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Groupe de travail d'analyse ultramétrique, tome 9, nº 3 (1981-1982), exp. nº J5, p. J1-J10
<http://www.numdam.org/item?id=GAU_1981-1982__9_3_A6_0>

Groupe d'étude d'Analyse ultramétrique
(Y. AMICE, G. CHRISTOL, P. ROBBA)
9e année, 1981/82, fasc. 3, n° J5, 10 p.
Journée d'Analyse p-adique
[1982. Marseille-Luminy]

septembre 1982

A NOTE ON THE p-ADIC GAMMA FUNCTION

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Let K be a universal p-adic domain, i. e. K is an algebraically closed field of characteristic zero complete under a valuation extending the p-adic valuation of $\mathbb{Q}$. This valuation is normalized by $|p| = 1/p$, and is denoted additively by $\operatorname{ord} x = - \log |x|/\log p$. We assume $p \neq 2$. Let $U = \mathbb{Q} \cap \mathbb{Z}_p - \mathbb{Z}$. For r real positive, $D(z , r^-)$ denotes the open disk $\{x ; |x - z| < r\}$. We shall use $W_r(\mathbb{Z})$ to denote the union of all disks $\{D(z , r^-)\}$, $z \in \mathbb{Z}$. Clearly this union may be replaced by a finite disjoint union of some of the indicated disks. For

$$r \geq 1 , \quad W_r(\mathbb{Z}) = D(0 , r^-) .$$

We shall avoid the symbol $W_r(\mathbb{Z})$ with $r \geq 1$. For $s \in \mathbb{N}$, let $(x)_s$ denote the polynomial $\Pi(x + i)$ the product being over $i \in [0 , s - 1]$ (and hence $(x)_0 = 1$). For $s \in \mathbb{N}$, we use $\Gamma(s + x)/\Gamma(x)$ to denote $(x)_s$ and $\Gamma(x - s)/\Gamma(x)$ to denote $1/(x - s)_s$. Let $\pi \in K$, $\pi^{p-1} = (- p)$. Let $e = p^{-1} + (p - 1)^{-1}$, $\rho = p^{-e}$ (so $1 > \rho > 1/p$). A basis $\{u_i\}_{i\in I}$ of a Banach space will be said to be O. N. if $\|\sum x_i u_i\| = \sup |x_i|$.

Let θ denote the function $\theta(X) = \exp(\pi(X - X^p))$, which has been used [Dw 1] to give an analytic description of additive characters of finite fields. By comparison with the function $\exp((\pi X)^{p^2}/p^2)$, it is known that the Taylor expansion

$$\theta(X) = \sum_{n=0}^{\infty} c_n X^n \tag{1}$$

satisfies

$$\operatorname{ord} c_n \geq n(p - 1)/p^2 \tag{2}$$

$$n^{-1} \liminf \operatorname{ord} c_n = (p - 1)/p^2 \tag{2'}$$

$$\operatorname{ord} c_n \geq \frac{n}{p - 1} - 2\left[\frac{n}{p^2}\right] - \operatorname{ord} \left[\frac{n}{p^2}\right] . \tag{3}$$

We recall the Morita p-adic gamma function, Γ_p, defined on $\mathbb{Z}$ by the initial condition and functional equation

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(4)
$$\begin{cases} \Gamma_p(0) = 1 \\ \Gamma_p(1+x)/\Gamma_p(x) = \begin{cases} -1 & \text{if } |x| < 1 \\ -x & \text{if } |x| = 1 . \end{cases} \end{cases}$$

The function Γ_p is extended to $W_\rho(\underset{\sim}{Z})$ by local analyticity as will be recalled below.

The intimate relation between θ and Γ_p has been examined several times ([Boy], [DW 2], [DW 3], [Ba]). The object of this note is to review this work and to examine more closely the method of BARSKY.

