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ORTHODOX BANDS OF MODULES

par Francis PASTIJN

Summary. - In this paper, we shall consider orthodox bands of commutative groups, together with a ring of endomorphisms. We shall generalize the concept of a left module by introducing orthodox bands of left modules ; we shall also deal with linear mappings, the transpose of a linear mapping and with the dual of an orthodox band of left modules.

We shall use the notations and terminilogy of [1](chap 2, § 1) and [2].

1. Definition.

Let $(R, +, \circ)$ be a ring with zero element 0 and identity 1. Let S be a semigroup and $R \times S \longrightarrow S$, $(\alpha, x) \longrightarrow \alpha x$ a mapping satisfying the following conditions :

(i) $\alpha(xy) = (\alpha x)(\alpha y)$ for every $\alpha \in R$ and every $x, y \in S$,

(ii) $(\alpha + \beta)x = (\alpha x)(\beta x)$ for every $\alpha, \beta \in R$ and every $x \in S$,

(iii) $(\alpha \circ \beta)x = \alpha(\beta x)$ for every $\alpha, \beta \in R$ and every $x \in S$,

(iv) $1x = x$ for every $x \in S$.

The so-defined structure will be called an orthodox band of left R-modules. Next theorem justifies our terminology.

2. THEOREM 1. - Let R, S and mapping $R \times S \longrightarrow S$ be as in 1. Then S is an orthodox band of commutative groups, and the maximal subgroups of S are left invariant by the elements of R.

Proof. - Let x be any element of S, and α any element of R ; we then have

$$(0x)(0x) = (0 + 0)x = 0x ,$$

$$(\alpha x)(0x) = (\alpha + 0)x = \alpha x = (0 + \alpha)x = (0x)(\alpha x) ,$$

$$(\alpha x)((-\alpha)x) = (\alpha - \alpha)x = 0x = (-\alpha + \alpha)x = ((-\alpha)x)(\alpha x) .$$

This implies that for any $\alpha \in R$ and any $x \in S$, αx belongs to the maximal subgroup of S with identity $0x$, the inverse of αx in this maximal subgroup must be $(-\alpha)x$. More specifically $1x = x$ belongs to the maximal subgroup of S with identity $0x$, and its inverse in this maximal subgroup must be $(-1)x$. We conclude that S must be a completely regular semigroup and that all maximal subgroup of S are left invariant by the elements of R.

For every $x, y \in S$ we have

$$(xy)(xy) = (1 + 1)(xy) = ((1 + 1)x)((1 + 1)y) = x^2 y^2 .$$

Let e, f be any idempotents of S, then the foregoing implies that

$$(ef)^2 = e^2 f^2 = ef \text{ , hence } E_S = \{x \in S \; ; \; x^2 = x\}$$

must be a subsemigroup of S. Let x and y belong to a same maximal subgroup of S, then the foregoing implies

$$xy = ((-1)x)x^2 y^2((-1)x)xyxy((-1)y)) = yx \text{ ,}$$

hence S is a union of commutative groups. We conclude that S is an orthodox union of commutative groups [3].

Let e and f be any idempotent of S, and $x \in H_e$, $y \in H_f$. We put $(-1)x = x'$ and $(-1)y = y'$. then

$$ef = (ef)^2 = (1+1)(ef) = (1+1)(x(x'f)) = x^2(x'f)^2$$
$$= x^2 x'fx'f = (xf)(x'f)$$

and analogously

$$ef = (x'f)(xf) \text{ .}$$

Since ef, $x'f$ and xf are elements of rectangular group D_{ef} [3], the foregoing implies that xf and $x'f$ are mutually inverse elements of maximal subgroup H_{ef}. Dually, ey and ey' are mutually inverse elements of maximal subgroup H_{ef}. Since $(xy)y' = xf$ and $(xf)y = xy$ we have $xy \,\mathcal{R}\, xf$, hence $xy \,\mathcal{R}\, ef$. Analogously, since $x'(xy) = ey$ and $x(ey) = xy$ we have $xy \,\mathcal{L}\, ey$, hence $xy \,\mathcal{L}\, ef$. We conclude that $xy \,\mathcal{H}\, ef$. Green's relation $\mathcal{H}$ must then be a congruence on S. Thus S is an orthodox band of commutative groups [3].

3. Remark.

Let S be an orthodox band of commutative groups. Then, by Yamada's theorem ([3] and [11]), there exists a band E, and a semilattice of commutative groups Q, both having the same structure semilattice Y, such that S is the spined product of Q and E over Y: $S = Q \times_Y E$. Let $Q = \bigcup_{\varkappa \in Y} G_\varkappa$ and $E = \bigcup_{\varkappa \in Y} E_\varkappa$, then S consists of ordered pairs $(x_\varkappa , e_\varkappa)$, $\varkappa \in Y$, $x_\varkappa \in G_\varkappa$, $e_\varkappa \in E_\varkappa$; multiplication is defined by

$$(x_\lambda , e_\lambda)(y_\mu , f_\mu) = (x_\lambda y_\mu , e_\lambda f_\mu)$$

for any $\lambda , \mu \in Y$, $x_\lambda \in G_\lambda$, $y_\mu \in G_\mu$, $e_\lambda \in E_\lambda$, $f_\mu \in E_\mu$. The identity element of $G_\varkappa$, $\varkappa \in Y$ will be denoted by $1_\varkappa$.

The following result will generalize a theorem of [4] about semilattices of left modules. In patching up next theorem and theorem 1, we actually get a characterization for orthodox bands of commutative groups.

4. THEOREM 2. - <u>Let S be any orthodox band of commutative groups, and let $\underset{\sim}{Z}$ be the ring of integers. Let e be any idempotent of S, and x and x' mutually inverse elements of maximal subgroup H_e. Define mapping $\underset{\sim}{Z} \times S \longrightarrow S$, $(k , x) \longrightarrow kx$ by</u>

$$kx = x^k \quad \text{if } k > 0 ,$$
$$= e \quad \text{if } k = 0 ,$$
$$= x'^{-k} \quad \text{if } k < 0 .$$

Then S is an orthodox band of left $\mathbb{Z}$-modules.

Proof. - Conditions (i), (ii), (iii) and (iv) of 1 are checked by some easy calculations.

5. Definitions and remarks.

Let S be an orthodox band of left R-modules, and τ a congruence on semigroup S. The natural homomorphism of S onto S, will be denoted by $\tau^{\natural}$. τ will be called R-stable if, and only if, $x \tau y$ implies $(\alpha x) \tau (\alpha y)$ for every $x , y \in S$ and every $\alpha \in R$. We can define a mapping $R \times (S/\tau) \longrightarrow S/\tau$, $(a , \bar{x}) \longrightarrow \alpha x = \overline{\alpha x}$. S/τ will then be an orthodox band of left R-modules.

