

# GROUPE DE TRAVAIL D'ANALYSE ULTRAMÉTRIQUE

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## **Families of Mumford curves**

*Groupe de travail d'analyse ultramétrique*, tome 9, n° 3 (1981-1982), exp. n° J12, p. J1-J5

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FAMILIES OF MUMFORD CURVES

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In this lecture, we consider the following :

$k$  algebraically closed,  $p$ -adic field with residue field  $\tilde{k}$ .

$\pi : X \rightarrow S$  family of Mumford curves, i. e. a proper, flat,  $k$ -analytic morphism such that all fibres  $X_s := \pi^{-1}(s)$  are Mumford curves with genus  $g(X_s)$ .

0. General facts

0.1. The Euler-Poincaré characteristic  $EP(\mathcal{O}, \pi, s) = g(X_s)$  is locally constant on  $S$  in the sense of Grothendieck topologies. If  $S$  is connected, then  $g(X_s) = g$  is constant on  $S$ .

0.2. If  $g = g(X_s) \geq 2$ , the relative-tricanonical linebundle  $\omega_{X/S}^{\otimes 3}$  is very ample on all fibres. The direct image  $\xi := \pi_* \omega_{X/S}^{\otimes 3}$  is a holomorphic vectorbundle on  $S$  of rank  $5g - 5$  and gives an embedding of the family

$$\begin{array}{ccc} X & \xrightarrow{\quad\quad\quad} & P(\xi) \\ & \searrow & \swarrow \\ & S & \end{array}$$

If  $g = 1$ , one gets the same, if one takes  $\mathcal{O}(-3D)$  instead of  $\omega_{X/S}^{\otimes 3}$ , where  $D \subset X$  is a divisor, finite over  $S$ .

0.3. Now by the GAGA-theorem in the relative case one knows, that for all affinoid subdomains  $U = \text{Sp}A \subset S$  the restricted family  $\pi : X|U \rightarrow U$  is an algebraic morphism.

I. Uniformization of families

All fibres  $X_s$  in the family have an uniformization. Now one wants to get a

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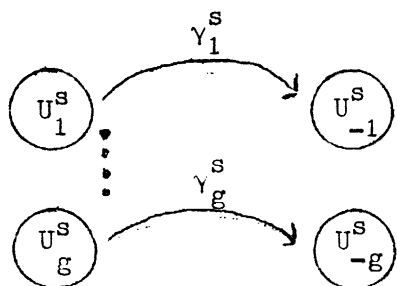
simultaneous uniformization of the family locally on the base  $S$ . To explain what I mean, I will repeat the definition of Mumford curve : In our situation, we know

$$X_s \simeq \Omega_s / \Gamma_s,$$

$$\Gamma_s = \langle \gamma_1^s, \dots, \gamma_g^s \rangle \subset \text{PGL}(2, k) \text{ free, discontinuous group,}$$

$$\Omega_s \subset \mathbb{P}^1 \text{ the set of ordinary points.}$$

Moreover one can assume that  $\gamma^s$  is a geometric base, i. e. there are closed balls  $U_i^s$  which are pairwise disjoint such that  $\gamma_i^s(\mathbb{P}^1 - U_i^s) = \dot{U}_{-i}^s$  for  $i = 1, \dots, g$ .



$$U_i^s = \{z \in \mathbb{P}^1 ; |z - a_i^s| \leq |r_i^s|\}$$

$$U_i^s = \{z \in \mathbb{P}^1 ; |z - a_i^s| < |r_i^s|\}$$

"Simultaneous Uniformization" means that all  $\gamma_i^s, a_i^s, r_i^s$  depend holomorphically on  $s \in S$ .

