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HARMONIC ANALYSIS ON p-TORSIONAL GROUPS

(after A. M. M. Gommers)

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The following is a presentation of results obtained by A. M. M. GOMMERS. Full details will appear in his forthcoming thesis prepared under the guidance of A. C. M. van ROOIJ.

1. The groups G that we consider are torsional, i. e. G satisfies the equivalent conditions :

(a) G is a commutative topological group, and G has a zero-dimensional open compact subgroup H such that G/H is a torsion group.

(b) G is a commutative, locally compact, zero-dimensional group such that every finite subset of G lies in a compact subgroup.

Let p be a prime number ; then G is called p-torsional (resp. p-free) when for any open compact subgroup H of G the group G/H is a p-torsion group (resp. has no p-torsion).

The field k is supposed to be a non-archimedean valued complete field with residue field $\bar{k}$ of characteristic p.

(1.1) LEMMA. - G has a unique decomposition as a topological product $G=G_1\times G_2$, where G_1 is p-torsonial and G_2 is p-free.

Proof. - For a compact zero-dimensional group G this decomposition is well known. In the general case, each open compact subgroup H of G has an unique decomposition $H_1 \times H_2$. Then $G_i = \bigcup\{H_i ; H \text{ open compact subgroup of } G\}$ $(i=1,2)$ provides the unique decomposition of G.

(1.2) Remarks. - On the part G_2 of G there exists a (k-valued) Haar measure μ. Let $C_\infty(G_2)$ denote the Banach space of the continuous functions $G_2 \to k$ which are "zero at ∞", provided with the supremum norm. On $C_\infty(G_2)$ we have a convolution

$$(f * g)(a) = \int f(b)\, g(a - b)\, d\mu(b)$$

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and $L(G_2)$ denotes $C_\infty(G_2)$ with the algebra structure given by the convolution.

Let us suppose, for convenience, that k is algebraically closed. Then the dual of G_2 is $\hat{G}_2$ = the continuous homomorphisms $G_2 \to k^*$, provided with the compact open topology. The Fourier theory ([2], [3]) states :

$$F : L(G_2) \to C_\infty(\hat{G}_2)$$

is an isometric isomorphism of Banach algebra's where the Fourier transform F is defined by :

$$F(f)(\chi) = \int f(b)\, \chi(-b)\, d\mu(b) \quad \text{with} \quad f \in L(G_2) \quad \text{and} \quad \chi \in \hat{G}_2 \; .$$

On the part G_1 of G there is (in general) no Haar-measure. So $L(G_1)$ is meaningless. One studies instead $M(G_1)$. In general, $M(G)$ = the Banach space of tight measures on $G = \text{inj lim}\{C_\infty(H)' \; ; \; H \text{ compact in } G\}$. In particular, if G is compact then $M(G) = C_\infty(G)'$ = the topological dual of the Banach space $C_\infty(G)$. On $M(G)$ the convolution is defined by

$$(\mu * \nu)(f) = \iint f(a + b)\, d\mu(a)\, d\nu(b) \; .$$

If G is p-free then $M(G) \xrightarrow{\sim} BUC(\hat{G})$ = the bounded uniformly continuous functions on $\hat{G}$. This isomorphism is given by :

$$\mu \mapsto \hat{\mu} \quad \text{and} \quad \hat{\mu}(\chi) = \int \chi(a)\, d\mu(a) \; , \text{ where } \; \mu \in M(G) \; ; \; \chi \in \hat{G} \; .$$

In general, the algebra $M(G)$ is (morally speaking) determined by $M(G_1)$ and $M(G_2)$. Since the part $M(G_2)$ is well known as an algebra, the remaining part $M(G_1)$ will have most of our attention.

We can formulate the connection between $M(G)$, $M(G_1)$, $M(G_2)$ as follows :

(1.3) PROPOSITION. - *If G_2 is compact then $M(G) \simeq M(G_1) \circ M(G_2)$ (as Banach algebra's)* .