Let S and T be orthodox bands of left R-modules. Mapping $\Phi : S \longrightarrow T$ will be called R-linear if, and only if,

(i) $\Phi(xy) = (\Phi x)(\Phi y)$ for every $x , y \in S$

(ii) $\Phi(\alpha x) = \alpha\Phi(x)$ for every $x \in S$ and every $\alpha \in R$.

$\Phi(S)$ will then be an orthodox band of left R-modules.

Subset A of S will be called R-stable if, and only if, $\alpha x \in A$ for every $x \in A$ and every $\alpha \in R$. If Φ is an R-linear mapping of S into T, $\Phi(S)$ will be an R-stable subsemigroup of T, and the kernel of Φ will be an R-stable subsemigroup of S. Any R-stable subsemigroup of an orthodox band of left R-modules must of course be an orthodox band of left R-modules. If τ is an R-stable congruence on S, the union of all τ-classes containing an idempotent will be an R-stable subsemigroup of S.

Mapping $\Phi : S \longrightarrow T$ will be R-linear if, and only if, $\Phi^{-1} \Phi$ is an R-stable congruence on S. Equivalence relation τ on S is an R-stable congruence if, and only if, $\tau^{\natural}$ is an R-linear mapping.

Mapping $\Phi : S \longrightarrow E_S$, $x \longrightarrow 0x$ is an R-linear mapping of S onto the band consisting of all idempotents of S; $\Phi^{-1} \Phi$ is then the R-stable congruence $\mathcal{H}$.

Let S be the spined product of semilattice of commutative groups Q and band E. We shall use the same notations as in 3. Q is the greatest inverse semigroup homomorphic image of S, and the mapping $\Delta : S \longrightarrow Q$, $(x_\varkappa , e_\varkappa) \longrightarrow x_\varkappa$ is a homomorphism of S onto Q. We shall put $\Delta^{-1} \Delta = \sigma$. This congruence σ is the minimal inverse semigroup congruence on S, and we will show that σ is R-stable. Let G be the greatest group homomorphic image of Q, and $\Gamma : Q \longrightarrow G$, $x_\varkappa \longrightarrow \tilde{x}_\varkappa$ be a homomorphism of Q onto G, $\Gamma^{-1} \Gamma$ being the minimal group congruence on Q. If x_λ and y_μ are any elements of Q, then $x_\lambda \Gamma^{-1} \Gamma y_\mu$ if, and only if, there exists a $\varkappa \in Y$, $\varkappa \leqslant \lambda \wedge \mu$, such that $x_\lambda 1_\varkappa = y_\mu 1_\varkappa$. We shall put $(\Gamma\Delta)^{-1}(\Gamma\Delta) = \rho$; this congruence ρ is the minimal group congruence on S , and we will show that ρ is R-stable.

6. THEOREM 3. - The minimal inverse semigroup congruence on an orthodox band of left R-modules is R-stable.

Proof. - Let $x_\varkappa$ be any element of Q , and let us take any two elements $(x_\varkappa , e_\varkappa)$ and $(x_\varkappa , f_\varkappa)$ in $\Delta^{-1}\Delta x$. Let α be any element of R . Since $\mathcal{H}$ is an R-stable congruence on S , $\alpha(x_\varkappa , e_\varkappa)$ belongs to the $\mathcal{H}$-class $G_\varkappa \times e_\varkappa$ of S containing $(x_\varkappa , e_\varkappa)$, hence,

$$\alpha(x_\varkappa , e_\varkappa) = (y_\varkappa , e_\varkappa) \text{ for some } y_\varkappa \in G_\varkappa .$$

Analogously,

$$\alpha(x_\varkappa , f_\varkappa) = (z_\varkappa , f_\varkappa) \text{ for some } z_\varkappa \in G_\varkappa .$$

Let $(1_\varkappa , g_\varkappa)$ be $\mathcal{L}$-related with $(1_\varkappa , e_\varkappa)$ and $\mathcal{R}$-related with $(1_\varkappa , f_\varkappa)$, and let $(1_\varkappa , h_\varkappa)$ be $\mathcal{R}$-related with $(1_\varkappa , e_\varkappa)$ and $\mathcal{L}$-related with $(1_\varkappa , f_\varkappa)$. Since, by the restriction of $R \times S \longrightarrow S$ to $R \times (G_\varkappa \times g_\varkappa)$, and $R \times (G_\varkappa \times h_\varkappa)$ respectively, $G_\varkappa \times g_\varkappa$ and $G_\varkappa \times h_\varkappa$ become left R-modules, we must have

$$\alpha(1_\varkappa , g_\varkappa) = (1_\varkappa , g_\varkappa) \text{ and } \alpha(1_\varkappa , h_\varkappa) = (1_\varkappa , h_\varkappa) .$$

Furthermore, we have

$$\begin{aligned}(z_\varkappa , e_\varkappa) &= (1_\varkappa , h_\varkappa)(z_\varkappa , f_\varkappa)(1_\varkappa , g_\varkappa) \\ &= (\alpha(1_\varkappa , h_\varkappa))(\alpha(x_\varkappa , f_\varkappa))(\alpha(1_\varkappa , g_\varkappa)) \\ &= \alpha((1_\varkappa , h_\varkappa)(x_\varkappa , f_\varkappa)(1_\varkappa , g_\varkappa)) \\ &= \alpha(x_\varkappa , e_\varkappa) = (y_\varkappa , e_\varkappa) ,\end{aligned}$$

hence $z_\varkappa = y_\varkappa$, and $\Delta(\alpha(x_\varkappa , e_\varkappa)) = \Delta(\alpha(x_\varkappa , f_\varkappa))$..

7. COROLLARY 1. - By mapping $R \times Q \longrightarrow Q$, $(\alpha , x_\varkappa) \longrightarrow \alpha x_\varkappa = \Delta(\alpha\Delta^{-1} x_\varkappa)$, Q becomes a semilattice of left R-modules, and Δ an R-linear mapping of S onto Q.

