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C^∞-ANTIDERIVATIVES OF p-ADIC C^∞-FUNCTIONS

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The purpose of this note is to prove the following theorem (for the definition of a C^∞-function see below).

THEOREM. - *Let* K *be a complete non-archimedean valued field with characteristic zero. Let* X *be a nonempty subset of* K *without isolated points and let* $f : X \longrightarrow K$ *be a* C^∞-*function. Then there is a* C^∞-*function* $X \longrightarrow K$ *whose derivative is* f.

First we quote some definitions and statements from [1] which are needed for the proof. Let K and X be as above.

Definition ([1], p. 8 and 75). - Let $f : X \longrightarrow K$. f is *differentiable* if its *derivative* $a \longmapsto f'(a) := \lim_{x\to a}(x - a)^{-1} (f(x) - f(a))$ $(a \in X)$ exists. For $n \in \mathbb{N}$, let $\nabla^n X := \{(y_1 , y_2 , \ldots , y_n) \in X^n ; \ y_i \neq y_j$ whenever $i \neq j\}$. The *difference quotients* $\Phi_n f : \nabla^{n+1} X \longrightarrow K$ $(n \in \{0 , 1 , 2 , \ldots\})$ are given inductively by

$$\Phi_0 f := f$$

and

$$\Phi_n f(y_1 , y_2 , \ldots , y_{n+1})$$

$$:= (y_1 - y_2)^{-1}(\Phi_{n-1} f(y_1 , y_3 , \ldots , y_{n-1}) - \Phi_{n-1} f(y_2 , y_3 , \ldots , y_{n+1}))$$

$((y_1 , y_2 , \ldots , y_{n+1}) \in \nabla^{n+1} X , \ n \in \mathbb{N})$.

f is a C^n-*function* $(f \in C^n(X \longrightarrow K))$ if $\Phi_n f$ can (uniquely) be extended to a continuous function $\overline{\Phi}_n f : X^{n+1} \longrightarrow K$.

f is a C^∞-function if $f \in C^\infty(X \longrightarrow K) := \bigcap_{n=0}^{\infty} C^n(X \longrightarrow K)$.

PROPOSITION ([1], p. 78, 86, 87, 110 and 123). - *Let* $f : X \longrightarrow K$. *For each* $n \in \mathbb{N}$ *the function* $\Phi_n f$ *is symmetric,* $C^{n-1}(X \longrightarrow K) \supset C^n(X \longrightarrow K)$, *if* $f \in C^n(X \longrightarrow K)$ *then* $f' \in C^{n-1}(X \longrightarrow K)$ *and* $\overline{\Phi}_n f(a , a , \ldots , a) = f^{(n)}(a)/n!$ *for each* $a \in X$,

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<u>if</u> $\lim_{x,y\to a}(x-y)^{-n}(f(x)-f(y)) = 0$ <u>for each</u> $a \in X$ <u>then</u> $f \in C^n(X \longrightarrow K)$ <u>and</u> $f' = 0$. <u>(Locally) analytic functions are</u> C^∞<u>-functions</u>

<u>Definition</u> ([1], p. 45 and 46). – Let $0 < \rho < 1$. For each $n \in \underset{\sim}{N}$, let R_n be a full set of representatives in X of the equivalence relation given by $|x - y| < \rho^n$ $(x, y \in X)$ such that $R_1 \subset R_2 \subset \dots$ Choose $x_0 \in R_1$. For each $x \in X$, $n \in \underset{\sim}{N}$, let x_n be determined by the conditions $x_n \in R_n$, $|x - x_n| < \rho^n$.

PROPOSITION ([1] Th. 11.2). – <u>Let</u> $n \in \underset{\sim}{N}$, $f \in C^{n-1}(X \longrightarrow K)$. <u>Set</u>

$$P_n f(x) := \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{f^{(j)}(x)_m}{(j+1)!} (x_{m+1} - x_m)^{j+1} \quad (x \in X) .$$

<u>Then</u> $P_n f$ <u>is a</u> C^n<u>-antiderivative of</u> f.

<u>Proof of the theorem</u>. – We shall use the terminology of above.

Let $j \in \{0, 1, 2, \dots\}$. $f^{(j)}$ is continuous hence locally bounded and there exists a partition of X into "closed" balls B_{ji} (relative to X) of radius < 1 where i runs through some indexing set I_j such that $f^{(j)}$ is bounded on each B_{ji}. For each $i \in I_j$, we can choose $m_{ji} \in \underset{\sim}{N}$ such that (recall that $0 < \rho < 1$)

(*) $\qquad \rho^{m_{ji}} \leq d(B_{ji}) < 1, \quad |f^{(j)}(x)|\, \rho^{m_{ji}} < |(j+1)!|\, \rho^j \quad (x \in B_{ji})$.

Define $F_j : X \longrightarrow K$ as follows. If $x \in X$, then $x \in B_{ji}$ for precisely one $i \in I_j$. Set

$$F_j(x) := \sum_{m \geq m_{ji}} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} .$$

We shall prove that $F := \sum_{j=0}^{\infty} F_j$ is a C^∞-antiderivative of f by means of the following steps.

(i) Each F_j is well defined.

(ii) For each $j \in \{0, 1, 2, \dots\}$ and for all $i \in I_j$,

$$|F_j(x)| \leq \rho^{jm_{ji}+j} \quad (x \in B_{ji})$$

so that F is well defined.

(iii) $\sum_{j=0}^{n} F_j$ is a C^n-antiderivative of f for each $n \in \underset{\sim}{N}$.

(iv) For each n, $\sum_{j=n+1}^{\infty} F_j$ is a C^n-function with zero derivative.

