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p-ADIC SIEGEL HALFSPACE

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Results about function theory on the Siegel halfspace H_n over an ultrametric field are given. It is proved that H_n is a Stein domain. Expansions for the analytic functions on H_n are obtained.

(1) Let K be field together with a multiplicative valuation $|\ |$. Denote by $H_n(K)$ the set of all symmetric $n \times n$ matrices $x = (x_{ij})$ whose entries $x_{ij} \in K_* := K - \{0\}$ and for which the associated real symmetric matrix $(-\log |x_{ij}|)$ is positive definite.

Example. - $K = \underset{\sim}{C}$ = field of complex numbers together with the usual absolute value. Let σ_n be the classical Siegel halfspace of all symmetric $n \times n$ matrices $z = (z_{ij})$ whose entries $z_{ij} \in \underset{\sim}{C}$ and for which the associated matrix $\operatorname{Im} z := (\operatorname{Im} z_{ij})$ is positive definite where $\operatorname{Im} z_{ij}$ is the imaginary part of z_{ij}, (see for instance [5], chapter I, § 6, p. 24).

Consider the mapping $e : \sigma_n \longrightarrow H_n$ given by $e(z_{ij}) := (\exp 2\pi \sqrt{-1}\, z_{ij})$. As

$$|\exp 2\pi \sqrt{-1}\,(\operatorname{Re} z_{ij} + \sqrt{-1}\ \operatorname{Im} z_{ij})| = \exp(-2\pi \operatorname{Im} z_{ij})$$

and

$$-\log |\exp 2\pi \sqrt{-1}\ z_{ij}| = -\log \exp(-2\pi \operatorname{Im} z_{ij}) = 2\pi \operatorname{Im} z_{ij} ,$$

we get that a symmetric matrix $z = z_{ij}$ is in σ_n if, and only if, $e(z) \in H_n(\underset{\sim}{C})$.

Moreover $e(z) = e(z')$ if, and only if, $z - z'$ has entries $\in \underset{\sim}{Z}$.

Thus we see that $H_n(\underset{\sim}{C}) = \sigma_n \bmod T_n$, where T_n is the group of all integral translations $z \longrightarrow t + z$ where $t = (t_{ij})$ is symmetrix, and all entries $t_{ij} \in \underset{\sim}{Z}$.

Remark. - Assume that K is complete. Let $x \in H_n(K)$. The multiplicative subgroup of K_*^n = n-fold product of the multiplicative group K_* generated by the columns of x is denoted by Λ_x.

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Λ_x is a lattice in K_*^n, and the quotient K_*^n/Λ_n is an analytic torus and an abelian variety over K (see i. e. [2], (VI 1.3) and (VI 6.1)).

x also determines a polarization given by the zeroes of the principal theta function

$$\theta(z_1 , \dots , z_n) = \theta(z) := \sum_{(k_1,\dots,k_n)\in \underset{\sim}{Z}^n} x[k]\, z_1^{2k_1} \dots z_n^{2k_n}$$

where

$$x[k] := \prod_{i,j=1}^{n} x_{ij}^{k_i k_j} .$$

Thus x determines a polarized abelian variety A_x over K.

The canonical projection $H_n(K) \times (K_*^n/\Lambda_x) \longrightarrow H_n(K)$ gives an analytic family of polarized abelian varieties.

$\underset{\sim}{(2)}$ Let $x = (x_{ij})$ be a $m \times n$ matrix with entries $x_{ij} \in K_*$, and $a = (a_{ij})$ be $n \times r$ matrix with entries $a_{ij} \in \underset{\sim}{Z}$.

We define

$$x^a := (y_{ij}) \text{ by } y_{ij} := \prod_{k=1}^{n} x_{ik}^{a_{kj}} .$$

x^a is a $m \times r$ matrix with entries $\in K_*$.