8. COROLLARY 2. - Let Q be any semilattice of left R-modules, and Y the structure semilattice of Q , let E be a band with the same structure semilattice Y , let $\bigcup_{\varkappa\in Y} G_\varkappa$ and $\bigcup_{\varkappa\in Y} E_\varkappa$ be the semilattice decompositions of Q and E respectively, let S be the spined product $Q \times_Y E$ of Q and E over Y . By mapping $R \times S \longrightarrow S$, $(\alpha , (x_\varkappa , e_\varkappa)) \longrightarrow (\alpha x_\varkappa , e_\varkappa)$ for every $\alpha \in R$, and every $\varkappa \in Y$, $x_\varkappa \in G_\varkappa$, $e_\varkappa \in E_\varkappa$, S become an orthodox band of left R-modules. Conversely, any orthodox band of left R-modules can be so constructed.

9. COROLLARY 3. - Let S be an orthodox normal band of left R-modules, and let $S = \bigcup_{\varkappa\in Y} S_\varkappa$ be the semilattice decomposition of S . For any $\lambda , \mu \in Y$, $\lambda \geqslant \mu$, the structure homomorphism $\Psi_{\lambda,\mu}$ is an R-linear mapping of orthodox rectangular band of left R-modules S_λ into orthodox rectangular band of left R-modules S_μ .

Proof. - In a semilattice of left R-modules the structure homomorphisms are R-linear [6]. The theorem now follows from corollary 2 and from a result about normal bands [10].

10. Remark.

Structure theorems for semilattices of left R-modules [6], together with corollary 2 yield structure theorems for orthodox bands of left R-modules.

11. THEOREM 4. - The minimal group congruence on an orthodox band of left R-modules is R-stable.

Proof. - Let $\tilde{x}_\lambda$ be any element of G , the greatest group homomorphic image of orthodox band of left R-modules S . Let us take any two elements x_λ and y_μ in $\Gamma^{-1}\tilde{x}_\lambda$. There exists a $\varkappa \in Y$, $\varkappa \leqslant \lambda \wedge \mu$, such that $1_\varkappa x_\lambda = 1_\varkappa y_\varkappa$. Let α be any element of R . From

$$(\alpha x_\lambda)1_\varkappa = (\alpha x_\lambda)(\alpha 1_\varkappa) = \alpha(x_\lambda 1_\varkappa) = \alpha(y_\mu 1_\varkappa) = (\alpha y_\mu)(\alpha y_\varkappa) = (\alpha 1_\mu)1_\varkappa ,$$

and $\alpha x_\lambda \in G_\lambda$, $\alpha y_\mu \in G_\mu$, we conclude that $\alpha y_\mu \in \Gamma^{-1}\Gamma(\alpha x_\lambda)$, and thus $\widetilde{\alpha x}_\lambda = \widetilde{\alpha y}_\mu$. This implies that the minimal group congruence $\Gamma^{-1}\Gamma$ on Q must be R-stable. Consequently, the minimal group congruence $(\Gamma\Delta)^{-1}\Gamma\Delta = \rho$ on S must be R-stable.

12. COROLLARY 4. - By mapping $R \times G \longrightarrow G$, $(\alpha , \tilde{x}_\varkappa) \longrightarrow \alpha\tilde{x}_\varkappa = \widetilde{\alpha x}_\varkappa$, G becomes a left R-module, and the mapping $\Gamma\Delta$ an R-linear mapping of S onto G .

13. Definitions.

An orthodox band of right R-modules S can be defined in an analogous way as an orthodox band of left R-modules. Condition (iii) of 1 must then be replaced by (iii)'. $(\alpha \circ \beta)x = \beta(\alpha x)$ for every $\alpha , \beta \in R$ and every $x \in S$. It will be more convenient to denote mapping $R \times S \longrightarrow S$, $(\alpha , x) \longrightarrow x\alpha$. (iii)', then, becomes

(iii)' $\quad x(\alpha \circ \beta) = (x\alpha)\beta$ for every $\alpha , \beta \in R$ and every $x \in S$.

If S is at the same time orthodox band of left R-modules, and orthodox band of right R-modules, then we shall say that S is an orthodox band of R-bimodules.

Let $R^\infty = R \cup \{\infty\}$, and define addition in R^∞ as follows. For any $\alpha , \beta \in R$, we put $\alpha + \beta = \gamma$ in R^∞ if, and only if, $\alpha + \beta = \gamma$ in R , and

$$\alpha + \infty = \infty + \alpha = \infty .$$

R^∞ will be a group with "zero" ∞ . We next define mapping $R \times R^\infty \longrightarrow R^\infty$ by

$$(\alpha , \beta) \longrightarrow \alpha\beta = \gamma \text{ if, and only if, } \alpha \circ \beta = \gamma \text{ in } R ,$$

and

$$(\alpha , \infty) \longrightarrow \alpha\infty = \infty .$$

We also define mapping $R \times R^\infty \longrightarrow R^\infty$ by

$$(\alpha , \beta) \longrightarrow \beta\alpha = \gamma \text{ if, and only if, } \beta \circ \alpha = \gamma \text{ in } R ,$$

and

$$(\alpha , \infty) \longrightarrow \infty\alpha = \infty .$$

By these two mappings R^∞ becomes a semilattice of R-bimodules, the structure semilattice being the two element semilattice. We shall use R^∞ later in this paper.

The next theorem generalizes a result of [9].

14. THEOREM 5. - Let S be an orthodox band of left R-modules, and T an orthodox band of right R-modules. Let $\mathfrak{F}_{S,T}$ be the set of all partial mapping of S into T. Define a multiplication in $\mathfrak{F}_{S,T}$ as follows : for every $\Phi , \Psi \in \mathfrak{F}_{S,T}$ $\operatorname{dom} \Phi\Psi = \operatorname{dom} \Phi \cap \operatorname{dom} \Psi$, and for every $x \in \operatorname{dom} \Phi\Psi$ we put $\Phi\Psi(x) = (\Phi x)(\Psi x)$. Define mapping $R \times \mathfrak{F}_{S,T} \longrightarrow \mathfrak{F}_{S,T}$, $(\alpha , \Phi) \longrightarrow \Phi\alpha$ by $\operatorname{dom}(\Phi\alpha) = \operatorname{dom} \Phi$ and $(\Phi\alpha)x = (\Phi x)\alpha$, for every $x \in \operatorname{dom} \Phi$. $\mathfrak{F}_{S,T}$ will then be an orthodox band of right R-modules if, and only if, T is a semilattice of right R-modules.

