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NUMBERS WITH ORDER $\leqslant 1$

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Let p be a prime integer, let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ for the p-adic absolute value $|.|$, let Ω_p be an algebraic closure of $\mathbb{Q}_p$ provided with the unique extension of the absolute value $|.|$, and let $\mathbb{C}_p$ be the (algebraically closed) completion of Ω_p .

In $\mathbb{C}_p$, we propose to study a notion of "transcendence order $\leqslant \alpha$ over $\mathbb{Q}_p$ " which looks a little like the transcendence type over $\mathbb{Q}$ ([2], [3], [4]).

Let $\|.\|$ be the Gauss norm on $\mathbb{Q}_p[X]$ defined by

$$\|\sum_{i=0}^{n} a_i X^i\| = \max_{0\leqslant i\leqslant n} |a_i| .$$

Let K be a transcendence degree 1 extension of $\mathbb{Q}_p$. Let $|.|$ be an extension to K of the absolute value defined on $\mathbb{Q}_p$, and let v be the valuation defined by $v(x) = -\log |x|$ where $\log$ is the logarithm function in base p . An element $x \in K$, transcendental over $\mathbb{Q}_p$, is said <u>to have order</u> $\leqslant \alpha$ $(\alpha > 0)$ if there exists a positive constant C_x such that every polynomial $P(X) \in \mathbb{Q}_p[X]$ satisfies

$$v(P(x)) \leqslant -\log \|P\| + C_x (\deg P)^\alpha .$$

No $x \in \mathbb{C}_p$ can have order $\leqslant \alpha$, if $\alpha < 1$, but with help of an example, we prove there exist some $x \in \mathbb{C}_p$ transcendental over $\mathbb{Q}_p$, with order $\leqslant 1$ (theorem 1). On the other hand, we know that order $\leqslant \alpha$ is stable by algebraic extension.

At the Oberwolfach meeting, we proved the existence of some $x \in \mathbb{C}_p$ with order $\leqslant 1 + \varepsilon$ whenever $\varepsilon > 0$. Here, we give a new kind of number, with order $\leqslant 1$.

THEOREM. - <u>There exist some numbers</u> $t \in \mathbb{C}_p$, <u>transcendental over</u> $\mathbb{Q}_p$, <u>of order</u> $\leqslant 1$.

In the proof of theorem, we will use the following notations.

For any finite set S , let $\operatorname{card}(S)$ be the number of elements of S .

Let H be an algebraic extension of $\mathbb{Q}_p$, and let $\operatorname{Aut}(H)$ be the group of the $\mathbb{Q}_p$-automorphisms in H . We know that every algebraic extension of $\mathbb{Q}_p$ is normal [1]

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hence it follows the classical result.

LEMMA. - Let H be an algebraic extension of $\mathbb{Q}_p$. Then $[H : \mathbb{Q}_p] = \mathrm{card}(\mathrm{Aut}(H))$

Proof of theorem. - Let (u_i), $i \in \mathbb{N}$, be a sequence in Ω_p such that $v(u_i)=0$ and $\mathbb{Q}_p[u_i]$ is the unramified extension of degree 2^i over $\mathbb{Q}_p$. Then we have

(1) $$v(u_i - \sigma(u_i)) = 0 \quad \text{for every } \mathbb{Q}_p\text{-automorphism } \sigma \text{ in } \Omega_p .$$

Let (f_i), $i \in \mathbb{N}$, be a strictly increasing sequence of positive integers such that the series $\sum_{i=1}^{\infty} f_i/2^i$ converges.

Set $a_i = u_i\, p^{f_i}$, and let $t = \sum_{i=1}^{\infty} a_i \in \mathbb{C}_p$. We are going to show that t has order ≤ 1.

Clearly, it is enough to prove the existence of some $C \in \mathbb{R}^+$ such that every irreducible polynomial $F(X) \in \mathbb{Q}_p[X]$, such that $\|F\| = 1$, satisfies

(2) $$v(F(t)) \leq C \deg F .$$

Consider such a polynomial F and set

$$F(X) = \lambda \prod_{i=1}^{d} (X - b_i) , \text{ where } b_1 , \dots , b_d \text{ are conjugate over } \mathbb{Q}_p .$$

We know that the $v(b_i)$, $1 \leq i \leq d$, are equal [1]. If the $v(b_i)$ are < 0, we have

$$v(F(t)) = v(\lambda) + d\, v(b_i) = - \log \|F\| = 0 .$$

So we can assume $v(b_i) \geq 0$ to prove the relation (2).

Then, by classical results [1], we have $v(\lambda) = 0$ hence we can assume that F is monic.

