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A Sketch theoretical survey Towards a typology of mathematical structures

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A Sketch theoretical survey

Towards a typology of mathematical structures

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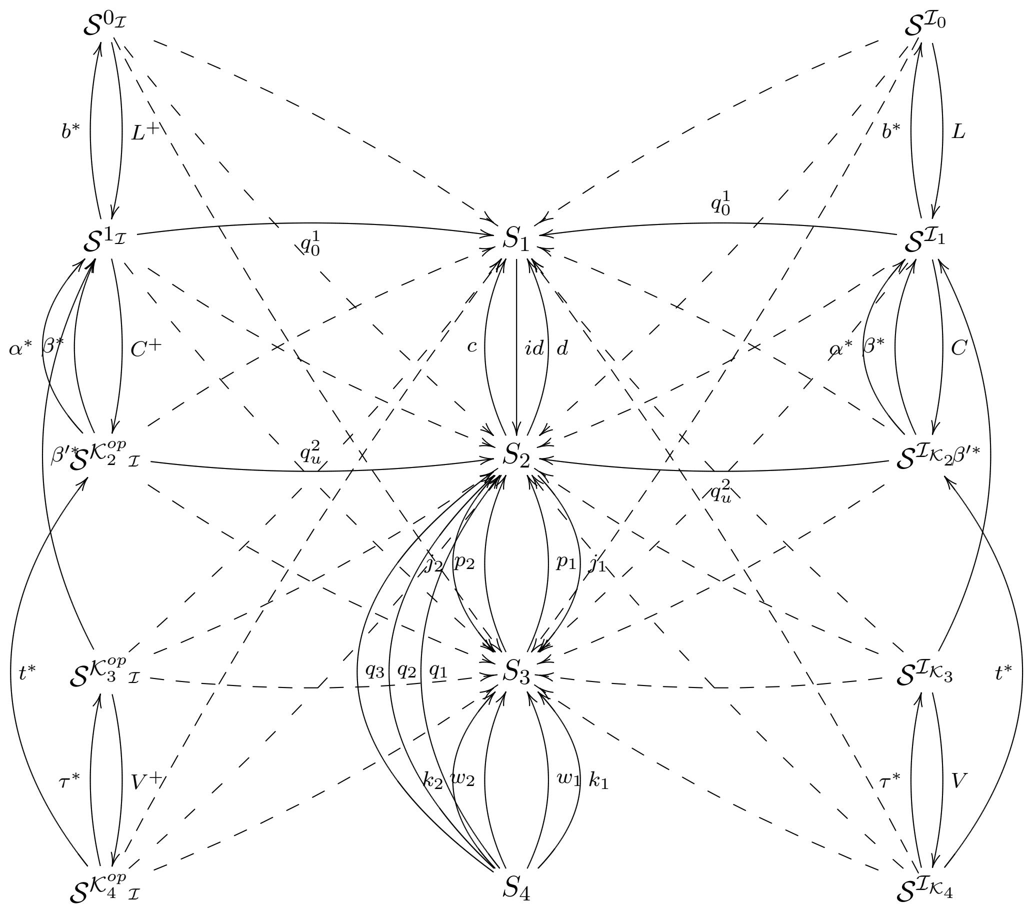


Masters Thesis for the Cand. Scient. degree in mathematics at the University of Copenhagen.

Abstract

We look at the advantages of adopting the sketch theoretical point of view when considering mathematical structures, i.e. when one considers mathematical theories as categories of models of a sketch and models as certain functors. We hereby get a formalization of the notion of a *type* of mathematical structure/theory, as well as a direct interplay between the sketch describing a theory (syntax) and the properties described (semantic). This leads to a fruitful application of *the generalized associated sheaf theorem* for certain *strict types* suggesting a general diagrammatic method for proving and discriminating, treating proofs as syntactic factorizations in the category of sketches, and discrimination as semantic investigations by comparison of model categories.
Using this application we then work out two examples of how to *typify* classical mathematical structures according to where and how they are defined.

Contents

1 Preface and Acknowledgements

The style of the present work is mostly influenced by the french school of category theory initiated by the french mathematician Charles Ehresmann (1905-1979) and then further developped by his students. My work is more directly related to this later work based on his ideas. I in particular want to thank Laurent Coppey (now retired french mathematician, Paris VII) for introducing me to sketches and category theory in general and for making the initial suggestions for my work in his article about sketches and types (Coppey, L. [1992]). Also for introducing me to René Guitart (Paris VII), to whom I owe my warmest acknowledgement for writing me a guideline presenting basic ideas about types, basic definitions and notation, and then for having been my unofficial (but indispensable) supervisor throughout the work. Finally I would like to thank Carsten Butz (IT-University of Copenhagen) for agreeing to supervise this Masters thesis and thereby have made it possible for me to write in a subject rather foreign to the mathematics department at the University of Copenhagen.

2 Introduction

The general motivation for such work, as here presented, has already been elegantly phrased by a great name in the history of mathematics, David Hilbert (1862-1943):

> *The question is urged upon us whether mathematics is doomed to the fate of those other sciences that have split up into several branches, whose representatives scarcely understand one another and whose connections become ever more loose. I do not believe this nor wish it. Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts. For with all the variety of mathematical knowledge we are still scaresly conscious of the similarity of the logical devises, the relationship of the ideas in mathematical theory and the numerous analogies in its different departments. We also notice that, the farther a mathematical theory is developed, the more harmoniously and uniformly does its construction proceed, and unsuspected relations are disclosed between hitherto separate branches of the science.[...]Every real advance [in mathematical science] goes hand in hand with the invention of sharper tools and simpler methods which at the same time assist in understanding earlier theories and cast aside older more complicated developments.*

(Quoted from "Mathematiche probleme"(1901), English translation BAMS 8,p. 478-79, by M.W. Newson).

Category theory, including sketch theory, provides a natural language (a logic (Guitart, R. [1981])) able to analyse contemporary mathematical work at an abstract level; taken out of specific contexts. Thus, at least some part of mathematical activity struggles against the above feared (and horrible) fate.
This thesis looks at how sketches, in describing mathematical theories/structures, can give direct diagrammatic ways of formulating usual mathematical problems on an abstract categorical level, and how this method makes it possible to consider different aspects of structures within a general frame (a category), using one and same (non set-theoretical) diagrammatic language (category theory).
Now why is this interesting? I here present seven *mottos*, that could tempt someone to adopt the sketch theoretical point of view:

1 Sketches furnish a nice and direct way of formalizing the notion of *a type of mathematical theory/structure.*

2 Sketching is proving and proving is sketching; proofs become objects in a category, simply defined by their relations to other objects. (Coppey, L. [1992])

3 Everything usually described by first order logic, can be described, more directly in terms of limits and co-limits, i.e. every first order theory is sketchable. (Guitart, R., Lair, C [1982])

4 We can apply the *generalized associated sheaf theorem* and profit from universal properties to talk about freely generated theories of a sketch and suggest general methods for proving and discriminating.

5 The application of the *generalized associated sheaf theorem* then also leads to joining the MacLane motto that "everything is Kan extensions".

6 By sketches, we get a double usage of universal properties: to study syntactic questions when specifying theories, but at the same time, to determine necessary semantics. Hence syntax and semantics becomes of the same, purely diagrammatic, nature.

7 Describing structures by sketches makes possible a formalization of how to discriminate between different ways of defining the same concept, avoiding false confusions of set theoretical definitions. (Coppey, L. [1992])

These mottos will all more or less directly find support in this thesis.
Now the more precise goal is to present a formalization of the notion *theoretical type* and then lay forth a basic understanding of what role these types play in the on the one hand syntactic and on the other semantic aspects of considering mathematical theories, emphasizing the double aspect of types in proving and discriminating.
The presentation is followed by two examples, the first one a rather simple example concerning two ways of sketching a field and the second example concerns two sketches of monoids. They are both examples of two different sketches with the same set theoretical models and we are then interested in finding out in what type of categories the two given sketches have the same models (equivalent model categories). This question turns out to have many faces, touching all of the above mottos. But our main objective is again two see how syntax and semantic become two sides of the same diagrammatic procedure.

Working out these two examples according to ideas indicated by René Guitart (partly based on the article Coppey, L. [1992]) has been my method of conduction. So the reader is invited to think of this thesis as an introduction to a general formalization of what mathematicians mean when they use the concept *a type of structure*, by working out two examples suggested in different articles (Coppey, L. [1992], Guitart, R. [1988]).

The main content consists of: an introduction of a category of sketches (section 5.) that will serve as the main frame, wherein the notion *theoretical type* is to be defined. Then we look at the syntax and semantic of sketches by discussing proving and discriminating in the the category of sketches limited to cases where we can apply *the generalized associated sheaf theorem*, which then leads to a general problem (section 6.). At last we present the two examples (section 7.) following this general problem.

3 Preliminaries

The reader is supposed to be familiar with category theory.
But since the style of this paper is very much influenced by the french school, I nevertheless, choose to briefly introduce some absolute basic concepts.

3.1 Categories, Functors, natural transformations.

• *Oriented Graph:* We say that $\mathcal{S} = (Ob_{\mathcal{S}}, Ar_{\mathcal{S}}, c, d, i)$ is an oriented graph if and only if:

- $Ob_{\mathcal{S}}$ is a set (set of objects).