If $x = (x_{ij})$ is a $n \times r$ matrix with entries $x_{ij} \in K_*$, and $a = (a_{ij})$ is a $m \times n$ matrix with $a_{ij} \in \underset{\sim}{Z}$, we define

$${}^a x := (z_{ij}) \text{ by } z_{ij} := \prod_{k=1}^{n} x_{kj}^{a_{ik}} .$$

${}^a x$ is a $m \times r$ matrix with entries $\in K_*$.

All formal rules of matrix manipulations hold also for these products. Especially the set $K_*^{n\times n}$ of all $n \times n$ matrices with entries in K_* is a left and a right module over the ring $\underset{\sim}{Z}^{n\times n}$ of all integral $n \times n$ matrices, and these two actions are compatible which means $({}^a x)^b = {}^a(x^b)$.

Denote by $S_n(K)$ the set of all symmetric $n \times n$ matrices $n = (x_{ij})$ with $x_{ij} \in K_*$. We consider $S_n(K)$ as a K-algebraic torus by identifying as usual $S_n(K)$ with $K_*^{n(n+1)/2}$. For any $a \in \underset{\sim}{Z}^{n\times n}$ denote by ϕ_a the mapping $S_n(K) \longrightarrow S_n(K)$ given by $\phi_a(x) := {}^{a^t} x^a$ where a^t is the transposed matrix of a. We obtain that ϕ_a is an algebraic finite covering of degree $|\det a|^{n+1}$ if $\det a \neq 0$ and that $\phi_a(H_n) \subseteq H_n$.

As $\phi_a \circ \phi_b = \phi_{ab}$ and $\phi_a = \phi_b$ if, and only if, $a = \pm b$, we get that $\Gamma_n := \{\phi_a ; a \in GL_n(\underset{\sim}{Z})\}$ is a transformation group on $S_n(K)$ isomorphic to $PGL_n(\underset{\sim}{Z})$.

<u>Remark</u>. – Let $x, x' \in H_n(K)$ and K be ultrametric. Then A_x is isomorphic to

$A_{x'}$ as polarized abelian varieties if, and only if, there exists $\Phi \in \Gamma_n$ such that $\Phi(x) = x'$.

This results is not true for the complex field $\underset{\sim}{C}$ (see [5], chapter III, § 6). It can be proved with the help of the lifting theorem in [3].

Thus we see that the orbit space $H_n(K)/\Gamma_n$ is a subset of the moduli space of all polarized abelian varieties. This motivates the following definitions.

Definition. – Let K be ultrametric and complete. $H_n(K)$ is called the Siegel halfspace over K, and the transformation group Γ_n on $H_n(K)$ is called the Siegel modular group.

(3) A K-valued function $f(x)$ on $H_n(K)$ is called K-analytic if the restriction of f onto any K-affinoid polyhedron P of $K_*^{n(n+1)/2}$ which is contained in $H_n(K)$ is analytic.

It means for K algebraically closed that f can uniformly on P be approximated by rational functions on $K_*^{n(n+1)/2}$ without poles on P.

In order to determine the analytic functions on $H_n(K)$, we introduce

$$M := \{k = (k_{ij}) \; ; \; k \text{ is } n \times n \text{ matrix} \; ; \; k_{ij} = k_{ji} = k_{ji} \in \tfrac{1}{2}\underset{\sim}{Z} \; ; \; k_{ii} \in \underset{\sim}{Z}\}$$

$$\langle x , k \rangle := \prod_{i,j=1}^{n} x_{ij}^{k_{ij}} = \prod_{i=1}^{n} x_{ii}^{k_{ii}} .$$

$\prod_{i<j} x_{ij}^{2k_{ij}}$ is a monomial in the variables $x_{11}, \ldots, x_{1n}, x_{22}, \ldots, x_{nn}$.

PROPOSITION 1. – The algebra of K-analytic functions on $H_n(K)$ coincides with the algebra of Laurent series

$$f(x) = \sum_{k \in M} c_k \langle x , k \rangle , \quad c_k \in K ,$$

which converge on all of $H_n(K)$.