For $y \in D(0, (p\rho)^-)$, $\mu \in \underset{\sim}{Z}$, we define

(5)
$$h_\mu(y) = \pi^{-\mu} \sum_{ps+\mu \geq 0} c_{ps+\mu}(-\pi)^{-s}\, \Gamma(y+s)/\Gamma(y) .$$

For $x \in W_\rho(\underset{\sim}{Z})$, $i \in \underset{\sim}{Z}$, let

(6)
$$g_i(x) = -\sum_{\ell=0}^{\infty} c_\ell\, \pi^{-\ell}\, \Gamma(-x+\ell+i)/\Gamma(-x) .$$

For $r \in [1/p, 1]$, $x \in W_r(\underset{\sim}{Z})$, it is known that $|(x)_s| \leqslant r^{[s/p]}$. This estimate together with (2) shows that aside from a possible finite set of poles at integral values of the argument if μ or i are negative, the function h_μ is analytic on $D(0, (p\rho)^-)$ and the function g_i is locally analytic of analyticity radius ρ on $W_\rho(\underset{\sim}{Z})$ (i. e. $g_i | D(z, \rho^-)$ is analytic for each $z \in \underset{\sim}{Z}$). The sums g_μ are by no means new. In lectures and articles since 1961, they have been associated with the calculation of Gauss sums.

For $x \in W_\rho(\underset{\sim}{Z})$, we define $\mathrm{Rep}(-x)$ to be element $\mu \in \{0, 1, \dots, p-1\}$ such that $|x + \mathrm{Rep}(-x)| < 1$. We then define $y \in D(0, (p\rho)^-)$ by the condition

(7)
$$x = -\mu + py .$$

As will again be explained below, with these definitions, we have

(8)
$$\Gamma_p(x) = h_\mu(y) .$$

This equation with $\mu = 0$ was used by BOYARSKY to show that $\Gamma_p | D(0, \rho^-)$ is an analytic function. The functional equation (4) then shows that Γ_p extends to a locally analytic function of analyticity radius ρ. Local analyticity with radius $|p|$ was known previously [Mo], but the improvement to ρ had not been previously reported.

The analyticity of Γ_p was subsequently studied by BARSKY using noncohomological methods. By his elementary methods one can show (cf. lemma 2 below), for $0 \leqslant i \leqslant p-1$,

$$(9)_i \qquad g_i(x) = \Gamma_p(1 + x)\cdot\chi_{D(i,\rho^-)}$$

where χ_A denotes the characteristic function of the subset A of K.

In particular, BARSKY examined the question of whether Γ_p has analyticity radius greater than ρ. Indeed, one may use either (3) or $(9)_0$ for this purpose. The point is that, for $r \geqslant 1$, the Banach space of bounded analytic functions on $D(0, r^-)$ have an O. N. basis deduced by normalization of the functions $\{(x)_s\}_{s\in\mathbb{N}}$ (cf. [Am]). Applying this to equation (8), we see that if Γ_p were to have analyticity radius greater than ρ then

$$\liminf_{s\to\infty} (ps + \mu)^{-1} \text{ ord } c_{ps+\mu} > (p - 1)/p^2$$

which according to (2') must be false for at least one $\mu \in \{0, 1, \dots, p - 1\}$.

For $r < 1$, the functions $\{(x)_s\}_{s\in\mathbb{N}}$ do <u>not</u> after normalization provide an O. N. basis for bounded analytic functions on $D(0, r^-)$. They do provide a basis [Am 1] for bounded locally analytic functions on $W_\rho(\mathbb{Z})$ with local analyticity radius ρ. Applying this, with $1 > r > \rho$, to Barsky's formula $(9)_0$, one again obtains a contradiction to (2'). (We here fill an omission of BARSKY, who neglected to evaluate g_0 on $D(i, \rho^-)$ for $i \not\equiv 0 \bmod p$. In the proof of his theorem 3, he put $x = py$, and incorrectly asserted $\{y \longrightarrow (py)_s\}_{s\in\mathbb{N}}$ to be a set of functions which after normalization provide an O. N. basis for the space of bounded analytic functions on $D(0, (p\rho)^-)$.) In this note, we explain $(9)_i$ by a simplified form of Barsky's method. We then show how it may be deduced cohomologically. We start by giving a rapid evaluation of the magnitude of $\Gamma_p(x)$ since this point has failed to recieve a careful explanation (cf. [Ba], theorem 3).

LEMMA 1. – $|\Gamma_p(x)| = 1$, $\forall x \in W_\rho(\mathbb{Z})$.

<u>Proof</u>. – We first observe that Γ_p has no zero in $W_\rho(\mathbb{Z})$ as if x_0 were a zero then, by (4), $x_0 + p^s$ would be a zero for each $s \in \mathbb{N}$ which, by analyticity on $D(x_0, \rho^-)$, would show that Γ_p is zero on $D(x_0, \rho^-)$, and then, by the functional equation Γ_p, would be zero or $D(0, \rho^-)$ contrary to the initial condition. If now $x_1 \in W_\rho(\mathbb{Z})$ then, by (4), there exists $i (= \text{Rep } x_1) \in D(x_1, \rho^-)$ such that $|\Gamma_p(i)| = 1$. If $|\Gamma_p(x_1)| \neq 1$, then, by a well known application of the newton polygon, Γ_p must have a zero in $D(x_1, \rho^-)$. This completes the proof of the lemma.