- $Ar_{\mathcal{S}}$ is a set (set of arrows) and we write $\mathcal{S}(A, B)$ or $\mathrm{Hom}(A, B)$ for the subset of arrows from A to B.
- $c : Ar_{\mathcal{S}} \longrightarrow Ob_{\mathcal{S}}$ is the map sending an arrow to its target (co-domain).
- $d : Ar_{\mathcal{S}} \longrightarrow Ob_{\mathcal{S}}$ is the map sending an arrow to its source (domain).
- and $id : Ob_{\mathcal{S}} \longrightarrow Ar_{\mathcal{S}}$ is a map that sends an object $C \in \mathcal{S}$ to its corresponding identity arrow 1_C, such that:

$$Ob_{\mathcal{S}} \; \overset{d}{\longleftarrow}\!\!\!\!\!\!\!\! \underset{c}{\longleftarrow} \xrightarrow{id} \; Ar_{\mathcal{S}} \quad : \quad c \cdot id = d \cdot id = 1_{Ob_{\mathcal{S}}}.$$

•*Multiplicative graph:* $\mathcal{S}^{\cdot} = (\mathcal{S}, Co_{\mathcal{S}}, k)$ is said to be a multiplicative graph, if and only if

- $\mathcal{S}$ is an oriented graph,
- $Co_{\mathcal{S}}$ is the set of composable pairs of arrows (i.e. a subset of all pairs of consecutive arrows $\{(g, f) \in Ar_{\mathcal{S}} \times Ar_{\mathcal{S}} \quad | \quad d(g) = c(f)\}$),
- $k : Co_{\mathcal{S}} \longrightarrow Ar_{\mathcal{S}}$ is a partial composition (multiplication) of arrows $k(g, f) = g \circ f$,

given subjected to the following axioms:

Unitarity: $\forall g \in Ar_{\mathcal{S}} : \quad (id \cdot c(g), g), (g, id \cdot d(g)) \in Co_{\mathcal{S}}$
et $k(id \cdot c(g), g) = k(g, id \cdot d(g)) = g$

Position: $\forall (g, f) \in Co_{\mathcal{S}} : \quad d(g \circ f) = d(f), c(g \circ f) = c(g)$

i.e. an oriented graph with a table of equations concerning a partial composition.

•*Category*: A category $\mathcal{C}$ is a multiplicative graph where $Co_{\mathcal{C}}$ is the set of all consecutive arrows and where the composition k is associative.
The notion "category" makes sense even if Ob et Ar are not sets. But then one of course has to be conscious of all use of ordinary set theory. In this thesis there will be certain size limitations, which will be discussed when appropriate.

•*Functor*: If $\mathcal{D}$ and $\mathcal{C}$ are multiplicative graphs, a (covariant) functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a pair of applications $F = (F_1, F_0)$: $F_0 : Ob_{\mathcal{C}} \longrightarrow Ob_{\mathcal{D}}$ et $F_1 : Ar_{\mathcal{C}} \longrightarrow Ar_{\mathcal{D}}$ such that:
$\forall (g, f) \in Co_{\mathcal{C}} : \quad (F(g), F(f)) \in Co_{\mathcal{D}}, \quad F(g \circ f) = F(g) \circ F(f); \quad \forall C \in Ob_{\mathcal{C}} : F(1_C) = 1_{F(C)}$
noting (abusively) F in stead of F_0 or F_1.

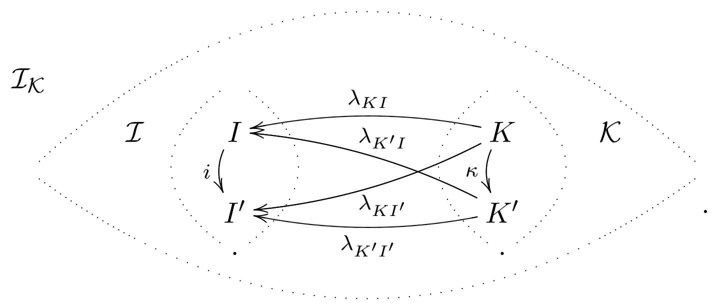
•*Natural transformation*: Take functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$, then a natural transformation $\tau : F \overset{\cdot}{\longrightarrow} G$ is defined by a family of maps ($\tau_C : F(C) \longrightarrow G(C)$)$_{C \in Ob_{\mathcal{C}}}$ such that for all arrows $f : C \longrightarrow D$ the following diagram commute:

$$\begin{array}{ccc} F(C) & \xrightarrow{\tau_C} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(D) & \xrightarrow[\tau_D]{} & G(D) \end{array}$$

If for all $C \in \mathcal{C}$ τ_C is an isomorphism, we say that F and G are naturally *isomorphic* and we write $F \cong G$. We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, and write $\mathcal{C} \sim \mathcal{D}$, if there exist functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that $F \circ G \cong 1_{\mathcal{D}}$, $\quad G \circ F \cong 1_{\mathcal{C}}$

• The *join of the graph* $\mathcal{K}$ with the graph $\mathcal{I}$, is denoted $\mathcal{I}_{\mathcal{K}}$ and contains:
-two subgraphs (supposed disjoint) isomorphic to $\mathcal{I}$ and $\mathcal{K}$ (and hence identified with these).
-In addition, for all pairs $(I, K) \in Ob_{\mathcal{I}} \times Ob_{\mathcal{K}}$ there is an arrow $\lambda_{KI} : K \longrightarrow I$ such that $j \cdot \lambda_{KI} = \lambda_{KI'}$ for all $K \in Ob_{\mathcal{K}}$ and for all $j : I \longrightarrow I' \in Ar_{\mathcal{I}}$ and such that $\lambda_{K'I} \cdot \kappa = \lambda_{KI}$ for all $I \in Ob_{\mathcal{I}}$ and for all $\kappa : K \longrightarrow K' \in Ar_{\mathcal{K}}$.

So if for example $\mathcal{I} = I \overset{i}{\longrightarrow} I'$ and $\mathcal{K} = K \overset{\kappa}{\longrightarrow} K'$ then we get $\mathcal{I}_{\mathcal{K}}$ as the multiplicative graph:



where all composites commute.
For a graph-homomorphism $f : \mathcal{K} \longrightarrow \mathcal{K}$ we associate the "natural" graph-homomorphism $f : \mathcal{I}_{\mathcal{K}} \longrightarrow \mathcal{I}_{\mathcal{K}'}$ (without changing its name) defined by:

$$\begin{aligned} f(i) &= i \quad \text{for} \quad i \in Ar_{\mathcal{I}} \\ f(\kappa) &= f(\kappa) \quad \text{for} \quad \kappa \in Ar_{\mathcal{K}} \quad (\text{ where the second f is } f : \mathcal{K} \longrightarrow \mathcal{K}' \,) \\ f(\lambda_{KI}) &= \lambda_{f(K)I} \quad \text{for} \quad I \in Ob_{\mathcal{I}}, \quad K \in Ob_{\mathcal{K}}. \end{aligned}$$

We will also use the notation $\mathcal{I}^-$ for the joint graph $\mathcal{I}_1$ (1 is the graph of one object and only the identity arrow) and the notation $\mathcal{I}^+$ for the joint graph $1_{\mathcal{I}}$,

i.e. just joining a vertex to a graph $\mathcal{I}$, such that $\mathcal{I}$ becomes the basis of a cone or co-cone.

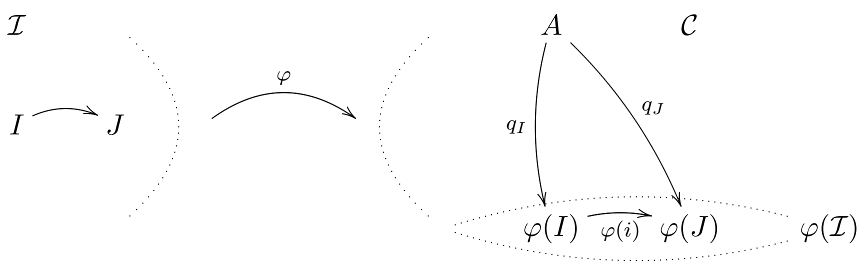
3.2 Projective and inductive limits, or limits and co-limits.

Let $\mathcal{I}$ be an oriented graph, $\mathcal{C}$ a category and $\varphi : \mathcal{I} \longrightarrow \mathcal{C}$ a functor (between oriented graphs).

•*Projective cones or just "cones"*: a projective cone (in $\mathcal{C}$) with basis φ is a couple (A,q) consisting of an object A from $\mathcal{C}$ and a family $(\ q_I : A \longrightarrow \varphi(I)\ )_{I\in Ob_{\mathcal{I}}}$ of arrows in $\mathcal{C}$, satisfying

$$\forall\ i : I \longrightarrow J\ \in Ar_{\mathcal{I}} : \quad \varphi(i)\circ q_I = q_J$$

i.e. all triangles in the cone commute. If $\mathcal{I} = \{\ I \overset{i}{\frown} J\ \}$, we can illustrate this as follows:

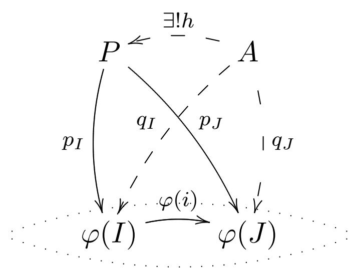


Let Λ_φ denote the set of projective cones (in $\mathcal{C}$ with basis φ).

•*Inductive cones or "co-cones"*: it is the dual notion, i.e. (B,ι) is a pair consisting of an object B and a family $(\ \iota_I : \varphi(I) \longrightarrow B\ )_{I\in Ob_{\mathcal{I}}}$ such that $\forall\ i : I \longrightarrow J\ \in Ar_{\mathcal{I}}$, we have $\iota_J \circ \varphi(i) = \iota_I$ and analogous we note V_φ the set of co-cones with basis φ.

•*Limits*: A cone (P,p) with basis φ is said to be a limiting cone (or limit of φ), written $\varprojlim\varphi$, if it fulfils the following *universal property*:

$$\forall(A,q)\in\Lambda_\varphi \quad \exists!\ h : A \longrightarrow P\ : (A,q) = (P,p)\circ h;$$



where $(P,p)\circ h$ is to be understood as the cone $(A, p\circ h)$ obtained by composing all arrows in p with h.
Likewise (dually) (S,s) is said to be a limiting co-cone with basis φ, written $\varinjlim\varphi$, if it fulfils the following universal property: $\forall(B,\iota)\in\mathrm{V}_\varphi \exists!\ k : S \longrightarrow B\ : (B,\iota) = k\circ(S,s)$.