Proof. - For any $\Phi , \Psi \in \mathfrak{F}_{S,T}$ and any $\alpha \in R$ we have

$$\operatorname{dom}(\Phi\Psi)\alpha = \operatorname{dom} \Phi\Psi = \operatorname{dom} \Phi \cap \operatorname{dom} \Psi = \operatorname{dom} \Phi\alpha \cap \operatorname{dom} \Psi\alpha = \operatorname{dom}(\Phi\alpha)(\Psi\Phi) ,$$

and for any $x \in \operatorname{dom}(\Phi\Psi)\alpha$ we have

$$((\Phi\Psi)\alpha)x = ((\Phi\Psi)x)\alpha = ((\Phi x)(\Psi x))\alpha = ((\Phi x)\alpha)((\Psi x)\alpha) = ((\Phi\alpha)x)((\Psi\alpha)x) = ((\Phi\alpha)(\Psi\alpha))x ,$$

hence $(\Phi\Psi)\alpha = (\Phi\alpha)(\Psi\alpha)$. For any $\Phi \in \mathfrak{F}_{S,T}$ and any $\alpha , \beta \in R$ we have

$$\operatorname{dom} \Phi(\alpha + \beta) = \operatorname{dom} \Phi = \operatorname{dom} \Phi\alpha \cap \operatorname{dom} \Phi\beta = \operatorname{dom}(\Phi\alpha)(\Phi\beta) ,$$

and, for any $x \in \operatorname{dom} \Phi(\alpha + \beta)$ we have

$$(\Phi(\alpha + \beta))x = (\Phi x)(\alpha + \beta) = ((\Phi x)\alpha)((\Phi x)\beta) = ((\Phi\alpha)x)((\Phi\beta)x) = (\Phi\alpha)(\Phi\beta)x ,$$

hence $\Phi(\alpha + \beta) = (\Phi\alpha)(\Phi\beta)$. Furthermore,

$$\operatorname{dom} \Phi(\alpha \circ \beta) = \operatorname{dom} \Phi = \operatorname{dom} \Phi\alpha = \operatorname{dom}(\Phi\alpha)\beta ,$$

and for any x dom $\Phi(\alpha \circ \beta)$ we have

$$(\Phi(\alpha \circ \beta))x = (\Phi x)(\alpha \circ \beta) = ((\Phi x)\alpha)\beta = ((\Phi\alpha)x)\beta = ((\Phi\alpha)\beta)x ,$$

hence $\Phi(\alpha \circ \beta) = (\Phi\alpha)\beta$. Finally, $\operatorname{dom} \Phi 1 = \operatorname{dom} \Phi$, and for any $x \in \operatorname{dom} \Phi 1$ we have

$$(\Phi 1)x = (\Phi x)1 = \Phi x ,$$

hence $\Phi 1 = \Phi$. We conclude that $\mathfrak{F}_{S,T}$ is an orthodox band of right R-modules.

From the definition of the multiplication in $\mathfrak{F}_{S,T}$ follows that $\mathfrak{F}_{S,T}$ is commutative if, and only if, T is commutative. From this, follows the last part of the theorem.

15. THEOREM 6. - Let S be an orthodox band of left R-modules, S' the set of R-linear mappings of S into R, and S' the set of R-linear mapping of S into R^∞. Then S' is an R-stable subsemigroup of $\mathfrak{F}_{S,R}$ and S^* is an R-stable subsemigroup of $\mathfrak{F}_{S,R^\infty}$.

Proof. - We show that S^* is an R-stable subsemigroup of $\mathfrak{F}_{S,R^\infty}$. The proof of the rest is quite the same. Let x^* and y^* be any elements of S^*. Since R^∞ is a semilattice of commutative groups, x^*y^* must be a homomorphism of S into R^∞. For any $x \in S$ and any $x^* \in S^*$ we shall from now put $x^*(x) = \langle x , x^* \rangle$. For any $x \in S$, any $\alpha \in R$ and any $x^* , y^* \in S^*$ we then have

$$\begin{aligned}\langle \alpha x , x^* y^* \rangle &= \langle \alpha x , x^* \rangle + \langle \alpha x , y^* \rangle \\ &= \alpha\langle x , x^* \rangle + \alpha\langle x , y^* \rangle \\ &= \alpha(\langle x , x^* \rangle + \langle x , y^* \rangle) \\ &= \alpha\langle x , x^* y^* \rangle .\end{aligned}$$

We conclude that for any $x^* , y^* \in S^*$, $x^* y^*$ must be an R-linear mapping of S into R^∞, hence $x^* y^* \in S^*$. S^* is a subsemigroup of $\mathfrak{F}_{S,R^\infty}$.

For any $x , y \in S$, any $x^* \in S^*$ and any $\alpha \in R$ we have

$$\begin{aligned}\langle xy , x^* \alpha \rangle &= \langle xy , x^* \rangle\alpha \\ &= (\langle x , x^* \rangle + \langle y , x^* \rangle)\alpha \\ &= \langle x , x^* \rangle\alpha + \langle y , x^* \rangle\alpha \\ &= \langle x , x^*\alpha \rangle + \langle y , x^*\alpha \rangle ,\end{aligned}$$

hence $x^* \alpha$ must be a homomorphism of S into R^∞. For any $x \in S$, any $x^* \in S^*$ and any $\alpha , \beta \in R$ we have

$$\begin{aligned}\langle \beta x , x^* \alpha \rangle &= \langle \beta x , x^* \rangle\alpha \\ &= \beta\langle x , x^* \rangle\alpha \\ &= \beta\langle x , x^* \alpha \rangle .\end{aligned}$$

We conclude that for any $x^* \in S^*$ and any $\alpha \in R$, $x^* \alpha$ must be an R-linear mapping of S into R^∞. Consequently S^* must be an R-stable subsemigroup of $\mathfrak{F}_{S,R^\infty}$.

16. COROLLARY 5. - S^* <u>is a semilattice of right</u> R-<u>modules. The structure semilattice of</u> S^* <u>is isomorphic with the semilattice of prime ideals of</u> S . <u>The mapping</u> $1^* : S \longrightarrow R^\infty , x \longrightarrow 0$ <u>is the identity of</u> S^* <u>and the mapping</u> $0^* : S \longrightarrow R^\infty , x \longrightarrow \infty$ <u>is the zero of</u> S^*.

Proof. - R^∞ is a semilattice of right R-modules, hence $\mathfrak{F}_{S,R^\infty}$ is a semilattice of right R-modules. Since S^* is R-stable in $\mathfrak{F}_{S,R^\infty}$, S^* must be a semilattice of right R-modules too.

