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ON NOWHERE DENSE ϑ-SETS

by Andrew D. POLLINGTON (*)

In this note, we show that, for every $\vartheta > 1$, there are real numbers k so that $X_k = \{\widehat{k\vartheta^n} ; n = 1 , 2 , \ldots\}$ is nowhere dense. Here $\hat{x}$ denotes the image of x in R/Z. This answers a question raised by CHOQUET in [1], and is a straightforward corollary of the following theorem.

THEOREM. - *Given $\varepsilon > 0$, $\vartheta > 1$ and a sequence of real numbers a_n, there is a sequence ε_n of positive real numbers, and a set of real numbers K of Hausdorff dimension at least $\frac{1}{2} - \varepsilon$ so that, if $k \in K$, then*

(1) $$\|k\vartheta^n - a_m\| > \varepsilon_m \text{ for all } m , n \in N .$$

Proof. - Let $\frac{1}{2} - \varepsilon < s < \frac{1}{2}$, and choose r so that

(2) $$\vartheta^r - 4r > \vartheta^{rs} .$$

Put

(3) $$\varepsilon_n = \vartheta^{-2^{n+2}r} , \quad t = [\vartheta^r - 1] , \quad M = t - 4r .$$

We will construct a nested sequence of sets of intervals $I_0 \supset I_1 \supset \ldots$ so that $k \in \bigcap I_j$ satisfies (1). These sets I_n will be chosen so that, if I is an interval of I_n, then $|I| = \vartheta^{-rn}$. I_{2^n} will be a union of M^{2^n} intervals each containing M^{2^n} intervals of $I_{2^{n+1}}$.

The construction. - Put $I_0 = [0 , 1]$.

For each interval I of I_{n-1}, we divide I into t equally spaced subintervals, each of length $\vartheta^{-r} |I|$. We will delete some of these intervals, those remaining will form the set I_n. We will delete the intervals according to a rule depending on n.

Suppose $n = 2^p + u\, 2^{p+1}$, so $2^p \| n$. If $u = 0$, no intervals are deleted when forming I_n. If $u > 0$, we distinguish two cases :

(a) There is some integer q for which

$$n - 2^{p+1} < 2^q < n .$$

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(b) There is no such q .

We will choose the intervals I so that, if $k \in I_n$, then

(4) $$\|k \theta^m - a_p\| > \epsilon_p \ , \quad m = 1 , 2 , \dots , (n - 2^{p+1}) r \ .$$

Delete J from the choices for intervals of I_n if $J \cap L \neq \emptyset$, where

(5) $$L = \{k \ ; \ \|k \theta^m - a_p\| \leqslant \epsilon_p \ , \quad (n - 2^{p+2}) r + 1 \leqslant m \leqslant (n - 2^{p+1}) r \ .$$

Case a. - By (3) and (5), for every interval I of I_{2^q} , we delete at most $2^{p+2} r$ intervals contained in I .

Case b. - As above, for every interval I of $I_{n-2^{p+1}}$, we delete at most $2^{p+2} r$ intervals contained in I .

It now only remains to verify the condition concerning the number of intervals in I_{2^n} . Suppose that I is an interval of I_{2^n} . Then I contains at least M^w intervals of I_{2^n+w} , $0 \leqslant w \leqslant 2^n$.

Thus we may choose 2^n intervals of $I_{2^{n+1}}$ in every interval of I_{2^n} . Put $J_n = I_{2^n}$, $n = 1 , 2 , \dots$. By (2) and a theorem of EGGLESTON [2], the dimension of the set of numbers satisfying (1) is at least s .

COROLLARY. - Given $\theta > 1$, the set of real numbers k for which X_k is nowhere dense has Hausdorff dimension at least $\frac{1}{2}$.

Proof. - Apply the theorem with a_n any dense sequence modulo 1 .

REFERENCES

[1] CHOQUET (G.). - Construction effective de suites $(k(3/2)^n)$. Étude des mesures (3/2)-stables, C. R. Acad. Sc. Paris, t. 291, 1980, série A, p. 69-74.

[2] EGGLESTON (H. G.). - Sets of fractional dimension which occur in some problems of number theory, Proc. London math. Soc., Series 2, t. 54, 1951/52, p. 42-93.