<u>Note</u>. – Alternate treatments use (2), or (3) together with either (8) or (9), to show $|\Gamma_p(x)| \leqslant 1$. This is combined with the duality relation

$$(10) \qquad \Gamma_p(x)\,\Gamma_p(1 - x) = -(-1)^{\text{Rep}(-x)}$$

to complete the alternate proof.

LEMME 2. – <u>For</u> $x \in W_\rho(\mathbb{Z})$, $0 \leqslant i \leqslant p$,

$$g_i(x) = \Gamma_p(1 + x) \cdot \chi_{D(i,\rho^-)} \cdot$$

<u>Proof</u> (Following BARSKY). – We show that, for $N \in \underset{\sim}{N}$,

(11)
$$g_i(N + i) = \begin{cases} 0 & \text{if } N \not\equiv 0 \mod p , \\ \Gamma_p(1 + N + i) & \text{if } N \equiv 0 \mod p . \end{cases}$$

The lemma then follows from the analyticity properties of the functions g_i (and indeed demonstrates that $\Gamma_p|\underset{\sim}{N}$ may be extended to a locally analytic function on $W_p(\underset{\sim}{Z})$ satisfying (4), the appeal to Mahler's theorem ([La], p. 82) in Lang's account of Barsky's method is quite superfluous).

By equation (1), replacing X by x/π,

(9)
$$\exp \frac{x^p}{p} = \exp(- x) \times \sum c_s \, x^s/\pi^s$$

and so comparing coefficients

$$\sum_{\ell+k=N} \frac{(-1)^\ell \, c_k}{\ell! \, \pi^k} = \begin{cases} 0 & \text{if } N \not\equiv 0(p) , \\ 1/(n! \, p^n) & \text{if } N = pn . \end{cases}$$

Multiplying by $(N + i)!$, we obtain

(12)
$$\sum_{\ell+k=N} (-1)^\ell \frac{(N + i)!}{\ell!} \frac{c_k}{\pi^k} = \begin{cases} 0 & \text{if } N \not\equiv (p) , \\ (pn + i)!/(n! \, p^n) & \text{if } N = pn . \end{cases}$$

The right side (12) is zero if $N \not\equiv 0$, and is $(-1)^{1+N+i} \Gamma_p(1 + N + i)$ if $N = pn$. On the other hand with $\ell + k = N$, we compute

$$(N + i)!/\ell! = (-1)^{k+i} (- N - i)_{k+i} = (-1)^{N+i-\ell} \, \Gamma(- N - i + k + i)/\Gamma(- N - i)$$

from which we recognize that the left side of (12) coincides with $(-1)^{N+i+1} g_i(N+i)$. This completes the proof of (11).

<u>Note</u>. – BARSKY stated ([Ba] equations (16), (25)]

$$\Gamma_p(1 + x) = g_0(x) + g_1(x) + \dots + g_{p-1}(x) , \quad \forall \, x \in W_p(\underset{\sim}{Z})$$

$$\Gamma_p(x) = g_0(x) , \quad \forall \, x \in D(0 , \rho^-)$$

Remark. – We have avoided the use of the Laplace transform since it seems to obscure the basic fact that $\exp x$ is the generating function of $1/\Gamma(1 + n)$ and that the purpose of equation (9) is to get the relations between $\Gamma(n)$ and $\Gamma([\frac{n}{p}])$, which indeed is approximately the role of $\Gamma_p(n)$.

In this regard, it may be useful to examine the connection between the Boyarsky matrix [Dw 3] for Bessel functions and the relation between the coefficients of the Laurent series

(13) $$\exp \frac{\lambda}{2}(t - \frac{1}{t}) = \sum_{n=-\infty}^{+\infty} J_n(\lambda)\ t^n ,$$

as deduced from

(14) $$\exp \frac{\lambda^p}{2^p\, p}(t^p - \frac{1}{t^p}) = \exp \frac{-\lambda}{2}(t - \frac{1}{t}).F ,$$

where $F(\lambda , t) = \theta_0(\frac{t\lambda}{2})\ \theta_0(-\frac{\lambda}{2t})$, $\theta_0(x) = \theta(x/\pi)$. Using estimate (2) and differentiating (13), one should be able by means of equation (14) to deduce relations between $(J_n(\lambda)$, $J'_n(\lambda)$ and $(J_{[n/p]}(\lambda^p)$, $J'_{[n/p]}(\lambda^p))$. This is our understanding of how Barsky's method should be interpreted and generalized.

