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ABOUT p-ADIC INTERPOLATION OF CONTINUOUS AND DIFFERENTIABLE FUNCTIONS

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0. Introduction.

In 1958, MAHLER proved that $\{\binom{x}{n} ; n \in \mathbb{N}\}$ form a normal base for $C(\mathbb{Z}_p , \mathbb{Q}_p)$. Since then, a number of different proofs of this theorem were given (cf. [1], [2], [4], [5], [8]).

In section 1, we show that the method used by Yvette AMICE [1] can be generalised to prove that $\{\binom{x}{n}^s ; n \in \mathbb{N}\}$ form a normal base, for each $s \in \mathbb{N}^*$. This leads to a generalisation of Mahler's formula (1.2). It is a remarkable fact that some polynomials (e. g. x) get an infinite expansion. So the linear space spanned by the $\binom{x}{n}^2$ lays dense in $C(\mathbb{Z}_p , \mathbb{Q}_p)$; however, it does not lay dense in $C^1(\mathbb{Z}_p , \mathbb{Q}_p)$.

In section 2, we prove that there exist polynomials $\tilde{R}_n$, with $\deg \tilde{R}_n = 2n + 1$, such that the polynomials $\gamma_n \binom{x}{n}^2$ together with the $\tilde{R}_n$ form a normal base of $C^1(\mathbb{Z}_p , \mathbb{Q}_p)$. A close relation with Van der Put's base, consisting of locally constant and locally linear functions should be noted.

1. Normal bases for $C(\mathbb{Z}_p , \mathbb{Q}_p)$.

1.1 THEOREM. - *For each* $s \in \mathbb{N}^*$, $\{q_n = \binom{x}{n}^s ; n \in \mathbb{N}\}$ *form a normal base of* $E = C(\mathbb{Z}_p , \mathbb{Q}_p)$.

Proof. - In view of [2] n° 3.1.5, or [7] lemme 1, it is sufficient to prove that $\{\bar{q}_n ; n \in \mathbb{N}\}$ form a vectorial base of $\bar{E} = C(\mathbb{Z}_p , \mathbb{F}_p)$. Let $\bar{E}_h$ be the space of $\mathbb{F}_p$-valued functions constant on each ball

$$B'_{p^{-h}}(a) = \{x \in \mathbb{Z}_p ; |x - a| \leqslant p^{-h}\} .$$

Since $\bar{E} = \bigcup \bar{E}_h$, our proof will be finished if we can show that $\{\bar{q}_i ; i < p^h\}$ form a base of $\bar{E}_h$.

For $i < p^h$ and $|x - y| < p^{-h}$, we have

$$|\binom{x}{i} - \binom{y}{i}| < 1 \quad ([2], 3.2.2.3) ,$$

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hence

$$|\binom{x}{i}^s - \binom{y}{i}^s| = |\binom{x}{i} - \binom{y}{i}|\ |\Sigma_{m=0}^{s-1} \binom{x}{i}^m \binom{y}{i}^{s-m-1}| < 1 ,$$

so $\bar{q}_i(x) = \bar{q}_i(y)$. It follows that $\bar{q}_i \in \bar{E}_h$, and

$$\bar{q}_i = \Sigma_{j=i}^{p^h-1} \bar{q}_i(j)\ \chi_j .$$

So the transition matrix form $\{\chi_j\ ;\ i < p^h\}$ to $\{\bar{q}_i\ ;\ i < p^h\}$ is triangular ; the desired result follows.

1.2 COROLLARY. - Let $s \in \underset{\sim}{N}^*$. Each continuous $f : \underset{\sim}{Z}_p \to \underset{\sim}{Q}_p$ can be written as a uniformily convergent series

$$f(x) = \Sigma_{n=0}^{\infty} a_n^{(s)} \binom{x}{n}^s$$

where

$$a_n^{(s)} = \Sigma_{k=0}^{n} (-1)^{n-k} \binom{n}{k}^s \beta_{n-k}^{(s)} f(k)$$

and

$$\beta_0^{(s)} = 1 ,\quad \beta_m^{(s)} = \sum_{\substack{(\ell_1 \cdots \ell_r) \\ \Sigma \ell_i = m \\ 1 \leq \ell_i \leq m}} (-1)^{r+m} \binom{m}{\ell_1 \ \cdots \ \ell_r}^s$$