Proof. – H_n is a connected Reinhardt domain (see [4], def. 1.8). For any $x^0 \in H_n$ one finds $\rho_{ij} < \rho'_{ij}$ $(\in |K_*|)$ such that the polyhedron

$$P := \{x \in H_n(K) \; ; \; \rho_{ij} \leqslant |x_{ij}| \leqslant \sigma'_{ij}\}$$

is contained in $H_n(K)$ and such that $x^0 \in P$.

Now P is the product of ring domains. One knows that any analytic function $f(x)$ on P has a Laurent expansion $\sum_{k \in M} c_k \langle x , k \rangle$. The coefficients c_k can not depend on P which gives the result.

COROLLARY. – $f(x) = \sum_{k \in M} c_k \langle x , k \rangle$ is Γ_n-invariant if, and only if, $c_k = c'_k$

<u>whenever</u> $k' = a^t ka$ <u>with</u> $a \in GL_n(\underset{\sim}{Z})$.

<u>Proof.</u> - $f({}^{a^t}x^{a}) = \sum_{k \in M} c_k \langle {}^{a^t}x^{a} , k \rangle$. Now

$$\langle x , k \rangle = tr(x^{k^t}) = tr({}^{k^t}x) \quad \text{where} \quad tr\, x := \prod_{i=1}^{n} x_{ii} \ .$$

Thus

$$\langle {}^{a^t}x^{a} , k \rangle = tr({}^{a^t}x^{ak^t}) = \langle {}^{a^t}x , ka^t \rangle = tr({}^{ak^t a^t}x) = \langle x , aka^t \rangle \ .$$

Thus

$$\sum c_k \langle {}^{a^t}x^{a} , k \rangle = \sum c_k \langle x , aka^t \rangle \ ,$$

which proves the corollary.

For $m \in M$, we denote by $\mathcal{O}_m$ the integral orthogonal group with respect to the quadratic form m . This means

$$\mathcal{O}_m = \{a \in \Gamma \ ; \quad a^t\, ma = m\} \ .$$

Let

$$\theta_m(x) := \sum_{a \in \mathcal{O}_m \backslash \Gamma} \langle x , a^t\, ma \rangle \ .$$

It is a formal Laurent series in the variables x_{ij} . Remark that for any representative $a' \in \mathcal{O}_m\, a$ one gets $a^t\, ma = (a')^t\, ma'$ because if $a' = b.a$, $b \in \mathcal{O}_m$, then

$$(ba)^t\, mba = a^t\, b^t\, ma = a^t\, ma \ .$$

Also if $a^t\, ma = (a')^t\, ma'$, then $a' \in \mathcal{O}_a$ because

$$(a'\, a^{-1})^t\, ma'\, a^{-1} = (a^t)^{-1}(a')^t\, ma'\, a^{-1} = (a^t)^{-1}\, a^t\, maa^{-1} = m \ .$$

This shows that each coefficient of the Laurent series has either the value 1 or the value 0 . In the complex case, one part of the following proposition is known as the theorem of Koecker (see [1], théorème 1).

PROPOSITION 2. - $\theta_m(x)$ <u>is an analytic function on</u> $H_n(K)$ <u>if, and only if,</u> m <u>is positiv semi-definite</u>.

<u>Proof.</u> - Let $s = \{ s \in M \ ; \ s \text{ positive semi-definite}\}$.

Let $x \in H_n(K)$ and $v := (-\log |x_{ij}|) =: (v_{ij})$. We will show that, for any given $\rho > 0$, one gets $\langle v , s \rangle \geq \rho$ for almost all s .

There is a real orthogonal matrix b such that $b^t\, vb = \lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ is a diagonal matrix. As v is positive definite all $\lambda_i > 0$.

Let $\lambda_1 \leq \lambda_i$ for all i .

Now

$\langle v , s\rangle = tr(v^t . s) = tr(b^{-1} vbb^{-1} sb) = tr(b^t vb . b^{-1} sb) = \langle \lambda , b^{-1} sb\rangle$, as $b^t = b^{-1}$.