• If $\varprojlim\varphi$, [resp. $\varinjlim\varphi$] exist, for all $\varphi : \mathcal{I} \longrightarrow \mathcal{C}$, we say that $\mathcal{C}$ has $\mathcal{I}$-limits [resp. $\mathcal{I}$-co-limits].
• If $\mathcal{C}$ has $\mathcal{I}$-limits [resp. $\mathcal{I}$-co-limits] and if for all $\varphi : \mathcal{I} \longrightarrow \mathcal{C}$ there has been chosen a limit $\underleftarrow{l}(\varphi)$ [resp. $\underrightarrow{l}(\varphi)$] of φ, we say that $(\mathcal{C}, \underleftarrow{l})$ [resp. $(\mathcal{C}, \underrightarrow{l})$] is a category with chosen $\mathcal{I}$-limits [resp. $\mathcal{I}$-co-limits].

4 Sketches

We here introduce the basic notions concerning sketches.

• a *projective sketch* is a pair $(\mathcal{S}, \mathfrak{P})$ with $\mathcal{S}$ a multiplicative graph, called the underlying graph of the sketch, and $\mathfrak{P}$ is a set of cones in $\mathcal{S}$, called the set of distinguished cones.

• A *mixed sketch* is then a triple $(\mathcal{S}, \mathfrak{P}, \mathfrak{I})$ with a supplementary set $\mathfrak{I}$ of distinguished co-cones.

•*Sketch Morphism:* a *sketch morphism* $s : \sigma \longrightarrow \sigma'$ is a graph morphism such that any distinguished cone (co-cone) in σ is mapped to a distinguished cone (co-cone) in σ'. i.e. if a distinguished cone c in σ has basis $B : \mathcal{I} \longrightarrow \sigma$ then $s(c)$ will be a distinguished cone with basis $s \circ B$.

4.1 Models and sketchability

We here introduce the sketch theoretical notion of a model and how a sketch then can be viewed as describing/sketching a theory.

• A *model* of a sketch $\sigma = (\mathcal{S}, \mathfrak{P}, \mathfrak{I})$ ($\mathfrak{I}$ perhaps empty), is a functor $R : \mathcal{S} \longrightarrow \mathcal{C}$ from the underlying graph to a category $\mathcal{C}$ that transforms all distinguished cones and co-cones in σ into limits and co-limits in $\mathcal{C}$.
• A category $\mathcal{C}$ is called *sketchable* in a category $\mathcal{D}$ if there exist a sketch σ, such that we have an equivalence of categories

$$\mathrm{Mod}(\sigma, \mathcal{D}) \sim \mathcal{C}.$$

where $\mathrm{Mod}(\sigma, \mathcal{D})$ is the category with objects all models $\sigma \longrightarrow \mathcal{D}$ and arrows all natural transformations beween models, we then view $\mathcal{C}$ as the *theory* sketched by σ in $\mathcal{D}$.

•For a sketch morphism $t : \sigma \to \sigma'$ we will call *forgetful functor* the corresponding functor between model categories:

$$U_t : \mathrm{Mod}(\sigma', \mathrm{Set}) \longrightarrow \mathrm{Mod}(\sigma, \mathrm{Set})$$
$$R \longmapsto R \circ t$$

Generalised Associated Sheaf Theorem
For applications later we briefly state a general result concerning the existence of a left adjoint to a forgetful functor:

> Generalised associated sheaf theorem: *Given a sketch morphism $t : \sigma \to \sigma'$ between two projective sketches, then if σ' is small the forgetful functor U_t has a left adjoint.*

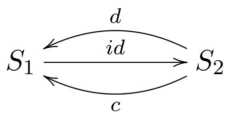
■

There is also a version of this result called *Kennison's Theorem* (stated in Barr and Wells's "Toposes, Triples and Theories", Springer 1985). The version here stated is from the french school of Charles Ehresmann and his students and the proof goes by a transfinite induction using Kan extensions.

The way of really understanding what a sketch is, is by sketching and sketching is what this thesis is all about. We start with some quite elementary examples which are also indispensable later on.

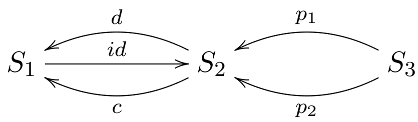
4.2 Sketching

Sketch of multiplicative graphs: when we consider the definition of a multiplicative graph, we see that there are the following four sets in play: *objects, arrows, consecutive arrows* and *composable arrows*, which indicate that a graphic version of "what a graph is" is to be modelled in Set and should contain four Objects: S_1 to be realized as the set of *objects*, S_2 to be realized as the set of *arrows* between these as well as the identity arrows. This is the oriented graph part which can be sketched as follows:

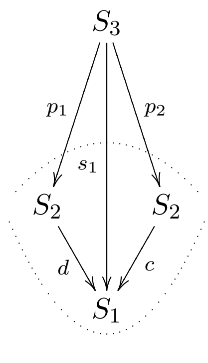


$$d \cdot id = c \cdot id = 1_{S_1}.$$

Then we add an object S_3 to be realized as the set of all consecutive arrows, the subset of all pairs of arrows (f, g) where the domain of f matches the co-domain of g. We get

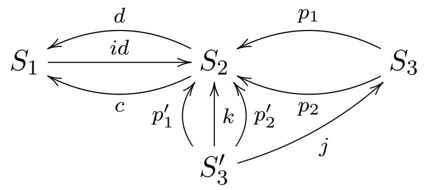


where p_1, p_2 will be the restrictions of the projections from the arrow product. If S_3 plays the wanted role, it should when modelled, by a model with underlying multiplicative graph $\mathcal{S}$, posses the universal property that for all arrows $f, g : A \longrightarrow Ar_{\mathcal{S}}$ with $d \cdot g = c \cdot f$ there exist a unique arrow $h : A \longrightarrow Ar_{\mathcal{S}} \times_{Ob_{\mathcal{S}}} Ar_{\mathcal{S}}$ such that $p_1 h = g, p_2 h = f$, where $Ar_{\mathcal{S}} \times_{Ob_{\mathcal{S}}} Ar_{\mathcal{S}}$ is to be understood as the fibered product over $Ob_{\mathcal{S}}$, the pullback of d, c modelled. This means that we need to distinguish the following cone:

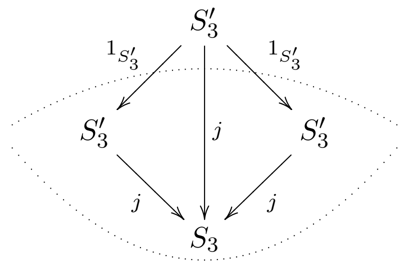


s_1 just naming the composites.
And at last we need an object S'_3 to be realized as the subset of the object of consecutive

arrows for which there is defined a composition. We get



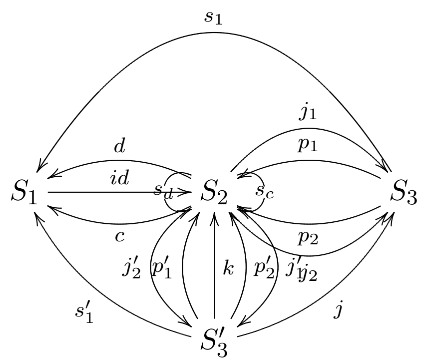
$p_1 j = p'_1, p_2 j = p'_2$ just further restricting. Now to make sure that S'_3 is realized as a subset of consecutive arrows, we distinguish the following cone



meaning that j is realized as a monomorphism (injection in Set). Now the composition is subject to some axioms (unitarity, position), giving us the need for maps $j_1, j_2 : S_2 \longrightarrow S_3$ intuitively sending an arrow to $(1_{c(f)}, f)$ resp. $(f, 1_{d(f)})$, i.e. we have equations:

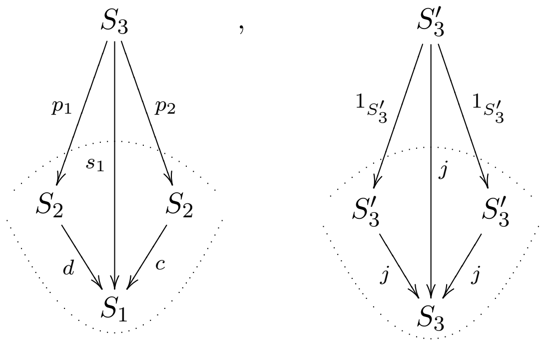
$$\begin{aligned} p_2 \cdot j_1 &= 1_{S_2}, \quad p_1 \cdot j_2 = 1_{S_2} \\ p_1 \cdot j_1 &= id \cdot c, \quad p_2 \cdot j_2 = id \cdot d \\ \text{and then } j'_1, j'_2 : S_2 \longrightarrow S'_3 : \quad j \cdot j'_1 &= j_1, \quad j \cdot j'_2 = j_2 \\ \text{unitarity:} \quad k \cdot j'_1 &= 1_{S_2} = k \cdot j'_2 \\ \text{position:} \quad d \cdot k &= d \cdot p'_2 \cdot j, \quad c \cdot k = c \cdot p'_1 \cdot j. \end{aligned}$$

