$p_1 \geqq 0, \qquad p_2 \geqq 0, \qquad p_1+p_2=1;\ u_1 \geqq 0, \qquad u_2 \geqq 0, \qquad u_1+u_2=1 \quad (10)$$

and m_{1d}, m_{2d}, n_{1d}, n_{2d} are positive integers.

Setting

$$m_{1d}=m_{2d}=m_d; \qquad u_1=1-u, \qquad u_2=u \quad \text{in}\,(9)$$

we get

$$\varphi^{-1}\left[(1-u)\ \varphi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{1d}+m_d}\right)+u\,\varphi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{2d}+m_d}\right)\right]$$
$$=\varphi^{-1}\left[(1-u)\,\varphi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{1d}}\right)+u\,\varphi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{2d}}\right)\right]+\log_2 \prod_{d=1}^{\infty} d^{m_d} \qquad (11)$$

for all $u\in[0,\ 1]$ and positive integral values n_{1d}, n_{2d} and m_d.

Now, let us take

$$\psi_m(x)=\varphi(x+m) \qquad (x\in(1,\ \infty[),$$

where

$$m=\log_2 \prod_{d=1}^{\infty} d^{m_d}.$$

Then, (11) gives

$$\varphi^{-1}\left[(1-u)\,\varphi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{1d}}\right)+u\,\varphi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{2d}}\right)\right]$$
$$=\psi_m^{-1}\left[(1-u)\,\psi_m\left(\log_2 \prod_{d=1}^{\infty} d^{n_{1d}}\right)+u\,\psi_m\left(\log_2 \prod_{d=1}^{\infty} d^{n_{2d}}\right)\right]$$

for all $u\in[0,\ 1]$ and arbitrary integers n_{1d} and n_{2d}.

Now refer Hardy, Littlewood and Pólya [6], there must be a linear relation in φ and ψ_m i. e.,

$$\psi_m(x)=\alpha(m)\,\varphi(x)+\beta(m) \qquad (12)$$

the constants $\alpha(m)$ and $\beta(m)$ may in general depend on m.
Thus, by (12) we have

$$\varphi(x+m)=\psi_m(x)=\alpha(m)\,\varphi(x)+\beta(m); \qquad x\in[1,\ \infty[. \qquad (13)$$

Now, there arise two different cases viz. for $\alpha(m)\equiv 1$ and $\alpha(m)\not\equiv 1$. In these cases (*cf.* Aczél [2]) we get an equation for φ of the form

$$\varphi(x+m)=a\,\varphi(x)\,\varphi(m)+b\,\varphi(x)+b\,\varphi(m)+c \qquad (14)$$

with

$$a=0, \qquad b=1 \text{ in the first case} \qquad (15)$$

and

$$a\neq 0, \qquad b=a\,B, \qquad c=a\,B^2-B \text{ in the second case} \qquad (16)$$

so, (9) gives

$$\begin{aligned}\varphi^{-1}\Bigg[a\Bigg\{p_1\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{m_{1d}}\Bigg)&\\+p_2\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{m_{2d}}\Bigg)\Bigg\}\Bigg\{u_1\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{n_{1d}}\Bigg)+u_2\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{n_{2d}}\Bigg)\Bigg\}&\\+b\Bigg\{p_1\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{m_{1d}}\Bigg)+p_2\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{m_{2d}}\Bigg)\Bigg\}&\\+b\Bigg\{u_1\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{n_{1d}}\Bigg)+u_2\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{n_{2d}}\Bigg)\Bigg\}+c\Bigg]&\\=\varphi^{-1}\Bigg[p_1\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{m_{1d}}\Bigg)+p_2\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{m_{2d}}\Bigg)\Bigg]&\\+\varphi^{-1}\Bigg[u_1\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{n_{1d}}\Bigg)+u_2\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{n_{2d}}\Bigg)\Bigg]&\quad(17)\end{aligned}$$

with the variables restricted only by (10).