We now give a cohomological explanation of equation (9). The underlying theory has discussed elsewhere ([Boy], [Dw 2], [Dw 3]) so we shall be brief.

For $a \in U = \underset{\sim}{Q} \cap \underset{\sim}{Z}_p - \underset{\sim}{Z}$, let Ω_a^0 denote the space of all products $\{X^a\, \xi\ ;\ \ \xi \in L_{0,\infty}\}$ where $L_{0,\infty}$ is the space of Laurent series converging in an annulus $\{X\ ;\ \epsilon_1 > |X| > \epsilon_2\}$, where ϵ_1 , ϵ_2 are unspecified real numbers $\epsilon_1 > 1 > \epsilon_2$. We define a differential operator D in Ω_a^0 by the formula

$$D(X^a\ \xi) = X^a(X \frac{d}{dX} + a + \pi X)\xi \ .$$

The factor space $\bar{\Omega}_a = \Omega_a^0 / D\Omega_a^0$ has dimension 1 with the image of X^a as a basis. The space Ω_a depends only upon a mod $\underset{\sim}{Z}$ but, for $m \in \underset{\sim}{Z}$, the image of X^{m+a} need not coincide with that of X^a , the relation being given by the change in basis formula

(15) $$X^{a+m} \equiv \frac{\Gamma(a + m)}{\Gamma(a)}\ (-\pi)^{-m}\ X^a \mod D\Omega_a^0 \ .$$

For $b \in U$, $pb \equiv a$ mod $\underset{\sim}{Z}$, we have the mapping α of Ω_a^0 into Ω_b^0 and a one side inverse β given by

$$\alpha\ :\ \ X^a\ \xi \longrightarrow X^b\ \psi(\xi X^{a-pb}\ \theta(X))$$

$$\beta\ :\ \ X^a\ \frac{1}{\theta(X)}\ X^{pb-a}\ \Phi\eta \longrightarrow X^b\ \eta$$

where Φ is the endomorphism $\eta(X) \longrightarrow \eta(X^p)$ of $L_{0,\infty}$ and ψ is the one-sided inverse defined by

$$(\psi\xi)(X) = p^{-1}\ \Sigma\ \xi(Y)$$

the sum being over all Y such that $Y^p = X$. From α and β , we deduce a pair of inverse mappings between $\bar{\Omega}_a$ and $\bar{\Omega}_b$. Letting $\gamma_p(a , b)$ denote the "matrix" (it is one by one) relative to the bases $\{X^a\}$, $\{X^b\}$ of the mapping induced by α , it follows from the definitions and the reduction formula (15) (with a **replaced** by b) that

$$\gamma_p(a\ ,\ b) = \pi^{pb-a}\ h_{pb-a}(b)\ . \tag{16}$$

A similar calculation for the matrix of the inverse mapping induced by β gives

$$(\gamma_p(a\ ,\ b))^{-1} = \Sigma_{s=0}\ (-\ 1)^s\ c_s(-\ \pi)^{-s-t}\ \Gamma(a + s + t)/\Gamma(a)\ , \tag{17}$$

where $t = pb - a$.

Furthermore using (15) as a change in basis formula, we obtain, for m , $n \in \underset{\sim}{Z}$,

$$\gamma_p(a + m\ ,\ b + n) = \gamma_p(a\ ,\ b)\ \frac{\gamma(a + m)}{\Gamma(a)}\ \frac{\Gamma(b)}{\gamma(b + n)}(-\ \pi)^{n-m}\ . \tag{18}$$