THEOREM I. - Let  $S$  be a reduced affinoid space,  $\pi : X \rightarrow S$  an analytic family of Mumford curves of genus  $g$  (with  $g + 1 < p = \text{char } \tilde{k}$  or  $0 = \text{char } \tilde{k}$ ), then there is a finite base extension  $S' \rightarrow S$  and a finite covering  $\{S'_1, \dots, S'_r\}$  of  $S'$  by affinoids  $S'_i$ , such that  $X'_i := X \times_S S'_i$  have simultaneous uniformizations, i. e. :

$$\mathbb{P}_{S'_i}^1 \supset \Omega'_i \rightarrow X'_i \text{ } S'_i\text{-morphism,}$$

$$\gamma^i : S'_i \rightarrow (\text{PGL}(2, k))^g \text{ holomorphic,}$$

such that

$\Gamma'_i := \langle \gamma^i \rangle$  act holomorphically, discontinuously on  $\Omega'_i$  by the canonical action  
 $\mathbb{P}_{S'_i}^1$ ,

$$\Omega'_i / \Gamma'_i \simeq X'_i \text{ isomorphic over } S'_i,$$

$$\gamma^i(s) \text{ is a geometric base for } \Gamma'_i(s).$$

To prove this theorem, one makes the following steps :

I.1. There is a finite base extension  $S' \rightarrow S$ , a finite covering  $\{S'_1, \dots, S'_r\}$  of  $S'$  by affinoids  $S'_i$ , and a finite covering  $\{X'_{ij}\}_j$  von  $S'_i$  with

$$X'_{ik} \simeq \{(s, z) \in S'_i \times \tilde{P}^1; |z - a_{ik}^0(s)|^{m_0} \leq |r_{ik}^0(s)|,\$$

$$|z - a_{ik}^v(s)|^{m_v} \leq |r_{ik}^v(s)| \text{ for } 1 \leq v \leq n_{ik}\}$$

where  $m_v \geq 1$  are natural members, and  $a_{ik}^v, r_{ik}^v \in \mathcal{O}(S'_i)$  with

$$|a_{ik}^v(s) - a_{ik}^0(s)|^{m_0} < |r_{ik}^0(s)| \text{ for } 1 \leq v \leq n_{ik}$$

$$|a_{ik}^v(s) - a_{ik}^\mu(s)|^{m_v} > |r_{ik}^v(s)| \text{ for } 1 \leq v \neq \mu \leq n_{ik}$$

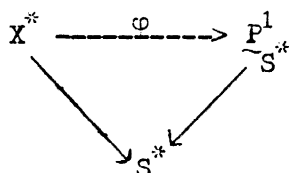
$$0 < |r_{ik}^v(s)|^{m_0} < |r_{ik}^0(s)|^{m_v} \text{ for } 1 \leq v \leq n_{ik}$$

For the components  $U_v$  of  $X_{ik} \cap X_{i\ell}$  one has :

$$U_v \simeq \{(s, z) \in S'_i \times \tilde{P}^1; |r_v^-(s)| \leq |z - a_v(s)|^{m_v} \leq |r_v^+(s)|\}$$

where  $r_v^+, r_v^-, a_v \in \mathcal{O}(S'_i)$  satisfy  $0 < |r_v^-(s)| < |r_v^+(s)|$  for all  $s \in S'_i$ .

Sketch of proof : After finite base extension  $S^* \rightarrow S$  one finds a section  $\sigma$  to  $\pi^* := \pi \times_S S^*$ . Then by the theorem of Riemann-Roch, one finds a finite morphism



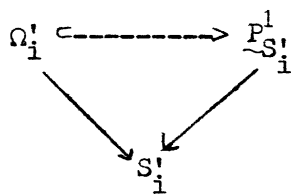
with  $\deg \varphi \leq g + 1 < p$ .

