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NON STANDARD ARITHMETIC AND APPLICATION TO HEIGHT FUNCTIONS

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Let K be a finite number field, and $^*K$ a non standard model of K (for example, take an ultra power of K modulo an ultra filter). $^*K$ is an extension field of K , which is elementarily equivalent with K , and which has the additional property that, for all left concurrent binary relations R , there is a $x \in {}^*K$ such that, for all $a \in K$, $R(a , x)$ is true. The arithmetic of K given by the set of places Σ_K of K induces a corresponding arithmetic of $^*K$ given by the set $^*\Sigma_K$ of (internal) valuations of $^*K$. The value groups of these valuations are contained in $^*\underset{\sim}{R}$, and by dividing out the isolated subgroup of finite elements of $^*\underset{\sim}{R}$, we get valuations $\dot{v} \in {}^{*\bullet}\Sigma_K$ with value group $^*\underset{\sim}{R}/\underset{\sim}{R}_{fin} = \dot{\underset{\sim}{R}}$. These valuations are external (and hence one can hope to get new informations about $^*K$), but they define an arithmetic that is not too far from the internal arithmetic. All valuations in $^{*\bullet}\Sigma_K$ are non archimedean and trivial on K . (For all this and application to the Siegel-Mahler finiteness theorem, cf. [5].)

As K is algebraically closed in $^*K$, the field $^*K$ together with the system of valuations $^{*\bullet}\Sigma_K$ behaves a little bit like a function field over K , and these analogies inspired E. KANI to define the Néron-Tate height on the $^*K$-rational points of Jacobians of curves $\mathcal{C}$ over K in a rather geometric way : If P_1 , P_2 are two different points of $\mathcal{C}(^*K)$, then KANI defines an intersection product $P_1 . P_2$ which is an element in the $^*\underset{\sim}{Z}$-module generated by the elements in $^*\Sigma_K$. The degree function maps this divisor into $^{*\bullet}\underset{\sim}{R}$, and this map can be extended to a mapping from the divisor class groups of $\mathcal{C}/{}^*K$ and is equivalent to the Néron-Tate height (cf. [3]).

This approach to height functions has the advantage that one can combine geometric and arithmetic methods. It seems to be possible to get results that are related to Mordell's conjecture stating the finiteness of K-rational points of curves of genus ≥ 2 .

As a first application, KANI gets a natural proof for Mumford's theorem about the growth of the height on K-rational points of such a curve in [4], and this can be used to reprove Manin-Demjanenko's theorem about K-rational points on $X_0(n)$ (modular curve belonging to elliptic curves with cyclic isogenies of degree n), if n

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has a lot of divisors. The tool is the study of arithmetic properties of Fricke involutions (cf. [2]). If one wants to replace $X_0(n)$ by twisted modular curves (for definition, cf. [1]), one has to study Hecke operators T_p. Using geometric methods like norm-conorm formulas, and adjunction formulas on the one hand, and the modular interpretation of points of these modular curve and the arithmetic of elliptic curves (Tate curve, complex multiplication) on the other hand, one can describe the arithmetic properties of T_p with respect to the Néron-Tate height, and using a version of Mumford's theorem one can prove :

THEOREM. - *If* $\mathcal{C}_0 \xleftarrow[X]{} \cdots \xleftarrow[X]{} \mathcal{C}_i \xleftarrow[X]{} \mathcal{C}_{i+1} \xleftarrow[X]{} \cdots$ *is a nontrivial tower of twisted modular curves over* K, *then there is a natural number* i_0 *such that Mordell's conjecture is true for all* $\mathcal{C}_i$ *with* $i \geqslant i_0$.

It follows from [1] that the same statement is true, if $\mathcal{C}_i$ are Fermat curves given by equations :

$$a_1 X^{n_i} - a_2 Y^{n_i} = a_3 Z^{n_i} \quad (n_i | n_{i+1}) \quad \text{and} \quad a_1 , a_2 , a_3 \in K^X .$$

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