<u>Proof of (i)</u>. – $f^{(j)}$ is bounded on B_{ji}, and $\lim_{m\to\infty} (x_{m+1} - x_m) = 0$.

<u>Proof of (ii)</u>. – Let $x \in B_{ji}$ and $m \geq m_{ji}$. Then by (*),

$$|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho^{m_{ji}} \leq d(B_{ji})$$

from which it follows that $x_m \in B_{ji}$ and $|x_{m+1} - x_m| \leq \rho^{m_{ji}}$. Applying the second formula of (*) with x replaced by x_m, we get

$$\left|\frac{f^{(j)}(x_m)}{(j+1)!}(x_{m+1} - x_m)^{j+1}\right| \leq \rho^j \rho^{-m_{ji}} \rho^{m_{ji}(j+1)} = \rho^{jm_{ji}+j} ,$$

and (ii) is proved.

Proof of (iii). - The function F_j and $x \longmapsto \sum_{m=0}^{\infty} f^{(j)}(x_m)(x_{m+1} - x_m)^{j+1}/(j+1)!$ differ (on each B_{ji}, hence globally) by a locally constant function. Summation from $j = 0$ to $j = n$ shows that $\sum_{j=0}^{n} F_j - P_{n+1} f$ is locally constant. By the second proposition

$$\sum_{j=0}^{n} F_j \in C^{n+1}(X \longrightarrow K) \subset C^n(X \longrightarrow K) \quad \text{and} \quad (\sum_{j=0}^{n} F_j)' = f .$$

Proof of (iv). - Set $H := \sum_{j=n+1}^{\infty} F_j$. We shall prove that $|H(x)-H(y)| \leq |x-y|^{n+1}$ for all $x , y \in X$ which, by the first proposition implies (iv). To obtain the inequality if suffices to prove

$$(**) \qquad |F_j(x) - F_j(y)| \leq |x - y|^{n+1} \quad (x , y \in X) \quad \text{for each } j \geq n + 1 .$$

We consider several cases.

(a) $x \in B_{ji}$, $y \in B_{ji'}$, where $i \neq i'$. Then by (*),

$$|x - y| \geq d(B_{ji}) \geq \rho^{m_{ji}} \quad \text{so that} \quad |x - y|^{n+1} \geq \rho^{m_{ji}(n+1)} .$$

By (ii),

$$|F_j(x)| \leq \rho^{jm_{ji}+j} .$$

As $jm_{ji} + j \geq (n + 1) m_{ji}$, we have $|F_j(x)| \leq |x - y|^{n+1}$. By symmetry, $|F_j(y)| \leq |x - y|^{n+1}$, and (**) follows.

(b) There is i such that $x , y \in B_{ji}$. We may assume $x \neq y$, there exists an $s \in \underset{\sim}{N} \cup \{0\}$ such that (recall that $d(B_{ji}) < 1$)

$$\rho^{s+1} \leq |x - y| < \rho^s .$$

Then $|x - y|^{n+1} \geq \rho^{(s+1)(n+1)}$. Consider two subcases.

(b.1) $s < m_{ji}$. Then by (ii),

$$|F_j(x)| \leq \rho^{jm_{ji}+j}$$

and since $jm_{ji} + j \geq (n + 1)(s + 1) + j \geq (s + 1)(n + 1)$, we have $|F_j(x)| \leq |x-y|^{n+1}$. By symmetry $|F_j(y)| \leq |x - y|^{n+1}$ and (**) follows.

(b.2) $s \geq m_{ji}$. Then since $x_0 = y_0 , \ldots , x_s = y_s$, we have, for $m=m_{ji},\ldots,s-1$,

$$\frac{f^{(j)}(x_m)}{(j+1)!}(x_{m+1}-x_m)^{j+1} = \frac{f^{(j)}(y_m)}{(j+1)!}(y_{m+1}-y_m)^{j+1}$$

so that

$$F_j(x) - F_j(y) = \sum_{m\geq s} \frac{f^{(j)}(x_m)}{(j+1)!}(x_{m+1}-x_m)^{j+1} - \sum_{m\geq s} \frac{f^{(j)}(y_m)}{(j+1)!}(y_{m+1}-y_m)^{j+1} .$$

If $m \geq s$, we have by (*) (observe that $x_m \in B_{ji}$)

$$\left|\frac{f^{(j)}(x_m)}{(j+1)!}(x_{m+1}-x_m)^{j+1}\right| \leq \rho^{j-m_{ji}+m(j+1)}$$

$$\left|\frac{f^{(j)}(y_m)}{(j+1)!}(y_{m+1}-y_m)^{j+1}\right| \leq \rho^{j-m_{ji}+m(j+1)}$$

and we find $|F_j(x) - F_j(y)| \leq \rho^{j-m_{ji}+s(j+1)}$. Using the fact that $j \geq n+1$ and our assumption $s \geq m_{ji}$, we obtain

$$j - m_{ji} + s(j+1) = (s+1)\,j + s - m_{ji} \geq (s+1)(n+1) .$$

By consequence

$$|F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x-y|^{n+1}$$

which finishes the proof.

Remark. - The above construction does not give us a linear antiderivation map $C^\infty(X \to K) \to C^\infty(X \to K)$, and it is somewhat doubtful whether there exists a linear antiderivation map $P : C^\infty(X \to K) \to C^\infty(X \to K)$ that is continuous with respect to a natural locally convex topology ([1], p. 119) on $C^\infty(X \to K)$.

REFERENCE

[1] SCHIKHOF (W. H.). - Non-archimedean calculus. - Mathematisch instituut, Nijmegen, 1978 (Lecture Notes. Report 7812).