All in all we get the underlying multiplicative graph of the sketch of multiplicative graphs:



$$\begin{aligned} \text{Equations:} \quad p_1 \cdot j &= p'_1 & c \cdot i = d \cdot i = 1_{S_1} \\ p_2 \cdot j &= p'_2 & d \cdot p'_1 = c \cdot p'_2 = s'_1 \\ j \cdot j'_1 &= j_1 & p_1 \cdot j_1 = id \cdot d = s_d \\ j \cdot j'_2 &= j_2 & p_2 \cdot j_2 = id \cdot c = s_c \\ p_2 \cdot j_1 &= 1_{S_2} & k \cdot j'_1 = 1_{s_2} = k \cdot j'_2 \\ p_1 \cdot j_2 &= 1_{S_2} \end{aligned}$$

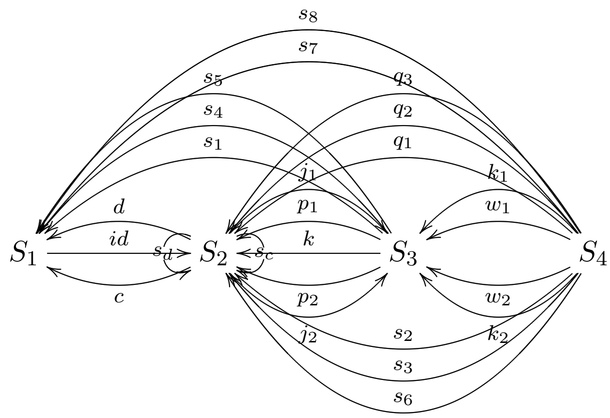
And distinguished projective cones:



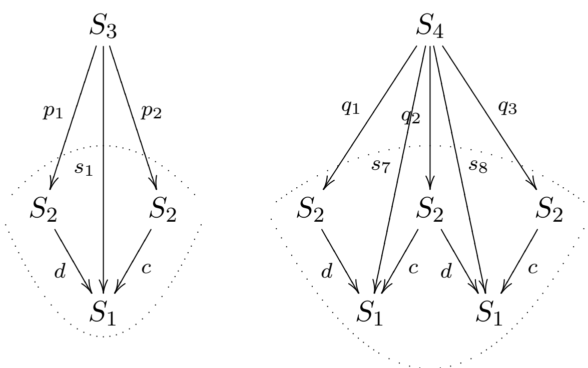
both indexed by the graph $\cdot \rightarrow \cdot \leftarrow \cdot$.
So if call this sketch $\mathfrak{g}$ we have $\mathrm{Mod}(\mathfrak{g}, \mathrm{Set}) \cong \mathrm{Graph}$.

Sketch of categories: We can continue the above sketch of graphs to obtain the sketch of categories by making all consecutive arrows composable ($S_3 = S'_3$) and adding a fourth object to be modelled as consecutive triples of arrows and then equations to describe the associativity of the composition. We get:

Underlying graph :



Distinguished projective cones :

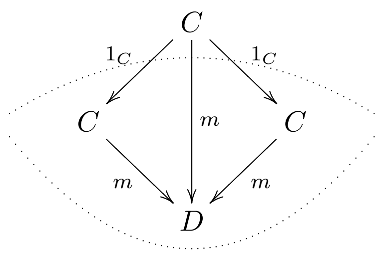


where the second and new cone is the cone assuring that, in any set theoretical model, S_4 will be mapped to the the set of consecutive triples of arrows (up to isomorphism). The arrows of the sort s_i are just naming the composites.

$$
\begin{aligned}
\text{Equations:} \qquad c \cdot i &= d \cdot i = 1_{S_1} \qquad d \cdot p_1 = c \cdot p_2 = s_1 \\
p_2 \cdot j_1 &= 1_{S_2} \qquad p_2 \cdot j_2 = i \cdot c = s_c \\
p_1 \cdot j_1 &= i \cdot d = s_d \qquad p_1 \cdot j_2 = 1_{S_2} \\
\\
p_1 \cdot w_1 &= q_1 \quad p_1 \cdot w_2 = q_2 \quad p_2 \cdot w_1 = q_2 \quad p_2 \cdot w_2 = q_3 \\
p_1 \cdot k_1 &= k \cdot w_1 = s_2 \qquad p_1 \cdot k_2 = q_1 \\
p_2 \cdot k_1 &= q_3 \qquad p_2 \cdot k_2 = k \cdot w_2 = s_3 \\
\\
c \cdot k &= c \cdot p_1 = s_4 \qquad k \cdot j_1 = k \cdot j_2 = 1_{S_2} \\
d \cdot k &= = d \cdot p_2 = s_5 \qquad k \cdot k_1 = k \cdot k_2 = s_6 \\
\\
d \cdot q_1 &= c \cdot q_2 = s_7 \qquad d \cdot q_1 = c \cdot q_3 = s_8
\end{aligned}
$$

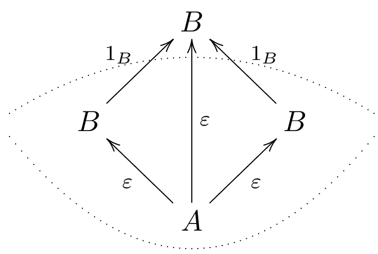
We will come back to the intuitive meaning of the arrows, in the underlying multiplicative graph, later when we in the examples make use of this sketch. Though a careful reading of the equations should suffice.

When sketching, we will often need to point out that certain arrows in the underlying multiplicative graph should be realized as monomorphisms (as we saw in the sketch of graphs) or epimorphisms. To avoid too much unnecessary repetition we introduce the notation $C \xrightarrow{m^{\rightarrowtail}} D$ to mean that m is distinguished as potential monomorphism, i.e. the following cone is distinguished:



meaning that any model/realisation will render this diagram a pullback, and hence as is well known and easy to see, the map m sketchs a monomorphism.

Likewise we introduce the notation $A \xrightarrow{\varepsilon^{\twoheadrightarrow}} B$ to distinguish the fact that ε is to be realized as an epimorphism, meaning that we distinguish the following co-cone:



then any model will render the diagram a push-out diagram and hence again, as is well known and easy to see, the arrow ε sketchs an epimorphism.

Furthermore, when we illustrate a sketch by a diagram we usually present a generating sample with an explanation. For example we normally don't include arrows signifying composites (arrows such as the s_i's in the above).

5 A category of sketches

In this section we show that the category of sketches is sketchable with a projective sketch, when we limit the size of the sketches by some large cardinal λ. The category of sketches thus arrived at, as the model category of set theoretical models of a projective sketch, will then serve as the frame for our actions.
We shall present the projective sketch of sketches and then look at how we then think of a mixed sketch as a set theoretical model of a projective sketch (within the size limitation).

As mentioned, the category of mixed sketches is sketchable projectively, at least, when we limit the size of the sketches by some cardinal λ, meaning that the sets of distinguished cones and of distinguished co-cones are smaller than λ, plus all index-graphs $\mathcal{I}$ should be λ-small, i.e. $Ar_{\mathcal{I}}$ smaller that λ. We will call the category of mixed λ-small sketches with sketch morphisms Esq_λ (Esq as in the french word for sketch "esquisse"),I could have chosen the notation "Sketch_λ", but it seemed natural to me to leave some french trace concerning the main ingredient.

Let $\epsilon_\lambda = (\mathcal{S}_{\epsilon_\lambda}, \mathfrak{P}_{\epsilon_\lambda})$ denote the projective sketch of mixed λ-small sketches for some arbitrary cardinal λ.
ϵ_λ is constructed to be modelled in Set, so the size limitation is a way of avoiding paradoxical notions such as "the set of all sets", i.e. we wish the result: for all cardinals λ there exists a sketch ϵ_λ such that there is an isomorphism:

$$\text{Mod}(\epsilon_\lambda, \text{Set}) \cong \text{Esq}_\lambda$$

Remark that we wish isomorphism and not just equivalence, we return to this in the next section. First the sketch of Esq_λ will be constructed.

5.1 Sketching sketches

I will go through the construction of ϵ_λ according to the way it is presented by C. Lair and L. Coppey in their "lecon 3. Esquisses" (Coppey, L., Lair, C. [1984]). In fact it is almost a direct translation of their presentation. But it is indispensable to the context of this paper to have this construction at hand, and in matching notation.
So take λ a cardinal and $\mathfrak{J}, \mathfrak{J}'$ two sets of multiplicative graphs such that:

$$\begin{array}{rrrrcl} & & |\mathfrak{J}|, & |\mathfrak{J}'| & < & \lambda \\ \text{and for all} & \mathcal{I} \in \mathfrak{J}, \quad \mathcal{I}' \in \mathfrak{J}': & |\mathcal{I}|, & |\mathcal{I}'| & < & \lambda \end{array}$$

then we set out to sketch the category of all small $\{\mathfrak{J}, \mathfrak{J}'\}$-mixed sketches, meaning $\mathfrak{J}$-projective and $\mathfrak{J}'$-injective.
To simplify notation and better understand the idea of the construction we first look at the sketch of small $\{\mathcal{I}\}$-projective sketches, for some multiplicative graph $\mathcal{I} \in \mathfrak{J}$. Let us denote

this (generating) sketch $\epsilon_{\mathcal{I}} = (\mathcal{S}_{\mathcal{I}}, \mathfrak{P}_{\mathcal{I}})$.
Now the underlying multiplicative graph $\mathcal{S}_{\mathcal{I}}$ will at least contain a copy of the sketch of multiplicative graphs, destined to determine the underlying graph of any $\{\mathcal{I}\}$-projective sketch (any model of $\epsilon_{\mathcal{I}}$ in Set). So the only thing needing specification is the choice of $\{\mathcal{I}\}$-projective distinguished cones, and to this we need:

- a special object; "object of projective $\mathcal{I}$-cones, thus denoted $\mathcal{S}^{\mathcal{I}^-}$, where $\mathcal{S}$ is meant to model the underlying multiplicative graph of an arbitrary $\{\mathcal{I}\}$-projective sketch.