Proof. - We have to calculate the interpolation coefficients $a_n^{(s)}$. They are determined by the formulas

$$a_n^{(s)} = f(0) ,\quad a_n^{(s)} = f(n) - \Sigma_{i=0}^{n-1} a_i^{(s)} q_i(n) .$$

We prove the formula using induction on n . Suppose true for $n \leq N$, then we have :

$$a_{N+1}^{(s)} = f(N+1) - \Sigma_{n=0}^{N} a_n^{(s)} \binom{N+1}{n}^s$$

$$= f(N+1) - \Sigma_{n=0}^{N} \Sigma_{k=0}^{n} (-1)^{n-k} \binom{n}{k}^s \beta_{n-k}^{(s)} f(k) \binom{N+1}{n}^s$$

$$= f(N+1) - \Sigma_{k=0}^{N} \Sigma_{n=k}^{N} (-1)^{n-k} f(k) \sum_{\Sigma_{i=1}^{r} \ell_i = n-k} (-1)^{r+n-k} \binom{n-k}{\ell_1 \ \cdots \ \ell_r}^s \binom{n}{k}^s \binom{N+1}{n}^s$$

$$= f(N+1) + \Sigma_{k=0}^{N} f(k) \Sigma_{n=k}^{N} \sum_{\Sigma_{i=1}^{r} \ell_i = n-k} (-1)^{r+1} \frac{(N+1)!}{\ell_1! \cdots \ell_r!\ k!\ (N+1-n)!} .$$

Putting $\ell_{r+1} = N + 1 - n$, we get

$$a_{N+1}^{(s)} = f(N+1) + \sum_{k=0}^{N} f(k) \sum_{\sum_{i=1}^{r+1} \ell_i = N+1-k} (-1)^{r+1} \binom{N+1-k}{\ell_1 \ldots \ell_r}^s \binom{N+1}{k}^s$$

$$= f(N+1) + \sum_{k=0}^{N} f(k)\ (-1)^{N+1-k}\ \beta_{N+1-k}^{(s)} \binom{N+1}{k}^s ,$$

this finishes the proof.

1.3 Note. – We can write down explicit formulas for the $\beta_m^{(s)}$:

$$\beta_0^{(s)} = \beta_1^{(s)} = 1 ,$$

$$\beta_2^{(s)} = 2^s - 1 ,$$

$$\beta_3^{(s)} = 6^s - 2.3^s + 1 ,$$

$$\beta_4^{(s)} = 24^s - 3.12^s + 6^s + 2.4^s - 1 .$$

It is easy to tabulate the $\beta_m^{(s)}$:

m \ s	1	2	3	4
0	1	1	1	1
1	1	1	1	1
2	1	3	7	15
3	1	19	163	1135
4	1	211	8993	271375

1.4 Note. – Comparing the case $s = 1$ with Mahler's formula, we get the following arithmetic formula :

$$\beta_m^{(1)} = \sum_{\substack{(\ell_1 \ldots \ell_r) \\ \sum \ell_i = m \\ 1 \leq \ell_i \leq m}} (-1)^{r+m} \binom{m}{\ell_1 \ldots \ell_p} = 1 .$$

1.5 Note (due to L. VAN HAMME). – One can determine the $\beta_m^{(s)}$ also, by using generating functions. One has the following identity between formal power series :

$$\left(\sum_{n=0}^{\infty} (-1)^n a_n \frac{Z^n}{(n!)^s}\right) \left(\sum_{n=0}^{\infty} (-1)^n b_n \frac{Z^n}{(n!)^s}\right) = \sum_{n=0}^{\infty} (-1)^n c_n \frac{Z^n}{(n!)^s} ,$$

if, and only if, $c_n = \sum_{k=0}^{n} \binom{n}{k}^s a_k b_{n-k}$. Put $b_n = 1$, $c_n = f(n)$, $a_n = a_n^{(s)}$.