Let e^* be any idempotent of S^*, then

$$V_{e^*} = \{x \quad S \; ; \; \langle x , e^* \rangle = \infty\}$$

is a prime ideal of S . For any $x \in S\backslash V_{e^*}$

$$\langle x , e^* \rangle \in R \text{ and } \langle x , e^* \rangle = \langle x , e^{*2} \rangle = \langle x , e^* \rangle + \langle x , e^* \rangle ,$$

hence $\langle x , e^* \rangle = 0$. Conversely, let P be any prime ideal of S , then we can define $e^*_P \in S^*$ by $\langle x , e^*_P \rangle = \infty$ for all $x \in P$, and $\langle x , e^*_P \rangle = 0$ for all $x \in S\backslash P$. Furthermore, if e^* and f^* are any two idempotents of S^*, we must have

$V_{e^*f^*} = V_{e^*} \cup V_{f^*}$. Consequently, the semilattice E_{S^*} consisting of the idempotents of S^* is isomorphic with the $\cup$-semilattice of all prime ideals of S . Since E_{S^*} is isomorphic with the structure semilattice of S^* , the result stated in the corollary follows.

17. COROLLARY 6. - S' <u>is a right</u> R-<u>module which is an</u> R-<u>stable subgroup of</u> S^* : S' <u>is the maximal submodule of</u> S^* <u>containing the identity</u> 1^* <u>of</u> S^* .

<u>Proof</u>. - All elements of S' are R-linear mappings of S into R , hence, they can be considered as R-linear mappings of S into R^∞ , and consequently $S' \subseteq S^*$. Since S' is R-stable in $\mathfrak{F}_{S,R}$, and since clearly $\mathfrak{F}_{S,R}$ is R-stable in $\mathfrak{F}_{S,R^\infty}$, S' must be R-stable in $\mathfrak{F}_{S,R^\infty}$; from this we imply that S' is R-stable in S^* .

It must be evident that $1^* : S \longrightarrow R^\infty$, $x \longrightarrow 0$ is the identity of S' . Let x^* be any element of S' , then $x^*(-1) \in S'$, and for any $x \in S$ we have

$$\langle x , x^*(x^*(-1))\rangle = \langle x , x^*\rangle + \langle x , x^*(-1)\rangle = \langle x , x^*\rangle + \langle x , x^*\rangle(-1) = 0$$

and analogously

$$\langle x , (x^*(-1))x^*\rangle = 0 ,$$

hence $x^*(x^*(-1)) = (x^*(-1))x^* = 1^*$. This shows that x^* and $x^*(-1)$ are mutually inverse elements of commutative group H_{1^*} , the maximal subgroup of S^* containing 1^* . For any element $y^* \in H_{1^*}$, we must have $V_{y^*} = \square$, hence any element $y^* \in H_{1^*}$ belongs to S' . We can conclude that $H_{1^*} = S'$.

18. THEOREM 7. - <u>Let</u> S <u>be an orthodox band of left</u> R-<u>modules and</u> τ <u>any</u> R-<u>stable congruence on</u> S . <u>The mapping</u> $\Phi : (S/\tau)^* \longrightarrow S^*$, $\bar{x}^* \longrightarrow \Phi\bar{x}^*$ <u>defined by</u> $\langle x , \Phi\bar{x}^*\rangle = \langle \tau^\natural x , \bar{x}^*\rangle$ <u>for every</u> $x \in S$ <u>is an</u> R-<u>isomorphism of</u> $(S/\tau)^*$ <u>into</u> S^*. <u>Whenever</u> $\iota_S \subseteq \tau \subseteq \sigma$, σ <u>being the minimal inverse semigroup congruence on</u> S , <u>this mapping</u> Φ <u>is a surjective</u> R-<u>isomorphism of</u> $(S/\tau)^*$ <u>onto</u> S^* .

<u>Proof</u>. - Let us suppose that $\bar{x}^*$, $\bar{y}^*$ are any elements of $(S/\tau)^*$, and x any element of S . We then have

$$\begin{aligned}\langle x , \Phi(\bar{x}^* \bar{y}^*)\rangle &= \langle \tau^\natural x , \bar{x}^* \bar{y}^*\rangle \\ &= \langle \tau^\natural x , \bar{x}^*\rangle + \langle \tau^\natural x , \bar{y}^*\rangle \\ &= \langle x , \Phi\bar{x}^*\rangle + \langle x , \Phi\bar{y}^*\rangle \\ &= \langle x , (\Phi\bar{x}^*)(\Phi\bar{y}^*)\rangle ,\end{aligned}$$

hence $\Phi(\bar{x}^* \bar{y}^*) = (\Phi\bar{x}^*)(\Phi\bar{y}^*)$. Let us suppose that $\bar{x}^*$ is any element of $(S/\tau)^*$, α any element of R and x any element of S , then

$$\begin{aligned}\langle x , \Phi(\bar{x}^* \alpha)\rangle &= \langle \tau^\natural x , \bar{x}^* \alpha\rangle \\ &= \langle \tau^\natural x , \bar{x}^*\rangle\alpha \\ &= \langle x , \Phi\bar{x}^*\rangle\alpha \\ &= \langle x , (\Phi\bar{x}^*)\alpha\rangle ,\end{aligned}$$

hence $\Phi(\bar{x}^* \alpha) = (\Phi\bar{x}^*)\alpha$. Since $\tau^\natural$ is an R-linear mapping of S onto S/τ ,

$\Phi\bar{x}^* \in S^*$ for any $\bar{x}^* \in (S/\tau)^*$. We conclude that Φ is an R-linear mapping of $(S/\tau)^*$ into S^*. Let us now suppose that $\bar{x}^*, \bar{y}^* \in (S/\tau)^*$, and $\Phi\bar{x}^* = \Phi\bar{y}^*$. If for some $\bar{x} \in S/\tau$ $\langle \bar{x}, \bar{x}^* \rangle \neq \langle \bar{x}, \bar{y}^* \rangle$, then for any $x \in (\tau^\natural)^{-1} x$ we should have

$$\langle x , \Phi\bar{x}^* \rangle = \langle \tau^\natural x , \bar{x}^* \rangle = \langle \bar{x} , \bar{x}^* \rangle \neq \langle \bar{x} , \bar{y}^* \rangle = \langle \tau^\natural x , \bar{y}^* \rangle = \langle x , \Phi\bar{y}^* \rangle ,$$

and this is impossible. We conclude that $\Phi\bar{x}^* = \Phi\bar{y}^*$ implies $\bar{x}^* = \bar{y}^*$, hence Φ is an isomorphism of $(S/\tau)^*$ into S^*.