We now explain the connection with Γ_p . Up to this point, Γ_p is a function of two variables a , $b \in U$, restricted by the condition $pb - a = t \in \underset{\sim}{Z}$. We obtain a function Γ^B of one variable a , by insisting that $t = \mathrm{Rep}(-\ a) \in \{0,1,\dots,p-1\}$. We then define $(b = (a + \mathrm{Rep}(-\ a))\ p^{-1})$,

$$\Gamma^B(a) = \gamma_p(a\ ,\ b)\ \pi^{-\mathrm{Rep}(-a)}\ . \tag{19}$$

(The factor $\pi^{-\mathrm{Rep}(-a)}$ serves to make Γ^B defined over $\underset{\sim}{Q}_p$ instead of over $\underset{\sim}{Q}_p(\pi)$.) Using (13) and the definition, we check that Γ^B satisfies the same functional equation as Γ_p

$$\frac{\Gamma^B(a + 1)}{\Gamma^B(a)} = \begin{cases} -\ 1 & \text{if } |a| < 1\ , \\ -\ a & \text{if } |a| = 1\ . \end{cases} \tag{20}$$

From equation (16), we deduce

$$\Gamma^B(a) = h_{\mathrm{Rep}(-a)}(b)\ , \tag{21}$$

and so Γ^B may be extended analytically on $W_p(\underset{\sim}{Z})$ satisfying the initial condition and functional equation of Γ_p as given by equation (4). Thus $\Gamma^B = \Gamma_p$. We now deduce from (17) that, for $a \in U$,

$$\frac{1}{\Gamma_p(a)} = \pi^{\mathrm{rep}(-a)}/\gamma_p(a\ ,\ b) = (-\ 1)^t\ \sum_{s=0}^{\infty}\ c_s(a)_{s+t}\ \pi^{-s}\ , \tag{22}$$

where $t = \mathrm{Rep}(-\ a)$. Replacing a by $-\ a$, t by $\mathrm{Rep}(a)$, and using (10) in the form

$$\Gamma_p(-\ a)\ \Gamma_p(1 + a) = -\ (-\ 1)^{\mathrm{Rep}\ a}\ , \tag{23}$$

we deduce

$$\Gamma_p(1 + a) = g_{\mathrm{Rep}(a)}(a)\ . \tag{24}$$

This gives a cohomological explanation of $(9)_i$ for $x \in D(i\ ,\ \rho^-)$. The assertion that $g_i(a) = 0$ for $a \notin D(i, \rho^-)$ reduces to the assertion that, for $a \not\equiv 0$ mod p, we have

(25) $$X^a/\theta(X) \in DX^a L_{0,\infty} .$$

Since formally $D = (\exp \pi X)^{-1} \circ X \frac{d}{dX} \circ \exp \pi X$, it suffices to show that

$$X^a \exp \pi X^p \in X \frac{d}{dX}(X^a \exp \pi X L_{0,\infty}) ,$$

or, equivalently, that

(26) $$X^a \exp \pi X^p = X \frac{d}{dX}(X^a \exp \pi X^p \xi)$$

has a solution ξ in $L_{0,\infty}$. The solution is

(27) $$\xi = a^{-1} \sum_{j=0}^{\infty} (-\pi)^j X^{pj}/(\frac{a}{p} + 1)_j ,$$

which clearly lies in $L_{0,\infty}$.

This completes our cohomological treatment of lemma 2.

The emphasis in our construction of the Boyarsky function, I^B (cf. (19)) has been its characterization by means of the functional equation (20) which is deduced from the change of basis formulae. BARSKY's point of view was to characterize the g_i be evaluation at a sufficient number of elements of $\underset{\sim}{Z}$. We now show how this can b done cohomologically, i. e. by a scientifically acceptable form of manipulation of integral formulae.

We first recognize g_i as a formal Mellin transform. Let

$$\theta_0(X) = \theta(X/\pi) = \exp(X + \frac{X^p}{p}) .$$

For $a \in U$, we have formally by equation (6)

$$- g_i(- a) = (\int_0^\infty e^{-x} x^{i+a} \theta_0(x) \frac{dx}{x})/\int_0^\infty e^{-x} x^a dx/x .$$

More precisely, for $a \in U$, $g_i(- a)$ is specified by the condition

(28) $$- g_i(- a) x^a e^{-x} dx/x \equiv \theta_0(x) e^{-x} x^{i+a} dx/x \mod d(e^{-x} x^a \hat{L}_{0,\infty})$$

where $\hat{L}_{0,\infty}$ is the image of $L_{0,\infty}$ under the substitution $X \longrightarrow X/\pi$. This is just a rearrangement of our cohomological treatment of g_1 and is based upon $X^{a+1} e^{-X} dX/X \equiv aX^a e^{-X} dX/X$. Since, $g_i(- a)$ is defined for $a \in \underset{\sim}{N}$ we may use equation (28) for this calculation provided we are dealing with a one dimensional space and provided $\nu \in \underset{\sim}{N}$ implies that