Then it follows that

$$f(n) = \sum_{k=0}^{n} \binom{n}{k}^s a_k$$

if, and only if,

$$\sum_{n=0}^{\infty} (-1)^n a^{(s)} \frac{Z^n}{(n!)^s} = \left(\sum_{n=0}^{\infty} \frac{\beta_n^{(s)} Z^n}{(n!)^s}\right) \left(\sum_{n=0}^{\infty} (-1)^n f(n) \frac{Z^n}{(n!)^s}\right)$$

if, and only if,

$$a_n^{(s)} = \sum_{k=0}^{n} \binom{n}{k}^s (-1)^{n-k} \beta_{n-k}^{(s)} f(k) ,$$

where

$$\sum_{n=0}^{\infty} \frac{\beta_n^{(s)} Z^n}{(n!)^s} = \frac{1}{\sum_{n=0}^{\infty} (-1)^n Z^n/(n!)^s}$$

this last condition determines the $\beta_n^{(s)}$.

1.6 Note. - Applying corollary 1.2, we can obtain a lot of p-adically convergent series, e. g.

$$x = x^2 - \frac{1}{2} x^2(x-1)^2 + \frac{1}{3} x^2(x-1)^2 (x-2)^2 + \dots$$

$$= x^3 - \frac{3}{4} x^3 (x-1)^3 + \frac{23}{36} x^3 (x-1)^3 (x-2)^3 + \dots$$

It is a remarkable fact that these series converge p-adically for each prime number p . Also note that derivation of the series yields an apparent contradiction after putting $x = 0$; this shows that the series do no converge in $C^1(\mathbb{Z}_p , \mathbb{Q}_p)$-norm The same phenomenon happens with Van der Put's base. We return to this problem in section 2.

1.7 Note. - The proof of theorem 1.1 is merely based on the proof of Mahler's theorem as given in [2].

One could try to adapt the proof given by BOJANIC [4], MAHLER [5] or VAN ROOŸ [8] to prove the theorem ; however, it seems that these kinds of argument do not work here.

We can generalise theorem 1.1 if we replace $\mathbb{Z}_p$ by regular compact part M of a local field K and the interpolation sequence $\mathbb{N}$ by a very well distributed sequence $u : \mathbb{N} \to M$. For more details about very well distributed sequences, we refer to the work of Yvette AMICE [1]. We denote, following the notations in [1],

$$P_n(X) = (X - u_0)(X - u_1) \dots (X - u_{n-1}) , \quad Q_n(X) = P_n(X)/P_n(u_n) .$$

Given $\alpha_1 , \alpha_2 , \dots$ in K , with $|\alpha_i| \leqslant 1$ and $\sum_{i=1}^{\infty} \alpha_i = 1$, we define $q_n = \sum_{i=1}^{\infty} \alpha_i Q_n^i$. We omit the proof of the following theorem, since it is merely the same as the proof of theorem 1 in [1], up to one modification as in theorem 1.1.

1.8 PROPOSITION. - If $u : \underset{\sim}{N} \to M$ is a very well distributed sequence in a regular compact part M of the local field K, and the q_n are defined as above, then $\{q_n ; n \in \underset{\sim}{N}\}$ form a normal base of $C(M, K)$.

2. A normal base for $C^1(\underset{\sim}{Z}_p, \underset{\sim}{Q}_p)$.

For details about p-adic differentiability, we refer to [6]. Recall that a function $f : \underset{\sim}{Z}_p \to \underset{\sim}{Q}_p$ is called C^1 (or continuously differentiable) it the difference quotient $\Phi_1 f$ defined by

$$\Phi_1 f(x, y) = (f(x) - f(y))/(x - y)$$

can be extended to a continuous function $\Phi_1 f$ on $\underset{\sim}{Z}_p^2$. The space of C^1-functions becomes the Banachspace $C^1 = C^1(\underset{\sim}{Z}_p, \underset{\sim}{Q}_p)$ under the norm

$$\|f\|_1 = \max\{|f(0)| , \sup\{|\Phi_1 f(x, y)| ; x \neq y\}.$$

It is known ([3], [6], [9]) that the following sets form normal bases for C^1:

$\{\gamma_n \binom{x}{n} ; n \in \underset{\sim}{N}\}$ (Mahler's base)

$\{\gamma_n \chi_n(x) ; n \in \underset{\sim}{N}\} \cup \{\chi_n(x)(x - n) ; n \in \underset{\sim}{N}\}$ (Van der Put's bases).