Now after finite base extension  $S^+ \rightarrow S^*$  the ramification set of  $\varphi \times_{S^*} S^+$  "essentially" splits into divisors of degree 1. So there are holomorphic sections  $\tau_i : S^+ \rightarrow \tilde{P}^1_{S^+}$  such that

$$\text{Ramif}(\varphi \times_{S^*} S^+) = \tau_1 S^+ \cup \dots \cup \tau_r S^+.$$

Then one constructs the coverings  $\{S'_i\}$  of  $S'$  and  $\{X'_{ij}\}$  of  $X'_i$  by using these sections  $\{\tau_\rho\}$  and by making a further finite base extension  $S' \rightarrow S^+$ . The main tool here is the method of constructing a stable reduction of an algebraic curve  $M$ , which admits a morphism  $\psi : M \rightarrow \tilde{P}^1$  with  $\deg \psi < p$ , by the ramification points of  $\psi$ .

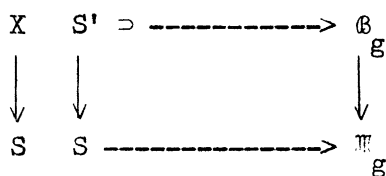
I.2. By the covering  $\{X'_{ij}\}_j$  one constructs a  $k$ -analytic  $S'_i$ -morphism  $\Omega'_i \rightarrow X'_i$  such that  $\Omega'_i(s)$  is the universal covering of  $X'_i(s)$  for all  $s \in S'_i$ . Next one proves an embedding theorem



By this embedding the group  $\text{Deck}(\Omega'_i/X'_i)$  can be regarded as a subgroup of  $\text{PGL}(2, \mathcal{O}(S'_i))$ . From this, one easily gets the theorem.

COROLLARY. - Let  $\mathcal{B}_g$  be the space of geometric bases for Mumford curves,  $\pi_g$  the space of Mumford curves and  $g \geq 2$ .

For every family  $X \rightarrow S$  of Mumford curves of genus  $g \geq 2$ , the canonical map  $S \rightarrow \pi_g$  admits locally a holomorphic lifting after finite base extension

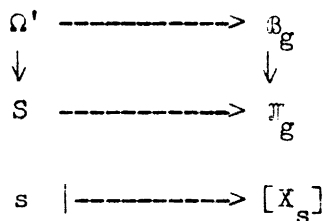


## II. Rigidity of algebraic families

An algebraic family of Mumford curves  $\pi : X \rightarrow S$  is a proper, flat, algebraic morphism where  $S$  is a connected scheme of finite type over  $k$ , such that all fibres  $X_s$  are Mumford curves. Let  $g$  be the genus of the fibres.

THEOREM II. - Every algebraic family of Mumford curves is constant, i. e. the map  $S \rightarrow \pi_g, s \mapsto [X_s]$ , is constant (if  $g + 1 < p = \text{char } \tilde{k}$  or  $0 = \text{char } k$ ).

Idea of the proof. - It is enough to consider affine, irreducible, algebraic curves for  $S$ . Similar as in § I, one can show, that there is a finite extension  $S' \rightarrow S$  by an algebraic curve such that  $X' = X \times_S S'$  has locally on  $S'$  a simultaneous uniformization. If  $\Omega' \rightarrow S'$  is the universal covering, then the canonical map can be lifted



Now  $\mathcal{B}_g$  is bounded in some sense, then  $\Omega' \rightarrow \mathcal{B}_g$  has to be constant by the :

PROPOSITION. - Bounded holomorphic functions on the universal covering of an algebraic curve are constant.

COROLLARY. - Analytic families  $X \rightarrow S$  over an connected, complete, algebraic curve are constant (GAGA).

Remarks.

(1) All considerations are made only for  $g \geq 2$ , because the case  $g = 0$  is trivial and, in the case  $g = 1$ , one gets rigidity for analytic families over connected, affine, algebraic curves by considering the  $j$ -invariants of the fibres as holomorphic function on the base.

(2) The assumption concerning the genus " $g + 1 < \text{char } \tilde{k}$ " should not be necessary, like a new research by the author has shown.

(3) Theorem II should be true for analytic families of Mumford curves over connected schemes of finite type over  $k$  as in the case  $g = 1$ .

This lecture is a very short version of my paper "Ein globaler Starrheitssatz für Mumford Kurven" which will come out in "Journal für reine und angewandte Mathematik" in the next months.

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