Proof. - The operation $\circ$ is a variant of the tensor product of Banach spaces. We define $\circ$ only for pairs (E , F') , where F' is the dual of some Banach space F .

Definition. - $E \circ F' = \text{proj lim}\{E \otimes F'_0 \; ; \; F_0 \text{ finite dimensional subspace of } F\}$. In our case, $M(G_2)$ is naturally given as the dual of $C_\infty(G_2)$. One easily verifies the formula when G_2 is finite (then $\circ$ and $\otimes$ agree). From this the general case follows.

(1.4) *Remarks.*

1° If G_2 is not compact then $M(G) \simeq \text{inj lim}\, M(G_1) \circ M(H_2)$, where H_2 runs in the set of all open compact subgroups of G_2 . The isomorphism is again an isomorphism of Banach algebras.

2° If G_2 is compact, and k is algebraically closed, then $M(G_2) = B(\hat{G}_2) =$

the bounded functions on $\hat{G}_2$. Proposition (1.3) yields

$$M(G) \simeq \prod_{\chi \in \hat{G}_2} M(G_1) * \chi \quad \text{and every} \quad M(G_1) * \chi \cong M(G_1) \; .$$

3° In many cases, one can show that there is a (1 - 1)-correspondance between the homomorphisms $\varphi : M(G) \to k$ and the pairs of homomorphisms

$$\varphi_i : M(G_i) \to k \quad (i = 1 , 2) \; .$$

This holds for instance if k is not locally compact.

2. <u>In this section, we assume that</u> G <u>is a</u> <u>p-torsional group</u>.

Let T denote the discrete p-torsion group $\mathbb{Q}_p/\mathbb{Z}_p$. If the field k has characteristic 0 and is algebraically closed then we can identify T with the subgroup of k^* consisting of the elements of order p^n $(n \geqslant 0)$.

For a p-torsional group G we define a dual G^* = the continuous homomorphisms $G \to T$, provided with the compact open topology.

G^* is again p-torsional ; $G \xrightarrow{\sim} G^{**}$; G is compact if, and only if, G^* is discrete.

There are two extreme cases for p-torsional groups :

<u>Type (1)</u> : G has no elements $(\neq 0)$ of finite order.

<u>Type (2)</u> : The elements of finite order are dense in G .

For compact G one has : G is of type (1) if, and only if, G^* is a p-divisible group ; G is of type (2) if, and only if, G^* has no p-divisible subgroups $\neq 0$. Further, if G is compact then $G = G_1 \times G_2$ where G_i is of type (i). This follows from $G^* = H_1 \times H_2$ where H_1 is a maximal p-divisible subgroup of G^* and so $G = H_1^* \times H_2^*$.

The compact groups G of type (1) are easily determined : G^* is p-divisible and (as is well known) it follows that $G^* = T^{(I)}$ for some index set I . Then $G \cong \mathbb{Z}_p^I$ since $T^* = \mathbb{Z}_p$.

The compact groups G of type (2) (or their duals G^*) are very complicated in general. One can however prove the following :

(2.1) PROPOSITION. - <u>Let</u> G <u>be compact, then there exists an exact sequence of topological groups</u>

$$0 \to \mathbb{Z}_p^I \to G \to \prod_{j \in J} \mathbb{Z}/p^{n_j} \to 0 \; .$$

<u>If</u> $\sup(x_j) < \infty$ <u>then the sequence splits topologically</u>.

Next, we have the following :

(2.2) PROPOSITION. - The following properties of the p-torsional group G are equivalent :

(a) G has no elements ($\neq 0$) of finite order (i. e. G of type (1)),

(b) $\underset{\sim}{Z}_p^I \subset G \subset \underset{\sim}{Z}_p^I \otimes_{Z_p} \underset{\sim}{Q}_p$ where $\underset{\sim}{Z}_p^I$, with the product topology, is an open compact subgroup of G,

(c) the norm on $M(G)$ is multiplicative,

(d) for any $\mu \in M(G)$, $\mu \neq 0$, one has :

$$\mu \text{ is invertible in } M(G) \Leftrightarrow \|\mu\| = |\mu(G)| .$$

Proof. - Since $M(G) = \text{proj lim}\{M(H) ; H \text{ open compact subgroup of } G\}$, it suffices to consider compact groups G. In this case, (b) can be replaced by (b') : $G \simeq \underset{\sim}{Z}_p^I$.