If $m_{1d}=n_{1d}=1$ and $m_{2d}, n_{2d}=2, 3, \ldots$; $p_2, u_2\in[0, 1]$ and if we take

$$v=p_1\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{m_{1d}}\Bigg)+p_2\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{m_{2d}}\Bigg)$$

and

$$w=u_1\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{n_{1d}}\Bigg)+u_2\,\varphi\Bigg(\log_2\prod_{d=1}^{\infty}d^{n_{2d}}\Bigg)$$

then (17) becomes

$$\varphi(x+y)=a\,\varphi(x)\,\varphi(y)+b\,\varphi(x)+b\,\varphi(y)+c,\qquad\forall\,x,\,y\in[1,\,\infty[,\qquad(18)$$

where

$$x=\varphi^{-1}(v)\quad\text{and}\quad y=\varphi^{-1}(w),\qquad\forall\,v,\,w\text{ in suitable domain.}$$

Now, if the constants are as given in (15), then by setting

$$f(x)=\varphi(x)+c;\qquad x\in[1,\,\infty[\qquad(19)$$

the functional equation (18) reduces to

$$f(x+y)=f(x)+f(y),\qquad\forall\,x,\,y\in[1,\,\infty[.\qquad(20)$$

Since φ is increasing, the function f is also increasing, and so, by Aczél [1], the solution of (20) is

$$f(x)=\gamma\,x,$$

where γ (>0) is an arbitrary constant, which gives

$$\varphi(x)=\gamma x+\delta \qquad (\gamma>0); \quad x\in[1, \infty[. \tag{21}$$

Again, when the constants are as in (16), we may set

$$g(x)=a[\varphi(x)+B]; \qquad x\in[1, \infty[; \quad a\neq 0 \tag{22}$$

and obtain the functional equation (18) in the form

$$g(x+y)=g(x)g(y), \qquad \forall x, y\in[1, \infty[. \tag{23}$$

From (22) we see that g is strictly increasing, because φ is strictly increasing. On the other hand, as (23) shows, if there exists an x_0 for which $g(x_0)=0$ then $g(x_0+y)=0$, $\forall y\in[1, \infty[$, which would contradict the strict monotonicity of g. Thus g is strictly monotonic and nowhere zero and then again from Aczél [1], we get

$$g(x)=2^{tx}; \qquad t\neq 0; \quad \forall x\in[1, \infty[$$

i. e.:

$$\varphi(x)=2^{tx}+\delta; \qquad t>0; \quad \forall x\in[1, \infty[. \tag{24}$$

The proof of the theorem now follows by considering the forms (21) and (24) in (4).

Q.E.D.

On the other hand, the functions given by (21) and (24) satisfy (8) for all $J>1$, $K>1$ [and all m_{jd}, n_{kd}, p_j, u_k ($j=1, 2, \ldots, J$; $k=1, 2, \ldots, K$) satisfying (6) and (8)], thus the arithmetic and exponential average charges (3) and (5) are always additive (7).

IV

In this section (*cf.* Campbell [3]), we will obtain a lower bound on the average charge of order t given in (5) for heterogeneous questionnaires.

THEOREM I: *Let n_{id} ($i=1, 2, \ldots, m$) satisfy*

$$\sum_{i=1}^{m} \prod_{d=1}^{\infty} d^{-n_{id}} \leqq 1. \tag{25}$$

Then we must have

$$E_P^t C(Q) \geqq H_\alpha(P), \tag{26}$$

where

$$\alpha=\frac{1}{1+t} \qquad \textit{and} \qquad H_\alpha(P)=\frac{1}{1-\alpha}\log\left(\sum_{i=1}^{m} p_i^\alpha\right);$$

$\alpha \neq 1$ *is the Rényi's Entropy.*

Proof: If $t=0$ and $\alpha=1$, the result is one proved by Duncan [5]. If $t=\infty$ and $\alpha=0$, then $E_P^\infty C(Q) = \max_{1 \leq i \leq m} \prod_{d=1}^{\infty} d^{n_{id}}$ and $H_0(P) = \log_2 m$.

If the n_{id} satisfy (25), then we must have

$$\prod_{d=1}^{\infty} d^{n_{id}} \leqq m^{-1}$$

for at least one i and hence for the max $\prod_{d=1}^{\infty} d^{n_{id}}$. It follows that

$$\max_{1 \leqq i \leqq m} \sum_{d=1}^{\infty} n_{id} \log_2 d \geqq \log_2 m.$$

Now, let $0 < t < \infty$. By Hölder's inequality,

$$\left(\sum_{i=1}^{m} x_i^p\right)^{1/p} \left(\sum_{i=1}^{m} y_i^q\right)^{1/q} \leqq \sum_{i=1}^{m} x_i y_i, \tag{27}$$

where $(1/p)+(1/q)=1$ and $p<1$. In (27), setting

$$x_i = p_i^{-1/t} \prod_{d=1}^{\infty} d^{-n_{id}}, \qquad y_i = p_i^{1/t}, \qquad p=-t \quad \text{and} \quad q=1-\alpha$$