- a special arrow $C_{\mathcal{I}} \xrightarrow{m^{\rightarrowtail}} \mathcal{S}^{\mathcal{I}^-}$ where the domain $C_{\mathcal{I}}$ will be modelled as the subobject of projective $\mathcal{I}$-cones, to be distinguished; m is the potential monomorhism that distinguishes cones.

Now $\mathcal{S}^{\mathcal{I}^-}$ is destined to describe the set of functors from $\mathcal{I}^-$ to $\mathcal{S}$ in some realisation R of $\epsilon_{\mathcal{I}}$ so $\mathcal{S}$ $(= R\mathcal{S}_{\mathcal{I}})$ is the underlying multiplicative graph of $R\epsilon_{\mathcal{I}}$. m is destined to be the injection choosing the distinguished cones among the objects of $\mathcal{S}^{\mathcal{I}^-}$, or more precisely choosing bases of cones to be distinguished. Thus we need distinguish the fact that m is to be realised as a monomorphism (injection in Set) and this is indicated in the manner described in the introduction, by writing $m^{\rightarrowtail}$.

Now we take a closer look at how the object $\mathcal{S}^{\mathcal{I}^-}$ is to be connected to the sketch of graphs, to complete $\epsilon_{\mathcal{I}}$.
We will do this with an arbitrary graph $\mathcal{I}$. The point is to sketch the object of functors from $\mathcal{I}$ to $\mathcal{S}$ (functors/homomorphisms of multiplicative graphs). The set $\mathcal{S}^{\mathcal{I}}$ is determined by the existence of three arrows:

$$Ob_{\mathcal{I}} \longrightarrow Ob_{\mathcal{S}}; \quad I \longmapsto S_I$$

$$Ar_{\mathcal{I}} \longrightarrow Ar_{\mathcal{S}}; \quad i \longmapsto s_i$$

$$Co_{\mathcal{I}} \longrightarrow Co_{\mathcal{S}}; \quad (i', i) \longmapsto t_{i',i}$$

satisfying:

$$\begin{array}{rrcl} \forall i \in Ar_{\mathcal{I}}: & c(s_i) & = & S_{c(i)} \\ & d(s_i) & = & S_{d(i)} \\ \forall (i', i) \in Co_{\mathcal{I}}: & t_{(i',i)} & = & (s_{i'}, s_i). \end{array}$$

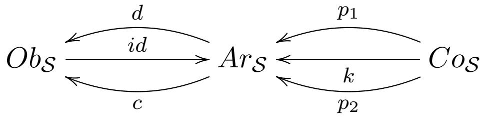
If k denotes the partial composition of $\mathcal{S}$, we have:

$$kt_{(i',i)} = s_{i'} \circ s_i = s_{i' \circ i} \quad \text{and of course} \quad S_{1_I} = 1_{S_I}.$$

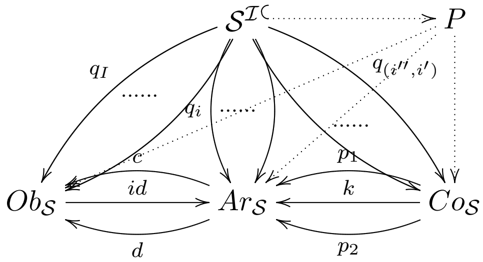
In this way $\mathcal{S}^{\mathcal{I}}$ can be viewed as a certain subset of the following product:

$$P = Ob_{\mathcal{S}}^{Ob_{\mathcal{I}}} \times Ar_{\mathcal{S}}^{Ar_{\mathcal{I}}} \times (Co_{\mathcal{S}})^{Co_{\mathcal{I}}}$$

So we can take the restrictions of the canonical projections from P evaluated at some object, some arrow and some composable pair, meaning that we get a triple of projections for all triples $(I, i, (i', i''))$ commuting with some arrows in

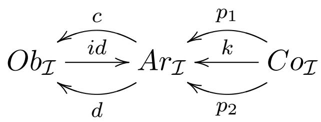


So in Set the cone will look like this:



then q_I is the restriction of the map $P \xrightarrow{\pi_1} Ob_{\mathcal{S}}^{Ob_{\mathcal{I}}} \xrightarrow{ev_I} Ob_{\mathcal{S}}$ (where π_1 is the canonical projection of the product P and ev_I is the evaluation in I of a map).
More precisely the cone will be indexed by the multiplicative graph $\S\S(\mathcal{I})$:

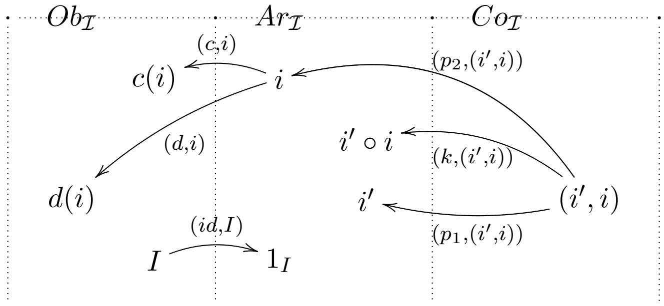
$Ob_{\S\S(\mathcal{I})}$:= disjoint union of $Ob_{\mathcal{I}}, Ar_{\mathcal{I}}$ and $Co_{\mathcal{I}}$

$Ar_{\S\S(\mathcal{I})}$:= pairs (z, x) $x \in Ob_{\S\S(\mathcal{I})}$ and z an arrow in 

such that $z(x)$ is defined; then x is the domain and $z(x)$ the co-domain.

$\S\S(\mathcal{I}) * \S\S(\mathcal{I})$:= pairs $((z', x'), (z, x))$ such that $z' \circ z$ is defined and $z(x) = x'$ then $(z', x') \circ (z, x) = (z' \circ z, x)$.

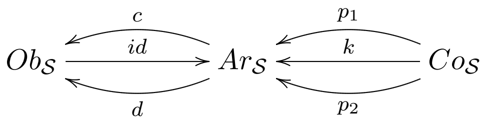
This can be illustrated by the following diagram of "generating samples":



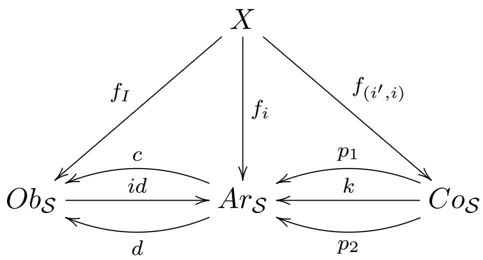
Now the above graph-homomorphism conditions can be described in terms of compositions of arrows:

$$\begin{array}{rrcl} \forall i \in Ar_{\mathcal{I}}: & c \circ q_i & = & q_{c(i)} \\ & d \circ q_i & = & q_{d(i)} \\ \forall (i', i) \in Co_{\mathcal{I}}: & p_1 \circ q_{(i',i)} & = & q_{p_1(i',i)} = q_{i'} \\ & p_2 \circ q_{(i',i)} & = & q_{p_2(i',i)} = q_i \\ & k \circ q_{(i',i)} & = & q_{i' \circ i} \\ \forall I \in Ob_{\mathcal{I}}: & id \circ q_I & = & q_{id(I)} = q_{1_I}. \end{array}$$

In fact $\mathcal{S}^{\mathcal{I}}$ is, when modelled in Set, the vertex of the limit with image of basis



since if one take any $f_I, f_i, f_{(i',i)}$ such that the diagram



commutes (with respect to the basis §§($\mathcal{I}$)), then there is a unique $h : X \longrightarrow \mathcal{S}^{\mathcal{I}}$ such that

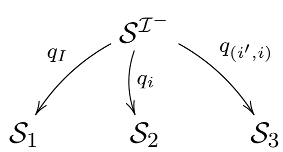
$$(\mathcal{S}^{\mathcal{I}}, q) \circ h = (X, f),$$

defined by:

$$h(x) = \big(I \longmapsto f_I(x), i \longmapsto f_i(x), (i', i) \longmapsto f_{(i',i)}(x)\big)$$

So we need distinguish a cone indexed by §§($\mathcal{I}$) to get our sketch $\epsilon_{\mathcal{I}}$, which will then be achieved by taking the sketch of graphs and add the following:

- the objects C and $\mathcal{S}^{\mathcal{I}^-}$; the arrows $C \xrightarrow{m^{\rightarrowtail}} \mathcal{S}^{\mathcal{I}^-}$ and

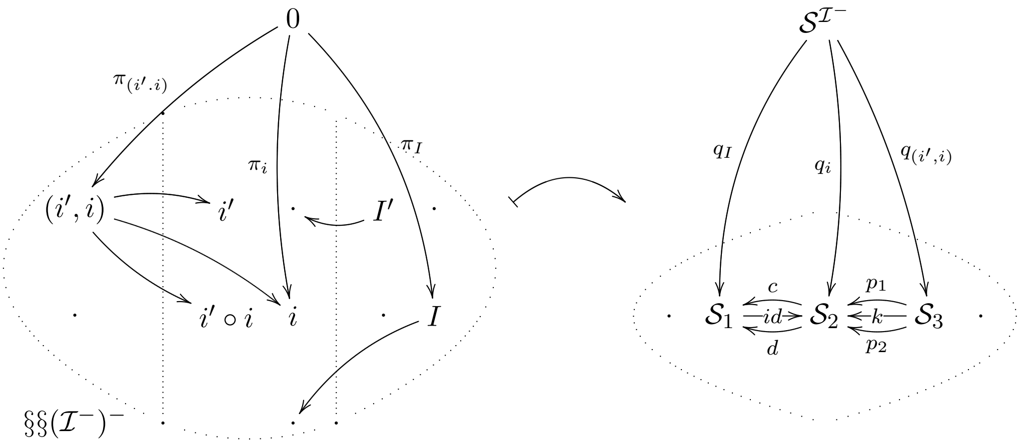


where $\begin{cases} I \in Ob_{\mathcal{I}^-} \\ i \in Ar_{\mathcal{I}^-} \\ (i', i) \in \mathcal{I}^- * \mathcal{I}^- \end{cases}$

- equations:

$$\begin{array}{ll} \forall i \in Ar_{\mathcal{I}^-}: & c \circ q_i = q_{c(i)} \\ & d \circ q_i = q_{d(i)} \\ \forall (i', i) \in \mathcal{I}^- * \mathcal{I}^-: & p_1 \circ q_{(i',i)} = q_{i'} \\ & p_2 \circ q_{(i',i)} = q_i \\ & k \circ q_{(i',i)} = q_{i' \circ i} \\ \forall I \in Ob_{\mathcal{I}^-}: & id \circ q_I = q_{1_I} \end{array}$$

- distinguished cone:



Hence we have established that $\epsilon_{\mathcal{I}}$ is a $\{\cdot \rightarrow \cdot \leftarrow \cdot, \S\S(\mathcal{I}^-)\}$-projective sketch.