Another argument shows that the general case will follows from the case where G is topologically finitely generated. Such a group has the form

$$G = \prod_{i=1}^{n} \underset{\sim}{Z}_p / p^{m_i} \underset{\sim}{Z}_p \quad \text{with } 0 < m_i \leq \infty .$$

In (2.3) and (2.4), $M(G)$ is explicitely given and one can verify (2.2).

(2.3) PROPOSITION. - Let $G = \underset{\sim}{Z}_p^n$ then $M(G) \simeq k\langle X_1, \ldots, X_n\rangle$ = the Banach algebra of all power series $\sum a_\alpha X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ with $\sup|a_\alpha| < \infty$.

Proof. - $C(\underset{\sim}{Z}_p^n)$ has the orthonormal base

$$\binom{X}{\alpha} = \binom{X_1}{\alpha_1} \ldots \binom{X_n}{\alpha_n} ,$$

considered as a function : $\underset{\sim}{Z}_p^n \to k$. The isomorphism of (2.3) is given by the map

$$\mu \longmapsto \sum_\alpha \mu(\tbinom{X}{\alpha}) X_1^{\alpha_1} \ldots X_n^{\alpha_n} .$$

(2.4) COROLLARY. - Let $G = \prod_{i=1}^{n} \underset{\sim}{Z}_p / p^{m_i} \underset{\sim}{Z}_p$ then $M(G) = k\langle X_1, \ldots, X_n\rangle / I$, where I is the ideal generated by $(X_i + 1)^{p^{m_i}} - 1$ (all i with $m_i \neq \infty$).

(2.5) Remark.- If G is compact then there exists a surjective map $\underset{\sim}{Z}_p^I \to G$. Hence $M(G)$ is a quotient of $M(\underset{\sim}{Z}_p^I) = k\langle X_i | i \in I\rangle$. If I is infinite then it is not clear what the kernel $M(\underset{\sim}{Z}_p^I) \to M(G)$ should be.

3. We suppose in this section that G is a compact p-torsional group.

(3.1) PROPOSITION. - Suppose that k has characteristic p. Then

(a) $M(G)$ has no idempotents $\neq 0, 1$,

(b) any character $\chi : G \to k^*$ with open kernel is trivial,

(c) if $\mu \in M(G)$ satisfies $\|\mu\| = |\mu(G)| \neq 0$, then μ is invertible and $\|\mu^{-1}\| = |\mu(G)|^{-1}$.

Proof. - Let $\mu = \mu^2 \in M(G)$, let H be an open compact subgroup of G , and let $\nu \in M(G/H)$ be the image of μ . Then $\nu^2 = \nu$ and $\nu = 0$ or 1 since $M(G/H)$ is a local ring. It follows easily that $\mu = 0$ or 1 . Statement (b) follows since k contains no p-th roots of unity. Statement (c) is easily seen for finite groups and follows from that special case.

Suppose now that k has characteristic zero (hence $k \supset Q_p$). If k is algebraically closed then we can identify G^* with the characters $\chi : G \to k^*$ with open kernel. Further $M(G)$ can contain idempotents $\neq 0 , 1$. Namely, let H be a finite subgroup of G , and let $\chi : H \to k^*$ be a character then

$$\mu_\chi = \frac{1}{p^n} \sum_{h\in H} \chi(-h)\ \delta_h \in M(H) \subset M(G) ,$$

where p^n is the order of H , is clearly an idempotent. For any finite set $E \subset H^*$ one can form

$$\mu_E = \sum_{\chi\in E} \mu_\chi .$$

In this way we have described all idempotents, with support in H . Now A. M. M. GOMMERS conjectures that there are no other idempotent elements in $M(G)$. We can state this as follows :

(3.2) CONJECTURE. - Every idempotent in $M(G)$ has finite support.