It will be sufficient to show that the mapping $\Phi : (S/\sigma)^* \longrightarrow S^*$, $x^* \longrightarrow \Phi\bar{x}^*$ defined by $\langle x , \Phi\bar{x}^* \rangle = \langle \sigma^\natural x , \bar{x}^* \rangle$ for every $x \in S$, will be an R-isomorphism of $(S/\sigma)^*$ onto S^*. Let x^* be any element of S^*, and $(x_\varkappa , e_\varkappa)$ and $(x_\varkappa , f_\varkappa)$ any two σ-related elements of S. Since $(x_\varkappa , e_\varkappa)$ and $(x_\varkappa , f_\varkappa)$ are $\mathcal{D}$-related in S, they generate a same principal ideal of S, and thus $\langle (x_\varkappa , e_\varkappa) , x^* \rangle = \infty$ if, and only if, $\langle (x_\varkappa , f_\varkappa) , x^* \rangle = \infty$. Let us suppose that $(x_\varkappa , e_\varkappa)$ and $(x_\varkappa , f_\varkappa)$ both belong to $S \backslash V_{x^*}$. Let $(1_\varkappa , g_\varkappa)$ be $\mathcal{L}$-related with $(x_\varkappa , e_\varkappa)$ and $\mathcal{R}$-related with $(1_\varkappa , f_\varkappa)$, and $(1_\varkappa , h_\varkappa)$ $\mathcal{R}$-related with $(x_\varkappa , e_\varkappa)$ and $\mathcal{L}$-related with $(1_\varkappa , f_\varkappa)$; $(1_\varkappa , g_\varkappa)$ and $(1_\varkappa , h_\varkappa)$ are both $\mathcal{D}$-related with $(x_\varkappa , e_\varkappa)$, and $(x_\varkappa , f_\varkappa)$. Hence $(1_\varkappa , g_\varkappa) , (1_\varkappa , h_\varkappa) \in S \backslash V_{x^*}$, since these two elements are idempotents of S, and since x^* is an homomorphism of $S \backslash V_{x^*}$ into R, we have

$$\langle (1_\varkappa , g_\varkappa) , x^* \rangle = \langle (1_\varkappa , h_\varkappa) , x^* \rangle = 0 .$$

From this follows that

$$\langle (x_\varkappa , e_\varkappa) , x^* \rangle = \langle (1_\varkappa , h_\varkappa)(x_\varkappa , f_\varkappa)(1_\varkappa , g_\varkappa) , x^* \rangle$$
$$= \langle (1_\varkappa , h_\varkappa) , x^* \rangle + \langle (x_\varkappa , f_\varkappa) , x^* \rangle + \langle (1_\varkappa , g_\varkappa) , x^* \rangle = \langle (x_\varkappa , f_\varkappa) , x^* \rangle .$$

In any case $(x^*)^{-1} x^* \supseteq \sigma$. Hence the mapping $\bar{x}^* \in (S/\sigma)^*$ defined by $\langle \sigma^\natural x , \bar{x}^* \rangle = \langle x , x^* \rangle$ for all $x \in S$ is well-defined, and we shall have $\Phi\bar{x}^* = x^*$. Thus, in this case Φ must be surjective.

19. COROLLARY 7. - <u>If</u> S <u>is an orthodox band of left</u> R-<u>modules, and</u> Q <u>the greatest inverse homomorphic image of</u> S, <u>then</u> S^* <u>and</u> Q^* <u>are</u> R-<u>isomorphic</u>.

20. THEOREM 8. - <u>Let</u> S <u>be an orthodox band of left</u> R-<u>modules and</u> τ <u>any</u> R-<u>stable congruence on</u> S. <u>The mapping</u> $\Psi : (S/\tau)' \longrightarrow S'$, $\bar{x}^* \longrightarrow \Psi(\bar{x}^*)$ <u>defined by</u> $\langle x , \Psi\bar{x}^* \rangle = \langle \tau^\natural x , \bar{x}^* \rangle$ <u>for any</u> $x \in S$ <u>is an</u> R-<u>isomorphism of</u> $(S/\tau)'$ <u>into</u> S'. <u>Whenever</u> $\iota_S \subseteq \tau \subseteq \rho$, ρ <u>being the minimal group congruence on</u> S, <u>this mapping</u> Ψ <u>is a surjective</u> R-<u>isomorphism of</u> $(S/\tau)'$ <u>onto</u> S'.

<u>Proof</u>. - It is clear that mapping Ψ must be the restriction of mapping Φ (of theorem 7) to maximal submodule $(S/\tau)'$ of $(S/\tau)^*$, hence Ψ is an R-isomorphism of $(S/\tau)'$ into S^*. Since for every $x \in S$, and every $\bar{x}^* \in (S/\tau)'$ we must have $\langle \tau^\natural x , \bar{x}^* \rangle \in R$. We conclude $\Psi\bar{x}^* \in S'$ for every $\bar{x}^* \in (S/\tau)'$, thus, Ψ is an R-isomorphism of $(S/\tau)'$ into S'.

It will be sufficient to show that the mapping $\Psi : (S/\rho)' \longrightarrow S'$, $\bar{x}^* \longrightarrow \Psi\bar{x}^*$

defined by $\langle x , \Psi\bar{x}^* \rangle = \langle \rho^{\natural} x , \bar{x}^* \rangle$ for every $x \in S$ will be an R-isomorphism of $(S/\rho)'$ onto S' . Let x^* be any element of S' . Since x^* must be a homomorphism of S into the additive group R , we have $(x^*)^{-1} x^* \supseteq \rho$. Hence the mapping $\bar{x}^* \in (S/\rho)'$ defined by $\langle \rho^{\natural} x , \bar{x}^* \rangle = \langle x , x^* \rangle$ for every $x \in S$ is well-defined, and we shall have $\Psi\bar{x}^* = x^*$. Thus, in this case Ψ must be surjective.