(29) $$\nu X^\nu e^{-X} dX/X \equiv X^{\nu+1} e^{-X} dX/X .$$

The formula

(30) $$\Gamma(n) = \int_0^\infty x^n e^{-x} dx/x ,$$

in particular, $\int_0^\infty e^{-x} dx = 1$ reminds us that we must not consider $d(e^{-x})$ to be exact. (Letting $\sigma_\nu = t^{-\nu} e^t\, dt$, the Hankel formula $2\pi i/\Gamma(\nu) = \int_{-\infty}^{(0+)} \sigma_\nu$ does not help here as $\sigma_\nu = \nu\sigma_{\nu+1}$.) With this hint, we let $\hat{L}_\infty$ denote the space of power series in X which lie in $L_{C,\infty}$, and we work with the factor space

$$\hat{L}_\infty\, e^{-X}\, dX/d(X\hat{L}_\infty\, e^{-X}) .$$

Putting $\omega_n(X) = e^{-X} X^n\, dX/X$, we have

$$n\omega_n \equiv \omega_{n+1} , \quad \forall\, n \geqslant 1 ,$$

and so

(31) $$\omega_n \equiv \Gamma(n)\, \omega_1 \mod d(X\hat{L}_\infty\, e^{-X}) .$$

Equation (28) now takes the form $(n > 1)$,

(32) $$- g_i(-n) \equiv \omega_{i+n} \exp(X + \frac{X^p}{p}) .$$

The left side is $- g_i(-n)\, \Gamma(n)\, \omega_1(x)$. The right side is $X^{i+n} \exp(X^p/p)\, dX/X$ which, for $i + n \not\equiv 0 \mod p$, we show to be of the form $d(\xi \exp \frac{X^p}{p})$ with $\xi \in X\hat{L}_\infty$ (cf. equation (26)). We now restrict our attention to the case $n = pm - i$ $(m > 1 ,\ 0 < i < p)$. The right side of (32) may be written, letting $- z = X^p/p$, as $(-1)^m p^{m-1} z^m e^{-z}\, dz/z \equiv (-1)^m p^{m-1} \Gamma(m)\, \omega_1(z)$. Thus,

(33) $$- g_i(i - pm)\, \Gamma(pm - i)\, \omega_1(X) \equiv (-1)^m p^{m-1} \Gamma(m)\, \omega_1(z) .$$

We observe that $\theta_0(X) = e^{X-z}$, and so

(34) $$\omega_1(X) - \omega_1(z) = d((\theta_0(X) - 1)\, e^{-X}) ,$$

and the point is that $\theta_0(X) - 1 \subset X\, \hat{L}_\infty$. Thus

$$\frac{1}{g_i(i - pm)} = \frac{\Gamma(pm - i)}{\Gamma(m)} (-p)^{m-1} .$$

On the other hand, by (10)

$$\frac{1}{\Gamma_p(1 + i - pm)} = \Gamma_p(pm - i)(-1)^{i+1} = \frac{(pm - i - 1)!}{(m - 1)!\, p^{m-1}} (-1)^{pm+1} .$$

This shows that equation (11) may be verified by calculation of Mellin transforms.

We note that h_μ is also a Mellin transform. We leave the details to the reader.

We are reminded by Yvette AMICE [Am 2] that contrary to our impression when writing 21.4.10 in [Dw 2], most of the results concerning radii of convergence may be deduced directly from the original formulae of MORITA [Mo] and DIAMOND [Di]. They showed that, for $x \in p\,\underset{\sim}{Z}$, we have

(35) $$\log \Gamma_p(x) = \sum b_s\, x^s ,$$

where

$$b_1 = \lim_{k\to\infty} p^{-k} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^k - 1} \log a$$

$$b_s = (-1)^s s^{-1} L_p(s, \omega^{1-s}) \quad (s \geq 2) .$$

Here ω denotes the Teichmüller character and L_p the Kubota-Leopoldt L-function. Using elementary properties of L_p and of Bernoulli numbers, one finds, for $s \geq 2$,

$$- L_p(s, \omega^{1-s}) = \lim_{\ell \to \infty} (1 - p^{n-1}) B_n/n ,$$

where $n = 1 - s + (p-1) p^{\ell}$. In fact, one shows that, $a_1 \in \mathbb{Z}_p$,

$$(36) \quad \begin{cases} a_s = 0 \text{ if } s \equiv 0 \mod 2 \\ sa_s \in \mathbb{Z}_p \text{ if } s \not\equiv 1 \mod (p-1) \\ |ps(s-1)a_s| = 1 \text{ if } s \equiv 1 \mod (p-1). \end{cases}$$