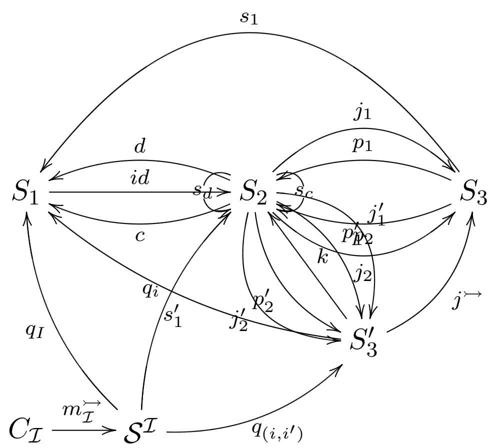
From the generality of the construction of projective cone-diagrams $\mathcal{S}^{\mathcal{I}^-}$ (or just cone diagrams), we see that introducing an object representing inductive cone-diagrams $\mathcal{S}^{\mathcal{I}^+}$ (or just co-cone diagrams) amounts to the same. Hence the general sketch ϵ_λ, is a matter of notation: to the sketch of multiplicative graphs we add

-Objects $\mathcal{S}^{\mathcal{I}^-}, C_{\mathcal{I}} \quad \forall \mathcal{I} \in \mathfrak{I}$ and $\mathcal{S}^{\mathcal{I}'^+}, C_{\mathcal{I}'} \quad \forall \mathcal{I}' \in \mathfrak{I}'$

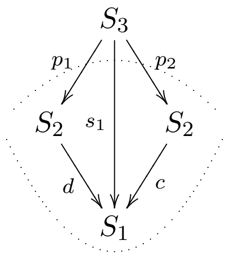
-Arrows $\{ \ C_{\mathcal{I}} \xrightarrow{m^{\rightarrow}_{\mathcal{I}}} \mathcal{S}^{\mathcal{I}^-} \ \}_{\mathcal{I} \in \mathfrak{I}}$ and $\{ \ C_{\mathcal{I}'} \xrightarrow{m^{\rightarrow}_{\mathcal{I}'}} \mathcal{S}^{\mathcal{I}^+} \ \}_{\mathcal{I}' \in \mathfrak{I}'}$

-Cones as above indexed by $\S\S(\mathcal{I}^-)$, $\S\S(\mathcal{J}^+)$ for each $\mathcal{I} \in \mathfrak{I}, \quad \mathcal{J} \in \mathfrak{I}'$.

A (generating) diagram of the underlying graph of the sketch ϵ_λ is thus:
Underlying graph:



Distinguished projective cone:



and for all $\mathcal{I} \in \mathfrak{I}$, $\mathcal{J} \in \mathfrak{I}'$ the projective cones with vertexes $\mathcal{S}^{\mathcal{I}^-}$ and $\mathcal{S}^{\mathcal{J}^+}$ as described above.
Equations:

$$
\begin{array}{rclc}
p_1 \circ j & = & p_1' & c \circ i = d \circ i = 1_{S_1} \\
p_2 \circ j & = & p_2' & d \circ p_1' = c \circ p_2' = s_1' \\
j \circ j_1' & = & J_1 & p_1 \circ j_1 = i \circ d = s_d \\
j \circ j_2' & = & j_2 & p_2 \circ j_2 = i \circ = s_c \\
p_2 \circ j_1 & = & 1_{S_2} & k \circ j_1' = 1_{s_2} = k \circ j_2' \\
p_1 \circ j_2 & = & 1_{S_2} &
\end{array}
$$

plus equations from the above additions concerning "choice" of distinguished cones, i.e. for all $\mathcal{I} \in \mathfrak{I}, \mathcal{J} \in \mathfrak{I}'$:

$$
\begin{array}{ll}
\forall i \in Ar_{\mathcal{I}^-} \cup Ar_{\mathcal{J}^+} : & c \circ q_i = q_{c(i)} \\
 & d \circ q_i = q_{d(i)} \\
\forall (i', i) \in Co_{\mathcal{I}^-} \cup Co_{\mathcal{J}^+} : & p_1 \circ q_{(i',i)} = q_{i'} \\
 & p_2 \circ q_{(i',i)} = q_i \\
 & k \circ q_{(i',i)} = q_{i' \circ i} \\
\forall I \in Ob_{\mathcal{I}^-} \cup Ob_{\mathcal{J}^+} : & id \circ q_I = q_{1_I}
\end{array}
$$

Now we obviously obtain an isomorphism of categories

$$
\boxed{\mathcal{R}_\lambda : \mathrm{Esq}_\lambda \xrightarrow{\cong} \mathrm{Mod}(\epsilon_\lambda, \mathrm{Set})}
$$

hence ϵ_λ is sketchable in Set projectively, as claimed.

Having done the construction of ϵ_λ, we can rather easily justify that ϵ_λ itself can be viewed as a member of Esq_λ: we have seen that ϵ_λ is a $\{\cdot \rightarrow \cdot \leftarrow\} \cup \{\S\S(\mathcal{I}^-)\}_{\mathcal{I} \in \mathfrak{I}} \cup \{\S\S(\mathcal{J}^+)\}_{\mathcal{J} \in \mathfrak{I}'}$-projective sketch, so the question is whether we can consider$\{\cdot \rightarrow \cdot \leftarrow \cdot\} \cup \{\S\S(\mathcal{I}^-)\}_{\mathcal{I} \in \mathfrak{I}} \cup \{\S\S(\mathcal{J}^+)\}_{\mathcal{I}' \in \mathfrak{I}'}$ to be included in $\mathfrak{I}$. We have:

$$
\begin{array}{rcl}
|Ob_{\S\S(\mathcal{I})}| & \leq & |Ob_{\mathcal{I}} \times Ar_{\mathcal{I}} \times Co_{\mathcal{I}}| < \lambda^3 \\
|Ar_{\S\S(\mathcal{I})}| & \leq & |\{z, x) | x \in Ob_{\S\S(\mathcal{I})}, z \in \{id, c, d, p_1, p_2, k\}\}| < \lambda \times 6 \\
|\S\S(\mathcal{I}) * \S\S(\mathcal{I})| & \leq & |Ar_{\S\S(\mathcal{I})} \times Ar_{\S\S(\mathcal{I})}| < \lambda^2
\end{array}
$$

so as long as we suppose λ to be infinite, there shouldn't be a problem, as well as there is no problem in supposing this.

5.2 Sketches as models of a projective sketch

We now look at how the above isomorphism $\mathcal{R}_\lambda$ will allow us to consider an object in Esq_λ (a mixed λ-small sketch) as an arrow i in Esq_λ.
We saw that ϵ_λ is a member of Esq_λ, so if we abusively consider the category Set as a member of Esq_λ; as the mixed sketch with underlying multiplicative graph Set and distinguished cones and co-cones all λ-small limits and co-limits in Set, then the above isomorphism of categories, saying that we can jump between sketches as objects in Esq_λ and corresponding models of the sketch of mixed λ-sketches into Set, can be viewed (locally) as taking place inside Esq_λ. This is abusive since we don't pretend to consider Set as a model of ϵ_λ in Set, which would obviously lead us into trouble. We rather consider Set as a very big sketch into which we would like to model sketches (the real members of Esq_λ). We join Set to the category Esq_λ remembering that Esq_λ with Set as extra object is not not sketchable by ϵ_λ.
We can perhaps think of Set as an added semi-final object in Esq_λ, since we always have a set theoretical model, i.e. always an arrow to Set (but it is rarely unique). Or just think of Set as the big and rich *reference category*, since we are often guided by set theoretical properties when sketching. But most importantly we just want to consider set theoretical models as arrows in Esq_λ.

In fact we can consider any λ-small category as a member of Esq_λ, by taking the category as underlying multiplicative graph and certain limits and co-limits as distinguished cones and co-cones, within the given size-frame.
By this view, the notation such as Set^σ will be understood as "hom-sets" in Esq_λ, i.e all sketch morphisms from σ to Set; all set theoretical models of σ, that we until now have written $\text{Mod}(\sigma, \text{Set})$. The point is that sketch morphisms into a category (viewed as a sketch) form the special sketch morphism we call models. Thus whenever we have a λ-small category $\mathcal{C}$ and a sketch $\sigma \in \text{Esq}_\lambda$ we can write the category of models as a hom-set $\text{Hom}_{\text{Esq}_\lambda}(\sigma, \mathcal{C})$ or as category of functors $\mathcal{C}^\sigma$.

6 Theoretical types

In this section we introduce the central notion *theoretical type* and then we go through the two main purposes of this notion, first on the syntactic side concerning the nature of proofs in sketches, and then the semantic side concerning the realization of proofs; when do we have proofs?

But first we will make precise what we mean by a *theoretical type*. As mentioned in the introduction the idea of theoretical types is essentially just a formal way of distinguishing mathematical theories according to their structural properties. Since we here can view mathematical theories as sketches (categories viewed as sketches), we let the notion of type concern all sketches in Esq_λ. Or one could consider this as an abstraction of the notion of a theory to a sketch, since we can consider sketch morphisms as realizations so in a way, any sketch is realizing some sketch (at least itself).