We remind of the fact that γ_n is defined by

$\gamma_0 = 1$

$\gamma_n = a_s p^s$ if $n = a_s p^s + a_{s-1} p^{s-1} + \ldots + a_0$, $a_s \neq 0$.

So $v(\gamma_n) = s$, and $|\gamma_n|^{-1} = \max\{|m|^{-1} ; 0 < m \leq n\}$.

χ_n is the characteristic function of $\{x ; |x - n| < |\gamma_n|\}$.

Define $R_n = \gamma_n \binom{x}{n}^2$; it will then follow from lemma 2.2 that $\|R_n\|_1 = 1$; however, the R_n do not form a base for C^1, as we allready know from 1.6. Can we choose polynomials $\tilde{R}_n$ such that $\deg \tilde{R}_n = 2n + 1$ and the R_{n-1} $\tilde{R}_n$ form a normal base for C^1 ? Inspired by Van der Put's base, we could try $\tilde{R}_n \sim R_n(x - n)$.

After normalisation, we get $\tilde{R}_n = \gamma_{n+1} \binom{x}{n} \binom{x}{n+1}$. Unfortunately, it turns out that $\{R_n , \tilde{R}_n ; n \in \underset{\sim}{N}\}$ are not orthogonal in C^1. This comes from the fact that

$$|\tilde{R}_n'(n)| = \left|\frac{\gamma_{n+1}}{n + 1}\right| < 1 \quad \text{for some } n.$$

An answer to our question is furnished by following theorem.

2.1 THEOREM. - Let

$$R_n = \gamma_n \binom{x}{n}^2 ,$$

$$\tilde{R}_n = \gamma_{n+1} \binom{x}{n} \binom{\frac{x - (n+1-\gamma_{n+1})}{\gamma_{n+1}}}{} \binom{x+1-\gamma_{n+1}}{n+1-\gamma_{n+1}}$$

then $\{R_n ; \tilde{R}_n ; n \in \underset{\sim}{N}\}$ form a normal base for $C^1(\underset{\sim}{Z}_p , \underset{\sim}{Q}_p)$.

Note that for $n = a_s p^s - 1$, $0 < a_s < p$, $\tilde{R}_n = \gamma_{n+1} \binom{x}{n} \binom{x}{n+1}$. We need some lemmas.

2.2 LEMMA. - $\|R_n\|_1 = \|\tilde{R}_n\|_1 = 1$.

Proof. - For all $x \neq y$, we have

$$\left|\frac{R_n(x) - R_n(y)}{x - y}\right| \leqslant \frac{|\gamma_n|}{|x - y|} \left|\binom{x}{n} - \binom{y}{n}\right| \max\left(\left|\binom{x}{n}\right| , \left|\binom{y}{n}\right|\right) \leqslant 1$$

because $\|\gamma_n \binom{x}{n}\|_1 = 1$ and $\|\binom{x}{n}\| = 1$.

Furthermore

$$\left|\frac{R_n(n) - R_n(n - \gamma_n)}{n - \gamma_n}\right| = 1$$

In quite a similar way, we prove that $\|\tilde{R}_n\|_1 \leqslant 1$; finally

$$\|\tilde{R}_n\|_1 \geqslant |R'_n(n)| = \left|\frac{\gamma_{n+1}}{\gamma_{n+1}}\right| = 1 .$$

2.3 LEMMA. - If $0 \leqslant m < n$, then $|\tilde{R}'_n(m)| < 1$.

Proof.

$$\tilde{R}'_n(m) = \gamma_{n+1} \frac{d}{dx} \binom{x}{n}\Big|_{x=m} \binom{m - (n+1-\gamma_{n+1})}{(n+1) - (n+1-\gamma_{n+1})} \binom{m - (\gamma_{n+1} - 1)}{n+1-\gamma_{n+1}} .$$

If $m \geqslant n + 1 - \gamma_{n+1}$, then $\tilde{R}_n(m) = 0$. Suppose $m < n + 1 - \gamma_{n+1}$. If $|\gamma_{n+1}| < |\gamma_n|$, the result follows easily from the fact that $\|\binom{x}{n}\|_1 = |\gamma_n|^{-1}$. So we can suppose that $\gamma_n = \gamma_{n+1} = a_s p^s$.