One has to work with G^* the characters of G to find a proof. The elements in G^* are linearly independent functions on G , but they are by no means orthogonal. This is the main difficulty in the verification of (3.2).

A. M. M. GOMMERS gives a proof of a special case :

(3.3) PROPOSITION. - For $G = (Z/p)^I$ every element $\mu \in M(G)$ with $\mu=\mu^2$ has finite support.

We give some comment on the conjecture. Let G be a group of order p^n . Let $E \subset G^*$ be given, then

$$\mu_E = \sum_{\chi\in E} \mu_\chi = \sum_{g\in G}\Big(\frac{1}{p^n} \sum_{\chi\in E} \chi(-g)\Big)\ \delta_g$$

is an idempotent.

It has the property $\hat{\mu}_E(\chi) = 1$ or 0 according to $\chi \in E$ or $\chi \notin E$. One sees that in general $\|\mu_E\| = p^n$. If μ_E has support in a subgroup H of G with order p^k , then $\|\mu_E\| \leq p^k$. This yields the following.

(3.4) CONJECTURE. - Let G be a group of order p^n , let $\mu \in M(G)$ be an idempotent with norm $\leq p^k$. If n is "large with respect to k " then μ has support in a proper subgroup of G .

We note that (3.4) implies (3.2). A first step towards (3.4) is estimating the absolute value of sums of p^d-th roots of unity. This is done in :

(3.5) LEMMA. - Let $\omega \in k$ be a primitive p^d-th root of unity and let $\ell, n_i \in \mathbb{Z}$; $\ell \geqslant 1$. Then equivalent are :

(a) $|\sum_{i=0}^{p^d-1} n_i \, \omega^i| \leqslant \frac{1}{p^\ell}$,

(b) For all $0 \leqslant i$, $j < p^d - 1$ with $i \equiv j(p^{d-1})$ one has $n_i \equiv n_j(p^\ell)$.

Proof. - (b) $\Longrightarrow$ (a) follows easily from the minimal equation $(X^{p^d}-1/X^{p^{d-1}}-1)$ satisfied by ω .

Further, we note that it suffices to show (a) $\Longrightarrow$ (b) for $\ell = 1$; $\ell > 1$ follows easily by induction.

We consider $\mathbb{Z}_p[\omega] = \mathbb{Z}_p[\xi]$ where $\omega = 1 + \xi$. This is a subring of k . Since $|\xi|^{p^{d-1}(p-1)} = \frac{1}{p}$, it follows that the elements in $\mathbb{Z}_p[\xi]$ with absolute value $\leqslant \frac{1}{p}$ form the ideal $I = p\mathbb{Z}_p[\xi]$. Dividing by this ideal one finds :

$$\mathbb{Z}_p[\omega]/I = \mathbb{F}_p[T]/(1 + T^{p^{d-1}} + T^{2p^{d-1}} + \dots + T^{(p-1)p^{d-1}})$$

where T has image ω . Hence $|\sum_{i=0}^{p^d-1} n_i \, \omega^i| \leqslant \frac{1}{p}$ implies $\sum \bar{n}_i \, t^i = 0$ (where $\bar{n}_i$ is the image of n_i in $\mathbb{F}_p$).

This means

$$\sum \bar{n}_i \, T^i = (a_0 + a_1 \, T + \dots + a_{p-1} \, T^{p-1})(1 + T^{p^{d-1}} + T^{2p^{d-1}} + \dots + T^{(p-1)p^{d-1}})$$

for certain $a_0 , a_1 , \dots , a_{p-1} \in F_p$. This is equivalent with statement (b) for $\ell = 1$.

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