we get

$$\left(\sum_{i=1}^{m} p_i \prod_{d=1}^{\infty} d^{tn_{id}}\right)^{-1/t} \left(\sum_{i=1}^{m} p_i^{\alpha}\right)^{1/(1-\alpha)} \leqq \sum_{i=1}^{m} \prod_{d=1}^{\infty} d^{-n_{id}}, \tag{28}$$

because the equation $(1/p)+(1/q)=1$ implies that $\alpha=(1+t)^{-1}$. Now, (28) can be rewritten as

$$\left(\sum_{i=1}^{m} p_i \prod_{d=1}^{\infty} d^{tn_{id}}\right)^{1/t} \geqq \frac{\left(\sum_{i=1}^{m} p_i^{\alpha}\right)^{1/(1-\alpha)}}{\sum_{i=1}^{m} \prod_{d=1}^{\infty} d^{-n_{id}}}$$

Using (25) and taking logarithms to the base 2 we get the required result.

Q.E.D.

It can be seen easily that equality holds in (26) iff

$$\prod_{d=1}^{\infty} d^{-n_{id}} = \frac{p_i^{\alpha}}{\sum_{j=1}^{m} p_j^{\alpha}}$$

or

$$\sum_{d=1}^{\infty} n_{id} \log_2 d = -\alpha \log_2 p_i + \log_2 \left(\sum_{j=1}^{m} p_j^{\alpha} \right). \tag{29}$$

We now prove the following generalization of the coding theorem:

THEOREM II: *Let* $\alpha = 1/(1+t)$. *It is possible to construct a valid heterogeneous questionnaire which determines sufficiently long sequences of elements of state space for which average charge of order t per state is as close to* $H_\alpha(P)$ *as desired.*

Proof: Let the state space $\Theta = \{\theta_1, \theta_2, \ldots, \theta_m\}$ have probability distribution $P = (p_1, p_2, \ldots, p_m)$. Consider a sequence of length M of the elements of Θ, say $s = (\theta_1, \theta_2, \ldots, \theta_M)$ in such a way that the probability of s is

$$P(s) = p_{i_1} p_{i_2}, \ldots, p_{i_M} \tag{30}$$

if $\theta_1 = x_{i_1}$, $\theta_2 = x_{i_2}$, $\ldots$, $\theta_M = x_{i_M}$. Let $n_d(s)$ be the number of questions of each resolution d required to determine the sequence s. The average charge of order t for the sequences s (whose number is m^M) of length M is

$$E_P^{t,M} C(Q) = \frac{1}{t} \log_2 \left[\sum P(s) \prod_{d=1}^{\infty} d^{tn_d(s)} \right], \tag{31}$$

where the summation extends over the m^M sequences s.

The entropy of order α of this product space is

$$H_{\alpha, M}(P^*) = M H_\alpha(P), \tag{32}$$

where

$$P^* = \sum [P(s)]^\alpha. \tag{33}$$

Let $n_d(s)$ be the integer which satisfies

$$-\alpha \log_2 P(s) + \log_2 P^* \leqq \sum_{d=1}^{\infty} n_d(s) \log_2 d < 1 - \alpha \log_2 P(s) + \log_2 P^*. \tag{34}$$

Now, if every $n_d(s)$ is equal to the left hand member of (34) then $E_P^{t,M} C(Q) = H_{\alpha, M}(P^*)$.

Now, (34) implies that

$$[P(s)]^{-\alpha t} P^{*t} \leq \prod_{d=1}^{\infty} d^{tn_d(s)} < 2^t [P(s)]^{-\alpha t} P^{*t}. \tag{35}$$

If we multiply each member of (36) by $P(s)$, sum over all s, and use the fact that

$\alpha t=1-\alpha$, we get

$$P^{*(1+t)} \leqq \sum P(s) \prod_{d=1}^{\infty} d^{t n_d(s)} < 2^t P^{*(1+t)}.$$

Now, taking logarithm, dividing by t and using the relations $1+t=\alpha^{-1}$ and $\alpha t=1-\alpha$, we get

$$H_{\alpha, M}(P^*) \leqq E_P^{t, M} C(Q) < H_{\alpha, M}(P^*)+1. \tag{36}$$

If we divide by M and use (33), we get

$$H_\alpha(P) \leqq \frac{E_P^{t, M} C(Q)}{M} < H_\alpha(P)+\frac{1}{M}. \tag{37}$$

The quantity $E_P^{t, M} C(Q)/M$ can be called the average charge of order t per state. By choosing M sufficiently large the average charge can be made as close to $H_\alpha(P)$ as desired. Thus we have proved the required result.

If $t=0$, it is just the ordinary coding theorem.

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