The formalization presented here will thus take place in Esq_λ, and we will think of types of theories, as a typification of sketches by extensions of their corresponding model of ϵ_λ (via R_λ) along a sketch morphism specifying the "typical properties" in question.

We consider a morphism of λ-small projective sketches

$$t : \epsilon_\lambda \longrightarrow \tau$$

we can think of t as a condition perhaps satisfied by a theory specified by some mixed sketch σ. Then the category of set theoretical models of τ in Set^τ is called Type_t, the category of t-types.

• *Theoretical type:* We say that a sketch σ in Esq_λ is of *type t* (or satisfies t) if the corresponding model $\mathcal{R}_\lambda(\sigma) : \epsilon_\lambda \longrightarrow \mathrm{Set}$ factorizes through t by a sketch morphism $\mathcal{R}_\lambda(\sigma)^t : \tau \longrightarrow \mathrm{Set}$, i.e. the following diagram commute:

$$\begin{array}{ccc} \epsilon_\lambda & \xrightarrow{t} & \tau \\ & {\scriptstyle \mathcal{R}_\lambda(\sigma)} \searrow \quad \swarrow {\scriptstyle \mathcal{R}_\lambda(\sigma)^t} & \\ & \mathrm{Set} & \end{array}$$

•*Strict theoretical type:* Then we say that σ is of *strict type t* if the above *factorization is unique.*
My work only considers these strict types, since they are much simpler to handle by way of the application of the generalized associated sheaf theorem, as we will see and they are in fact not so terribly restrictive.

• We call *model of type t*, or *t-model,* a model $R : \sigma \longrightarrow \mu$ of a sketch σ into a category μ of type t.

Now, any type t gives rise to the forgetful functor

$$U_t : \mathrm{Set}^\tau \longrightarrow \mathrm{Set}^{\epsilon_\lambda}; \quad \mathrm{Type}_t \longrightarrow \mathrm{Esq}_\lambda \quad .$$
$$R \longmapsto R \circ t; \quad R(\tau) \longmapsto R \circ t(\epsilon_\lambda)$$

Since t is a morphism between projective λ-small sketches we get, by the generalized associated sheaf theorem, that U_t has a left adjoint

$$L_t : \mathrm{Set}^{\epsilon_\lambda} \longrightarrow \mathrm{Set}^\tau$$

and the unit of this adjunction

$$\eta_t : 1_{\mathrm{Set}^{\epsilon_\lambda}} \longrightarrow U_t \circ L_t$$
$$\eta_t(\sigma) : \mathcal{R}_\lambda(\sigma) \longrightarrow L_t(\mathcal{R}_\lambda(\sigma)) \circ t$$

determines for each sketch $\sigma \in \mathrm{Esq}_\lambda$ a realization $L_t(\mathcal{R}_\lambda(\sigma)) \circ t : \epsilon_\lambda \longrightarrow \mathrm{Set}$. We denote the corresponding sketch $T_t(\sigma)$ (i.e. $\mathcal{R}_\lambda(T_t(\sigma)) = L_t(\mathcal{R}_\lambda(\sigma)) \circ t$), $T_t(\sigma)$ is clearly of type t since we have

$$\begin{array}{ccc} \epsilon_\lambda & \xrightarrow{t} & \tau \\ & {\scriptstyle \mathcal{R}_\lambda(T_t(\sigma))} \searrow \quad \swarrow {\scriptstyle L_t(\mathcal{R}_\lambda(\sigma))} & \\ & \mathrm{Set} & \end{array}$$

The isomorphism $\mathcal{R}_\lambda : \mathrm{Esq}_\lambda \cong \mathrm{Set}^{\epsilon_\lambda}$ permits us to write the unit as $\eta_t(\sigma) : \sigma \longrightarrow T_t(\sigma)$ which we will refer to as the constructor of the theoretical t-type of σ. Since $(L_t\mathcal{R}_\lambda(\sigma), \eta_t(\sigma))$ is often called the U_t-free structure generated by σ, we call $T_t(\sigma)$ the freely generated theoretical t-type of σ (a sort of t-completion of σ). Clearly σ is of type t if and only if $\eta_t(\sigma)$ is an isomorphism.
Furthermore, if t is strict, it extends (set theoretical) models of ϵ_λ uniquely and we have that U_t is an embedding, hence strict types Type_t for t strict, are full sketchable subcategories of Esq_λ. And, for all sketches μ, of a strict type t, we have the natural isomorphism

$$(\bigstar) \qquad \mathrm{Hom}(T_t(\sigma), \mu) \cong \mathrm{Hom}(\sigma, \mu)$$

by the universal property of the adjunction. Meaning in particular that two sketches have isomorphic freely generated t-types for a strict type t if and only if they have isomorphic model categories for models of type t. This means that asking if two sketches have the equivalent model categories for models of a certain strict type t, is the same as asking wether the two sketches have equivalent freely generated sketches (theories) of this type.

There are two aspects of this application (supporting our mottos), one on the syntactic level: we get a description of theorems and proofs in Esq_λ, and on the other on the semantical level: we get a way to discriminate in Esq_λ telling us when a given theorem is valid; when does it have a proof.

6.1 Types and syntax: proofs

We here look at the first aspect of the above application of *generalized associated sheaf theorem* concerning the purpose of strict types on the syntactic level. The application leads us to a purely diagrammatic conception of proofs, as well as envisaging that *sketching is proving and proving is sketching.*
The isomorphism ($\bigstar$) allows us to consider a proof of a theorem in a theory specified by a sketch σ, as a *progression towards a type.* Meaning that if we wish to proof a theorem for all t-models of σ for a strict type t, it becomes a question of enlarging σ (sketching towards $T_t(\sigma)$) in a way such that the enlarged sketch σ' has the same t-models as σ and such that it is evident (!) that the theorem is valid in all t-models of σ'. We then say that σ' contains the general proof of a theorem relative to σ and the progression from σ to σ' is proving this theorem for all t-models of σ. More precisely:
Let Θ be a theorem relative to a sketch σ, that we wish to prove for all t-models for some strict type t, then this proof π can be achieved in an *environment* (a sketch $T_t(\sigma)_{\pi_N}$, like σ' in the above) in between σ and $T_t(\sigma)$ by N finite steps

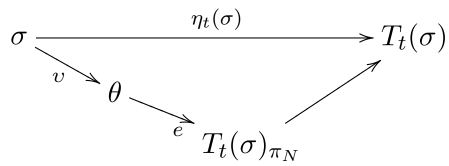
$$\sigma \longrightarrow T_t(\sigma)_{\pi_1} \longrightarrow \cdots \longrightarrow T_t(\sigma)_{\pi_N} \longrightarrow T_t(\sigma).$$

Meaning that obtaining a proof in N steps, becomes a question of obtaining a factorization

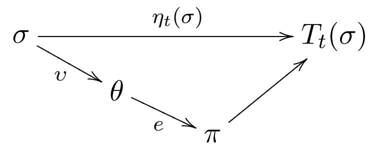
$$\begin{array}{ccc} \sigma & \xrightarrow{\eta_t(\sigma)} & T_t(\sigma) \\ & \searrow \quad \nearrow & \\ & T_t(\sigma)_{\pi_N} & \end{array}$$

such that $T_t(\sigma)_{\pi_N}$ evidently(!) satisfies Θ. Because the above application of *generalized associated sheaf theorem* gives that all realizations of σ factor uniquely through a realization of

$T_t(\sigma)$, thus also through a realization of $T_t(\sigma)_{\pi_N}$. Hence all realisations of σ satisfy Θ.
Take a theorem Θ relative to σ such that Θ can be formulated via a sketch morphism $\upsilon : \sigma \longrightarrow \theta$, for a model $R : \sigma \longrightarrow \mu$ of strict type t, in the following manner: R factorizes through υ, i.e. there exist a realization $R^{\upsilon} : \theta \longrightarrow \mu$ such that $R = R^{\upsilon}\upsilon$. Then, establishing a proof amounts to constructing a sort of captivation of υ by $\eta_t(\sigma)$; a factorization:



e is the *evidence* arrow, the *finite* progression from θ to $T_t(\sigma)_{\pi_N}$ making sure that the sketch $T_t(\sigma)_{\pi_N}$ evidently(!) satisfies the theorem Θ (indicated by the sketch morphism υ), for models of type t.
At last we simply put $\pi = T_t(\sigma)_{\pi_N}$ and consider a proof, of a theorem via $\upsilon : \sigma \longrightarrow \theta$ relative to a sketch σ, as a factorization



where the arrow $e \circ \upsilon$ should be finite; the progressing from σ to π via θ should be achieved by a finite series of sketch mutations towards a sketch with the same free t-type. So proving is sketching: *Progressing (sketching) towards a sketch (π) from a sketch (σ), via a sketch morphism (υ), in a restricted manner (factorizing $\eta_t(\sigma)$).* This idea of general proof by syntactic progression should be more clear in view of the examples in section 7. below.

We could now be tempted to call any sketch morphism $\sigma \longrightarrow \kappa$ *a conjecture* relative to the sketch σ, since, by the above, any sketch morphism can be viewed as a potential theorem. It is just a matter of investigating what semantics are necessary in order to achieve a theorem, e.g if σ and κ have the same t-model categories for some strict type t, then any property evidently satisfied in all t-models of κ gives rise to a theorem relative to σ for t-models. Whether this is interesting of course depends on the sketch κ. We return to this in the example section.
First we look some more at this other purpose of types, concerning the semantic level and the interplay between semantics and syntax in Esq_{λ}.