Let $S' = \{b^{-1} sb ; s \in S\}$, and S'_r all matrices from S' whose entries have absolute value $\leq r$.

Then S'_r is finite, and if $t = (t_{ij}) \in S'$, $\notin S'_r$ then there is an i with $t_{ii} > r$. Because if $|t_{12}| > r$, $t_{11} \leq r$, $t_{22} \leq r$, then t is not positive semi-definite as

$$(1 , \pm 1 , 0 , \dots , 0) \times t \times \begin{pmatrix} 1 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = t_{11} + t_{22} \pm 2t_{12} < 0$$

for + or − . This means that

$$\langle \lambda , t\rangle \geq r . \lambda_1 , \text{ for any } t \in S' , t \in S'_r .$$

From this one gets that $\sum_{a \in S} \langle x , a\rangle$ is convergent on $H_n(K)$ as well as that any $\theta_s(x)$, $s \in S$, is analytic on $H_n(K)$.

The convers can be proved as in the complex case (see [1], p. 4–04).

Let $\bar{S} := S/\Gamma_n$. One gets $\theta_s(x) = \theta_{s'}(x)$ if s' is in the Γ_n-orbit of s which means that we can write $\theta_{\bar{s}}(x)$ instead of $\theta_s(x)$.

COROLLARY. – Let $f(x)$ be an analytic modular (= Γ_n-invariant) function on $H_n(K)$. Then $f(x)$ has an expansion

$$f(x) = \sum_{\sigma \in \bar{S}} c_\sigma \theta_\sigma(x) \quad \text{with} \quad c_\sigma \in K .$$

Example. – Let $s = (s_{ij})$ be given by $s_{ij} = 0$ for all $(i , j) \neq (1 , 1)$, and $s_{11} = 1$. Then

$$\theta_s(x) = \sum_{k \in \mathbb{Z}^n} x[k] \quad \text{where} \quad x[k] = \prod_{i,j=1}^n x_{ij}^{k_i k_j} .$$

Problem. – Determine the coefficients of the powers of the modular function $\sum_{\sigma \in S} \theta_\sigma(x) = \sum_{s \in S} \langle x , a\rangle$.

(4) For any $\rho > 0$, define

$$H_n(\rho) := \{x \in \mathbb{S}_n ; |x[k]| \leq \rho^{\|k\|^2} \text{ for all } k \in \mathbb{Z}^n\}$$

where $\|k\| = (\sum_{i=1}^n k_i^2)^{1/2}$ is the euclidean norm of k .

Then $H_n = \bigcup_{\rho>0} H_n(\rho)$.

Proof. – Let $x \in H_n$ and $v := (-\log |x_{ij}|)$. The function $f(y) := y^t vy$ for $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ is positive for $y \neq 0$.

As $S_{n-1} = \{y \in \underset{\sim}{R}^n ; \ \|y\| = 1\}$ is compact, there is a constant $\rho > 0$ such that $f(y) \geq \rho$ for all $y \in S_{n-1}$. But $f(y) = \|y\|^2 f(y/\|y\|)$ which shows that $s \in H_n(\rho)$.

LEMMA. - Given $0 < \varepsilon < 1$, $0 < \rho < \rho' < 1$. There exists an r which depends on ε, ρ, ρ', such that

$$X_r(\rho , \varepsilon) := \{x \in \mathfrak{S}_n ; \ \varepsilon \leq |x_{ij}| \leq \varepsilon^{-1} \ \text{for all} \ i , j$$

and

$$x[k] \leq \rho^{\|k\|^2} \ \text{for all} \ k = (k_1 , \dots , k_n) \in \underset{\sim}{Z}^n \ \text{with} \ |k_i| \leq r\}$$

is contained in $H_n(\rho') \subseteq H_n$.