21. CORALLARY 8. - *If S is an orthodox band of left R-modules, Q the greatest inverse homomorphic image of S, and G the greatest group homomorphic imgage of S, then S' and Q' are both R-isomorphic with right R-module G' which is the dual of left R-module G.*

22. THEOREM 9. - *Let S be an orthodox band of left R-modules, and $S = \bigcup_{\varkappa \in Y} - \bigcup_{\varkappa} S = \bigcup_{\varkappa \in Y} G_{\varkappa} \times E_{\varkappa}$ its semilattice decomposition. For any $\lambda \in Y$, mapping $1^*_{\lambda} : S \longrightarrow R^{\infty}$ defined by $\langle x , 1^*_{\lambda} \rangle = 0$ if, and only if, $x \in \bigcup_{\varkappa \geq \lambda} S_{\varkappa}$, and $\langle x , 1^*_{\lambda} \rangle = \infty$ otherwise, is an idempotent of S^* . The maximal submodule $H_{1^*_{\lambda}}$ of S^* containing 1^*_{λ} is R-isomorphic with $(\bigcup_{\varkappa \geq \lambda} S_{\varkappa})'$ and with right R-module G'_{λ} , the dual of left R-module G_{λ} .*

Proof. - For any $\lambda \in Y$, $\bigcup_{\varkappa \geq \lambda} S_{\varkappa}$ is an R-stable subsemigroup of S , and G_{λ} will be the greatest group homomorphic image of $\bigcup_{\varkappa \geq \lambda} S_{\varkappa}$. From corollary 8 follows that $(\bigcup_{\varkappa \geq \lambda} S_{\varkappa})'$ and G'_{λ} are R-isomorphic right R-modules. It is easy to show that $S \setminus (\bigcup_{\varkappa \geq \lambda} S_{\varkappa})$ is a prime ideal of S . From results in the proof of corollary 5 then follows that 1^*_{λ} must be an idempotent of S^* . We remark that for any $x^* \in S^*$, $s^* \in H_{1^*_{\lambda}}$ if, and only if,

$$V_{x^*} = \{x \in S \ ; \ \langle x , x^* \rangle = \infty\} = S \setminus (\bigcup_{\varkappa \geq \lambda} S_{\lambda}) .$$

Hence the mapping $H_{1^*_{\lambda}} \longrightarrow (\bigcup_{\varkappa \geq \lambda} S_{\varkappa})'$, $x^* \longrightarrow x^* \in \bigcup_{\varkappa \geq \lambda} S_{\varkappa}$ is an R-isomorphism of $H_{1^*_{\lambda}}$ onto $(\bigcup_{\varkappa \geq \lambda} S_{\varkappa})'$.

23. COROLLARY 9. - *We use the same notations as in 22. Let Q be the greatest inverse semigroup homomorphic image of S and $Q = \bigcup_{\varkappa \in Y} G_{\varkappa}$ its semilattice decomposition. For any $\lambda , \mu \in Y$, $\lambda \geq \mu$, let ${}^t\Phi_{\lambda,\mu}$ be the structure homomorphism of Q , and $\Phi_{\lambda,\mu}$ its transpose. Then $1^*_{\mu} \geq 1^*_{\lambda}$ in S^* . Let $\Phi^*_{\mu,\lambda} : H_{1^*_{\mu}} \longrightarrow H_{1^*_{\lambda}}$ be the structure homomorphism of S^* . For any $\lambda \in Y$ the mapping $\Psi_{\lambda} : H_{1^*_{\lambda}} \longrightarrow G'_{\lambda}$, $x^* \longrightarrow \Psi_{\lambda} x^*$, defined by*

$$\langle (x_{\varkappa} , e_{\varkappa}) , x^* \rangle = \langle \Phi_{\varkappa,\lambda} x_{\varkappa} , \Psi_{\lambda} x^* \rangle \ \text{ for all } \ (x_{\varkappa} , e_{\varkappa}) \in \bigcup_{\varkappa \geq \lambda} S_{\varkappa} ,$$

is an R-isomorphism of $H_{1^*_{\lambda}}$ onto G'_{λ} , and the following diagram is commutative.

$$\begin{array}{ccccc} & H_{1^*_{\mu}} & \xrightarrow{\Psi_{\mu}} & G'_{\mu} & \\ \Phi^*_{\mu,\lambda} \rightarrow & \Big\downarrow & & \Big\downarrow & \leftarrow {}^t\Phi_{\lambda,\mu} \\ & H_{1^*_{\lambda}} & \xrightarrow[\Psi_{\lambda}]{} & G'_{\lambda} & \end{array}$$

Proof. – The mapping $\bigcup_{\varkappa\geq\lambda} S_\varkappa \longrightarrow G_\lambda$, $(x_\varkappa , e_\varkappa) \longrightarrow \Phi_{\varkappa,\lambda} x_\varkappa$ is an homomorphism of $\bigcup_{\varkappa\geq\lambda} S_\varkappa$ onto its greatest group homomorphic image G_λ . Ψ_λ must then be an R-isomorphism of $H_{1^*_\lambda}$ onto G'_λ by theorem 8.

Let x^* be any element of $H_{1^*_\mu}$, and x_λ any element of G_λ . We proceed to show that

$$\langle x_\lambda , {}^t\Phi_{\lambda,\mu} \Psi_\lambda x^*\rangle = \langle x_\lambda , \Psi_\lambda \Phi^*_{\mu,\lambda} x^*\rangle .$$

Indeed

$$\begin{aligned}
\langle x_\lambda , {}^t\Phi_{\lambda,\mu} \Psi_\mu x^*\rangle &= \langle \Phi_{\lambda,\mu} x_\lambda , \Psi_\mu x^*\rangle \\
&= \langle x_\lambda 1_\mu , \Psi_\mu x^*\rangle \\
&= \langle (x_\varkappa , e_\varkappa) , x^*\rangle \\
&\qquad \text{for all } \varkappa \geq \mu , \quad \Phi_{\varkappa,\mu} x_\varkappa = x_\lambda 1_\mu , \quad e_\varkappa \in E_\varkappa \\
&= \langle (x_\lambda , e_\lambda) , x^*\rangle \text{ for all } e_\lambda \in E_\lambda , \\
&= \langle (x_\lambda , e_\lambda) , x^* 1^*_\lambda\rangle \text{ for all } e_\lambda \in E_\lambda \\
&= \langle (x_\lambda , e_\lambda) , \Phi^*_{\mu,\lambda} x^*\rangle \text{ for all } e_\lambda \in E_\lambda \\
&= \langle x_\lambda , \Psi_\lambda \Phi^*_{\mu,\lambda} x^*\rangle .
\end{aligned}$$

We conclude that ${}^t\Phi_{\lambda,\mu} \Psi_\mu = \Psi_\lambda \Phi^*_{\mu,\lambda}$.

24. COROLLARY 10. – We use the same notations as in 22 and 23. Let the structure semilattice of S be a lattice. Consider $V = \bigcup_{\varkappa\in Y} G'_\varkappa$, and define multiplication in V by the following. For any x' , $y' \in V$, $x' \in G'_\lambda$, $y' \in G'_\mu$, put

$$x'y' = ({}^t\Phi_{\lambda\vee\mu,\lambda} x')({}^t\Phi_{\lambda\vee\mu,\mu} y') .$$

Define mapping $R \times V \longrightarrow V$, $(\alpha , x') \longrightarrow x'\alpha$ in the usual way. Then V is a semilattice of right R-modules, and there exists an R-isomorphism of V into S^* . If Y satisfies the minimal condition, V must be R-isomorphic with S^* .