As noted by AMICE, this is sufficient to show that $f(x) \overset{\text{def}}{=} \exp \sum b_s x^s$ is analytic for $\operatorname{ord} x > \rho = \frac{1}{p} + \frac{1}{p-1}$. Since $\Gamma_p(x) \equiv 1 \mod p$, for $x \in p\mathbb{Z}$, it follows that f is analytic for $\operatorname{ord} x > \rho$, and coincides with Γ_p on $p\mathbb{Z}$. This shows that Γ_p may be extended to a function analytic on the disk $\operatorname{ord} x > \rho$. This gives the correct lower bound for the radius of analyticity. It is not clear that the upper bound may be verified in this way. Of course, a second proof of lemma 1 may be immediately deduced.

It is well known that, for fixed $a \mod p-1$, the mappings $s \longrightarrow L_p(s, \omega^a)$ is analytic (or meromorphic) on the disk $D(0, |p/\pi|^-)$. One may be tempted to use this property to deduce the analytic continuation of the right side of (35) into the region $d(x, \mathbb{Z}_p^*) > |p/\pi|$. It is however better to use the fact that for x close to zero $\log \Gamma_p(x)$ coincides with Diamond's $G_p^*(x)$. Briefly, for $x \in \mathbb{Z}_p$ [Di], with $\ell(x) = x \log x - x$,

$$(36) \qquad G_p(x) = \lim_{k \to \infty} p^{-k} \sum_{n=0}^{p^k - 1} \ell(x+n)$$

and for $x \notin \mathbb{Z}_p^*$

$$G_p^*(x) = \lim_{k \to \infty} p^{-k} \sum_{n=0, p \nmid n}^{p^k - 1} \ell(x+n) .$$

Diamond's version of the Gauss multiplication formula gives, for $r \geq 1$,

$$(37) \qquad G_p(x) = \sum_{a=0}^{p^r - 1} G_p\left(\frac{x+a}{p^r}\right) ,$$

and hence, for $x \notin \mathbb{Z}_p^*$, we have

$$(38) \qquad G_p^*(x) = G_p(x) - G_p\left(\frac{x}{p}\right) = \sum_{a=1, p \nmid a}^{p^r} G_p\left(\frac{x+a}{p^r}\right) .$$

Thus if $d(x, \mathbb{Z}_p^*) > |p|^r$ by Diamond's Stirling formula for G_p, we have

$$(39) \qquad G_p^*(x) - \ell_r(x) = \sum_{s=1}^{\infty} B_s s^{-1} (s+1)^{-1} p^{rs} \sum_{a=1, p \nmid a}^{p^r - 1} (x+a)^{-s} ,$$

where B_s denotes the s-th Bernoulli number, and

$$(40) \qquad \ell_r = \sum_{a=1, p \nmid a}^{p^r-1} \{(\frac{x+a}{p^r} - \frac{1}{2}) \log(\frac{x+a}{p^r}) - \frac{x+a}{p^r}\}$$

log being the Iwasawa logarithm. These formulae reduce all questions of analyticity of G_p^* to question concerning ℓ_r. The analytic continuation of $\frac{d}{dx} G_p^*$ has been discussed by KOBLITZ [Ko], but his results and conjectures do not go beyond these earlier results of DIAMOND. In particular, it follows from equation (39) that, if $\alpha \in \underset{\sim}{Z}_p^*$, then $x \longmapsto G_p^{*\prime}(x) - G_p^*(\alpha x)$ is an analytic function (in the sense of KRASNER, naturally) on the set $K - \underset{\sim}{Z}_p^*$.

We observe that for the analysis of $\Gamma_p(x)$ for $x \in D(0, 1^-)$ along the lines of equation (35), it is better to use equation (39) with $r = 1$, recognize that the right side is bounded by $|p|$ for $|x| < 1$, and so reduce the analysis of $\Gamma_p(x)$ to that of $\exp \ell_1(x)$ for x close to zero. This procedure should again establish ρ as the precise radius of analyticity of Γ_p.

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