Proof. - Assume the lemma is not true. Then we find for any r a matrix $x^{(r)} \in x_r(\rho , \varepsilon)$ such that $X^{(r)} \notin H_n(\rho')$. Let $v_r := (- \log |x_{ij}^{(r)}|)$. The entries of v_r are bounded by $\log \varepsilon^{-1}$. We thus get a point of accumulation v^* of the sequence (v_r) which is again a symmetric $n \times n$ matrix which satisfies

$$k^t \ v^* \ k \geq C.\|k\|^2 ,$$

where $C = - \log \rho$, for all $k \in \underset{\sim}{Z}^n$ because $k^t \ v^* \ k$ is a point of accumulation of the sequence $(k^t \ v_r \ k)$, $r \geq 1$, and for large r we have $k^t \ v_r \ k \geq C.\|k\|^2$.

Let now $\rho < \rho'' < \rho'$, and let D be the set of all symmetric real $n \times n$ matrices $v = (v_{ij})$ which satisfy $k^t \ vk > C'' \ \|k\|^2$ with $0 < C'' = - \log \rho'' < C$ for all $k \in \underset{\sim}{R}^n$.

We claim that D is open in the space $\underset{\sim}{R}^{n(n+1)/2}$ of all symmetric real $n \times n$ matrices. Let $v \in D$ and $\varepsilon < 0$ be small such that

$$n^2 \ \varepsilon < \Big(\inf_{0 \neq k \in \underset{\sim}{R}^n} \frac{k^t \ vk}{\|k\|^2} - C''\Big)$$

and, if $w = (w_{ij})$ is a symmetric real matrix with $|w_{ij}| < \varepsilon$ for all ij, we obtain

$$k^t \ wk = \Sigma_{i_j=1}^n \ w_{ij} \ k_i \ k_j \leq \Sigma |w_{ij}| \ |k_i \ k_j| \leq \varepsilon \, \Sigma_{i,j=1}^{n^2} \ |k_i| \ |k_j| < n^2 \ \varepsilon \ \|k\|^2 .$$

Thus

$$k^t(v + w) \ k = k^t \ vk + k^t \ wk > C'' \ \|k\|^2$$

which means that $v + w \in D$. This proves D open.

As now $v^* \in D$, we get that infinitely many v_r are also in D as D is open. If $v_r \in D$ then $x^{(r)} \in H_n(\rho')$ which is a contradiction.

Remark. - One can choose

$$r = [n^2 \log \tfrac{\rho}{\varepsilon}] + 1 \quad \text{for} \ \rho' = 1 \quad \text{where} \ H_n(1) := H_n .$$

THEOREM. - $H_n(K)$ <u>is a Stein domain on which</u> Γ_n <u>acts discontinuously</u>.

<u>Proof</u>. - Let $0 < \quad < 1$, $\rho_m = \sqrt[m]{\delta}$, $\rho'_m = \sqrt[(m+1)]{\delta}$, $\epsilon_m = \delta^m$.

By the lemma, we find r_m such that

$$P_m := X_{r_m}(\rho_m , \epsilon_m) \subseteq H_n(\rho'_m) \subsetneq H_n .$$

P_m is analytic polyhedron in $\mathbb{S}_n(K)$ and $H_n = \bigcup_{m=2}^{\infty} P_m$.

Also P_m is in the interior of P_{m+1} . This proves that H_n is a Stein domain (see [6], § 2).

Let $\Gamma_n(m) := \{\Phi \in \Gamma_n ; \ \Phi(P_m) \cap P_m \neq \emptyset\}$. We clain the $\Gamma_n(m)$ is finite. It can be deduced from the fact that for any given $C > 0$, there are only finitely many $\Phi \in \Gamma$ such that each column vector of Φ has euclidean norm $\leq C$. This proves that Γ_n acts discontinuously.

Let me mention a few open questions :

1° Define the analytic quotient H_n/Γ_n , and prove that it is a Stein space.

2° Find the algebraic relations between the $\theta_\sigma(x)$ and its connection with the Satake compactification.

3° Are the Chow coordinates in the sense of Shimura (see [7]), analytic functions on H_n ?

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