We introduce the notation

$$\text{Schiff}(a_s p^s + a_{s-1} p^{s-1} + \dots + a_0) = a_s + a_{s-1} + \dots + a_0 .$$

We remind of the fact that

Schiff m + Schiff$(n - m - 1) + 1 -$ Schiff $n \leqslant (p - 1)\, v(\gamma_n)$, for $0 \leqslant m < n$.

This follows from the fact that $\|\binom{x}{n}\|_1 = 1$, but it can also be proved directly.

Now, let $n = a_s p^s + \dots + a_0$, then $m < a_{s-1} p^{s-1} + \dots + a_0$, and $n-m-1 \geqslant a_s p^s$. We have

$$v(|\frac{d}{dx}\binom{x}{n}|_{x=m}) = v(\frac{m!\,(n-m-1)!}{n!})$$

$$= (p-1)^{-1}\,(\text{Schiff}(n) - \text{Schiff}(m) - \text{Schiff}(n-m-1) + 1)$$

$$= (p-1)^{-1}\,(a_s + \text{Schiff}(n - a_s\,p^s) - \text{Schiff}(m) - \text{Schiff}(n-m-1-a_s\,p^s) - a_s + 1)$$

$$\geqslant - v(\gamma_n - a_s\,p^s) > - v(\gamma_n)\ ,$$

hence

$$|\frac{d}{dx}\binom{x}{n}|_{x=m} < \frac{1}{|\gamma_n|} = \frac{1}{|\gamma_{n+1}|}\ ;$$

the result follows.

Proof of theorem 2.1. – The polynomials form a dense subspace of C^1 (cf. Mahler's base). Since the R_n and $\tilde{R}_n$ generate the polynomials, it only remains to show that $\{R_n\ ,\ \tilde{R}_n\ ;\ n \in \underset{\sim}{N}\}$ form an orthogonal system.

Using [8], 5.1.(ε), it is sufficient to show that for each $m \in \underset{\sim}{N}$:

R_n is orthogonal to the linear hull of $\{\tilde{R}_n\ ,\ R_{n+1}\ ,\ \tilde{R}_{n+1}\ ,\ \ldots\}$

$\tilde{R}_n$ is orthogonal to the linear hull of $\{R_{n+1}\ ,\ \tilde{R}_{n+1}\ ,\ R_{n+2}\ ,\ \ldots\}$.

This follows from the fact that for all $\alpha_j\ \beta_j \in K$, we have

$$\|R_n - \textstyle\sum'_{j>n} \alpha_j\, R_j - \sum'_{j\geq n} \beta_j\, \tilde{R}_j\|_1$$

$$\geqslant |\Phi_1\,(R_n - \textstyle\sum'_{j>n} \alpha_j\, R_j - \sum'_{j\geq n} \beta_j\, \tilde{R}_j)(n\ ,\ n-\gamma_n)| = 1 = \|R_n\|_1\ ,$$

and

$$\|\tilde{R}_n - \textstyle\sum'_{j>n} \alpha_j\, R_j - \sum'_{j>n} \beta_j\, \tilde{R}_j\|_1$$

$$> |\tilde{R}'_n(n) - \textstyle\sum'_{j>n} \alpha_j\, R'_j(n) - \sum'_{j>n} \beta_j\, \tilde{R}'_j(n)| = |\tilde{R}'_n(n)| = 1 = \|\tilde{R}_n\|_1\ ,$$

using the fact that $R'_j(n) = 0$,

$$|R'_j(n) < 1 \quad \text{for} \quad j > n\ .$$

2.4 Note. – Our proof is merely inspired by Van Rooÿ's proof of Mahler's theorem ([8], 5.27). It is also possible to give a proof using the residue class space (as in 1.1), which is, however, considerably longer.

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