25. Remarks.

Corollaries 9 and 10 show that S^* could well be named the dual of S . If Y is a lattice, the structure semilattice of V is the $\vee$-semilattice Y . Results of [6] make the connections between structure theorems for S and structure theorems for V more explicit.

Theorem 7 is quite analogous with a result in [5] (§ 5) about the character semigroup of a commutative semigroup, and theorem 9, corollary 9 and corollary 10 are in a certain way analogous with results of [7] and [8] (see also [2], chapter 5).

Next theorem generalizes the concept of the transpose of an R-linear mapping.

26. THEOREM 10. – Let S and T be orthodox bands of left R-modules, and $\Theta : S \longrightarrow T$ an R-linear mapping. The mapping ${}^T\Theta : T^* \longrightarrow S^*$, $t^* \longrightarrow {}^T\Theta t^*$, defined by $\langle x , {}^T\Theta t^*\rangle = \langle \Theta x , t^*\rangle$ for all $x \in S$, must be an R-linear mapping of T^* into S^* , and ${}^T\Theta(T^*)$ is embeddable in $(S/\Theta^{-1}\Theta)^* \cong (\Theta S)^*$.

Proof. – It must be clear that for any $t^* \in T^*$, we must have ${}^T\Theta t^* \in S^*$, since

Θ is R-linear. It is not difficult either to show that ${}^T\Theta$ is R-linear.

Let t^* and v^* be any elements of T^*, then $t^*|\Theta S$ and $v^*|\Theta S$ are both elements of $(\Theta S)^*$, since ΘS is an R-stable subsemigroup of T. From the definition of ${}^T\Theta$ we have that ${}^T\Theta t^* = {}^T\Theta v^*$ if, and only if, $v^*|\Theta S = t^*|\Theta S$. This implies that the mapping ${}^T\Theta(T^*) \longrightarrow (\Theta S)^*$, ${}^T\Theta t^* \longrightarrow t^*|\Theta S$ is an R-isomorphism of ${}^T\Theta(T^*)$ into $(\Theta S)^*$.

27. COROLLARY 11. - *Let* S, T *and* Θ *be as in theorem 10. The mapping* ${}^t\Theta : T' \longrightarrow S'$, $t^* \longrightarrow {}^t\Theta t^*$, *defined by* $\langle x , {}^t\Theta t^*\rangle = \langle \Theta x , t^*\rangle$ *for all* $x \in S$, *must be an* R-*linear mapping of* T' *into* S', *and* ${}^t\Theta(T')$ *is embeddable in* $(S/\Theta^{-1}\Theta)' \cong (\Theta S)'$.

28. COROLLARY 12. - *We use the same notations as in* 26 *and* 27. *Let* ρ_S *and* ρ_T *be the minimal group congruences on* S *and* T *respectively. Let* $\Psi_S : (S/\rho_S)' \longrightarrow S'$, $\bar{x}^* \longrightarrow \Psi_S\,\bar{x}^*$, *be the* R-*isomorphism defined by* $\langle x , \Psi_S\,\bar{x}^*\rangle = \langle \rho_S^\natural x , \bar{x}^*\rangle$ *for all* $x \in S$, *and* $\Psi_T : (T/\rho_T)' \longrightarrow T'$, $\bar{t}^* \longrightarrow \Psi_T\,\bar{t}^*$, *defined by*

$$\langle t , \Psi_T\,\bar{t}^*\rangle = \langle \rho_T^\natural t , \bar{t}^*\rangle \quad \textit{for all} \quad t \in S .$$

Then there exists an R-*linear mapping* $\Lambda : (S/\rho_S) \longrightarrow (T/\rho_T)$ *such that the following diagrams are commutative* :

$$\begin{array}{ccc} S & \xrightarrow{\ \Theta\ } & T \\ {\scriptstyle \rho_S^\natural}\downarrow & & \downarrow{\scriptstyle \rho_T^\natural} \\ S/\rho_S & \xrightarrow[\ \Lambda\ ]{} & T/\rho_T \end{array} \qquad\qquad \begin{array}{ccc} S' & \xleftarrow{\ {}^t\Theta\ } & T' \\ {\scriptstyle \Psi_S}\uparrow & & \uparrow{\scriptstyle \Psi_T} \\ (S/\rho_S)' & \xleftarrow[\ {}^t\Lambda\ ]{} & (T/\rho_T)' \end{array}$$

Proof. - Since $\rho_T^\natural\,\Theta$ is an R-linear mapping of S into left R-module T/ρ_T, $(\rho_T^\natural\,\Theta)^{-1}(\rho_T^\natural\,\Theta)$ must be an R-stable group congruence on S, and, since ρ_S is the minimal group congruence on S, we must have $\rho_S \subseteq (\rho_T^\natural\,\Theta)^{-1}(\rho_T^\natural\,\Theta)$. This implies that Λ is a well-defined R-linear mapping of S/ρ_S into T/ρ_T. Λ is then an R-linear mapping of $(T/\rho_T)'$ into $(S/\rho_S)'$ which is defined by

$$\langle \rho_S^\natural x , {}^t\Lambda\bar{t}^*\rangle = \langle \Lambda\rho_S^\natural x , \bar{t}^*\rangle \quad \text{for all} \quad x \in S \text{, and all} \quad \bar{t}^* \in (T/\rho_T)' .$$

But since $\Lambda\rho_S^\natural = \rho_T^\natural\,\Theta$, we then have

$$\begin{aligned} \langle \rho_S^\natural x , {}^t\Lambda\bar{t}^*\rangle &= \langle \rho_T^\natural\,\Theta x , \bar{t}^*\rangle \\ &= \langle \Theta x , \Psi_T\,\bar{t}^*\rangle \\ &= \langle x , ({}^t\Theta\Psi_T)\bar{t}^*\rangle \\ &= \langle \rho_S^\natural x , (\Psi_S^{-1}\,{}^t\Theta\Psi_T)\bar{t}^*\rangle \end{aligned}$$

for all $x \in S$ and all $\bar{t}^* \in (T/\rho_T)'$, hence ${}^t\Lambda = \Psi_S^{-1}\,{}^t\Theta\Psi_T$.

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