Now, by reordering the b_i, we can obviously assume that

$$v(t - b_1) \geq v(t - b_i) \quad \text{for every } i = 2 , \dots , d .$$

Set $b = b_1$ and let N be the unique integer such that

$$f_N < v(t - b) \leq f_{N+1} .$$

Write $b = a_1 + \dots + a_N + a'$; then $v(a') \geq v(t - b) > f_N$.

Let $L = \mathbb{Q}_p[a_1 , \dots , a_N , a']$. Obviously, $\mathbb{Q}_p[b] \subseteq L$.

We are going to prove $\mathbb{Q}_p[b] = L$. Let $\sigma \in \mathrm{Aut}(L)$ be different of the identical automorphism $\underset{\sim}{1}$ and let us prove that $\sigma(b) \neq b$. First suppose there exist integers $h \leq N$ such that $\sigma(a_h) \neq a_h$, and let i be the lowest of these integers h. By hypothesis (1), we have $v(\sigma(a_i) - a_i) = v(a_i) = f_i$. It follows that $v(\sigma(b)-b)=f_i$ since the f_N are strictly increasing, and $v(a') > f_N$. Similarly, suppose now i does not exist ; then

$$\sigma(b) - b = \sigma(a') - a' .$$

But necessarily $\sigma(a') \neq a'$ since $\sigma \neq 1$ and $\sigma(a_i) = a_i$, $\forall\, i = 1, \ldots, N$.

Thus we see that the $\sigma(b)$ are all different, hence card(Aut $\mathbb{Q}_p[b]$ $\geq$ card(Aut L) ; then by lemma 1, $[\mathbb{Q}_p[b] : \mathbb{Q}_p] \geq [L : \mathbb{Q}_p]$ and finally $\mathbb{Q}_p[b] = L$.

Now, for each $j = 1, \ldots, d$, b_j is some $\sigma(b)$ (with $\sigma \in \mathrm{Aut}\,\Omega_p$).

Then $b_j = \sigma(a_1) + \ldots + \sigma(a_N) + \sigma(a')$. Consider

$$b - b_j = \sum_{i=1}^{N} a_i - \sigma(a_i) + \sum_{i=N+1}^{\infty} a_i - \sigma(a') .$$

By (1), we have $v(a_i - \sigma(a_i)) = v(a_i) = f_i$, and

$$v(\sum_{i=N+1}^{\infty} a_i - \sigma(a')) > f_N \quad \text{since} \quad v(\sum_{N+1}^{\infty} a_i) \geq f_{N+1} \quad \text{and} \quad \sigma(a') > f_N .$$

Thus the $v(b - b_j)$ are necessarily in the form f_ℓ with $\ell \leq N$, when they are not strictly superior to f_N . Then it **is the same thing for the $v(t - b_j)$** .

For each integer $\ell \leq N$, let

$$E_\ell = \{b_j \ ; \ v(t - b_j) = f_\ell\} ;$$

then it follows card E_ℓ is equal to the number A_ℓ of the $\sigma \in \mathrm{Aut}(L)$ which fix $a_1, \ldots, a_{\ell-1}$. Then by definition of the a_n (from u_n), we see clearly that every $\sigma \in \mathrm{Aut}(L)$ which fixes a_i also fixes $a_1, \ldots, a_{i-1}$, hence $A_\ell = d/2^\ell$ and $\mathrm{card}(E_\ell) = d/2^\ell$. So, we have

$$\sum_{\substack{b_j \in F_\ell \\ 1<\ell<N}} v(t - b_j) = \sum_{i=1}^{N} \frac{df_i}{2^i}$$

Similarly the set $\tilde{E}_N = \{b_j \ ; \ v(t - b_j) > f_N\}$ has $d/2^N$ elements. Since, for any $j \leq d$, we have $v(t - b_j) \leq v(t - b_1) \leq f_{N+1}$, we see that

$$\sum_{b_j \in \tilde{E}_N} v(t - b_j) \leq \frac{df_{N+1}}{2^N} .$$

Finally

$$v(F(t)) = \sum_{j=1}^{d} v(t - b_j) \leq (\sum_{i=1}^{N} \frac{df_i}{2^i}) + \frac{df_{N+1}}{2^N} .$$

Set $C = \sum_{i=1}^{\infty} f_i/2^{i-1}$; with greater reason, $v(F(t)) \leq C\,d$.

Thus, every irreducible polynomial F such that $|F| = 1$ satisfies $v(F(t)) \leq C \deg F$, hence by preliminary remarks, theorem 1 is proved.

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