6.2 Types and semantics: discrimination

This section is about the other important purpose of theoretical types, that when we sketch to describe a certain theory we at the same time envisage the properties needed in order to realize our sketch (as when we sketch set theoretically in the previous sections). This leads to certain types, hence a category Type_t for a strict type t is often arrived at as the category of possible semantics for a certain sketch. Or, perhaps more interestingly, arrived at as the type of semantic in which a theorem via a sketch morphism is valid, which amount to a type in which two sketches (σ and π in the previous section) have the same model categories.
So we are interested in at least two situations:

- Given two different sketches of the same structure, find out what type of models "ignore" this difference, i.e. give rise to equivalent model categories.
- Given a sketch σ and a theorem Θ via a sketch morphism $\upsilon : \sigma \longrightarrow \theta$, find out what type of models "ignore" the difference between σ and the sketch π containing a general proof of Θ.

This leads to the introduction of two concepts.

Confusion and discrimination of sketches: Given two sketches σ, θ, if there exists a third sketch η and a sketch-morphism $\upsilon : \sigma \longrightarrow \theta$, such that

$$\eta^{\upsilon} : \eta^{\theta} \longrightarrow \eta^{\sigma}$$
$$R \longmapsto R \circ \upsilon$$

is an equivalence of categories, then we will say that, via υ, η *confuses* σ and θ. If not we will say that, via υ, η *discriminate* between σ and θ.

Remark: The discrimination will rather depend on the sketch η, than the morphism υ, meaning that the difference between two sketches rather depends on the structural richness/poverty of the type of category we wish to model in. Said differently the discrimination takes place on a semantical level, but then again it depends on υ how easy it is to determine whether η confuse or discriminate.
We are thus interested in the following general problem.

General problem: Given sketches σ and θ in Esq_{λ}, find a strict type $t : \epsilon_{\lambda} \longrightarrow \tau$ such that all sketches of the type t confuses σ and θ. Then find at least one sketch ν discriminating σ and θ, meaning in particular that ν is not of type t; ν and $T_t(\nu)$ (its freely generated t-type) are non-isomorphic. Then one can amuse oneself in finding out what $T_t(\nu)$ resembles, either by calculating the Kan extension or by more ad hoc methods, to see how gravely it differs.

A third situation permitted by the application of the generalized associated sheaf theorem and that might be interesting is: given two sketches to be compared (perhaps via sketch morphism indicating a theorem wishing for a proof) and a "favorite" Semantics, then if our favorite semantics can be represented by a strict type t, we can check if models of this type confuse the two sketches, by comparing their freely generated t-types. If the two sketches have isomorphic t-types (or just equivalent when the types are categories), they are confused by the type t (we will have proofs of the theorem perhaps indicated, in all models of type t). This procedure involves calculating Kan extensions, which can be rather heavy (at least notationally) if the sketches are complicated. But in theory it is possible and perhaps fruitful and then again by the application of *generalized associated sheaf theorem* we perhaps don't need to calculate the entire free t-type, if we using categorical gymnastics can find a sketch with same model categories as both of the sketches we wish to compare, for our favorite type (the example of monoids in the section of examples below, contains such a procedure).
We hence get a general method for *checking certain conjectures*; assuring the existence of proofs in certain types of categories without going in to the concrete details of the concrete proofs.

7 Examples

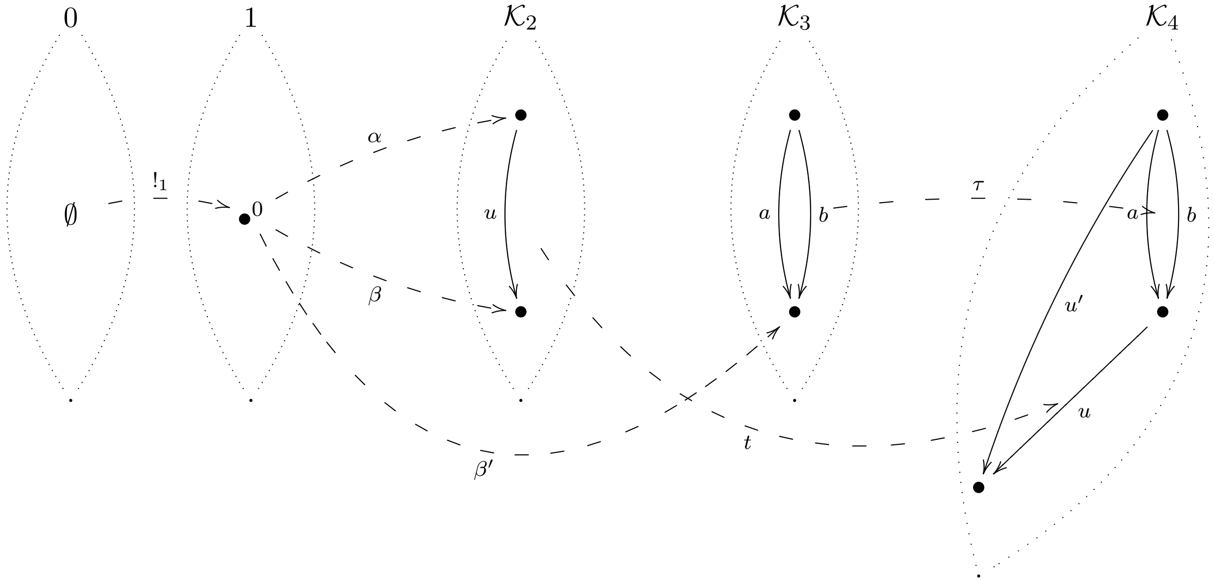
We will give two examples of the general problem. First a rather simple one concerning the sketch of fields, where we will compare the standard sketch of a field, following the set-theoretical definition, with the sketch of fields slightly modified, in order to get topological fields, when modelled in the category Top of topological spaces and continuous maps.
Then we look at an example where we compare two sketches of monoids each following a different standard set theoretical definition of what is meant by a monoid. This example will show that the apparent difference in language for two ways of defining the same structure, is not always trivially formalized, meaning that we don't see right away what the actual structural differences amount to, when we put on stronger glasses than just the set theoretical ones.
In short, we present two examples where two different sketches of the same classical mathematical structure have the same set theoretical models, but not the same model categories for any type of model. We then, following the general problem stated in the previous section, determine a type confusing the two sketches in question and at last give an example of a category discriminating them.
Before we present the examples we take a look at a basic type, that will be very helpful in the search of strict types confusing two given sketches, this counts in particular for the examples chosen in this thesis.

7.1 The classical type t_{cl_f}

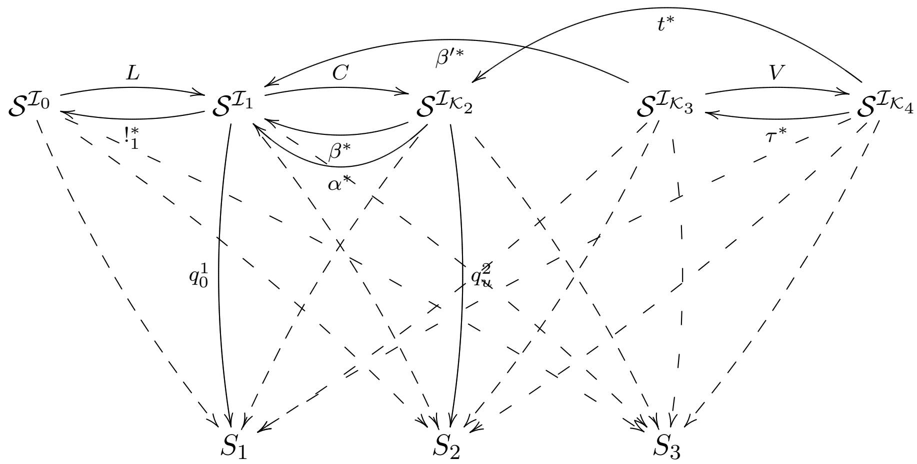
In the search for sketches playing η's role in the general problem above, a type of interest is the *classical type* of *categories with all finite limits and co-limits*, since if η is a category then η^θ and η^σ are (model-) categories (as required in general problem).
Sketches of interest (sketches of classical mathematical structures) normally have, at least, a number of finite distinguished cones and (perhaps) co-cones. So claiming that η is a category with all finite limits and co-limits, is a sort of minimal claim to avoid pathologies.
We will sketch this classical type, meaning that we will construct a sketch τ_{cl_f} such that the category of set-theoretical models of τ_{cl_f} is isomorphic to the category of categories with all finite limits and co-limits. Then we adjust the sketch in order for it to signify a strict type, i.e. in order for it to give rise to a strict type-morphism. This will turn out quite useful in the examples below, furnishing a sort of basic strict type.

Construction of the sketch τ_{cl_f}: We set out to sketch the category of categories with all finite limits and co-limits, so obviously we depart from the sketch of categories, as constructed earlier (p.12).
Now we wish to enlarge this sketch such that any model (category) will have all finite limits and co-limits. As in the construction of ϵ_λ, we add elements representing the cones we want to distinguish, only this time we want these cones to be limiting cones, so further structure is added to specify the universal property for the, chosen, limiting cones. For a model of the sketch to have all $\mathcal{I}$-limits for a chosen graph $\mathcal{I}$, we need, as in construction of ϵ_λ, to add the object $\mathcal{S}^{\mathcal{I}^-}$ to the sketch of categories and then specify the limiting cones for all cones of base $\mathcal{I}^-$ in the underlying multiplicative graph $\mathcal{S}$. We consider the following five multiplicative

graphs with graph-inclusions:



and then construct the joint graphs, as introduced in the preliminaries (p. 7), of the graphs $0, 1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$ with the graph $\mathcal{I}$ and get $\mathcal{I}_0, \ldots, \mathcal{I}_{\mathcal{K}_4}$ with graph inclusions inherited.
Now we add the objects $\mathcal{S}^{\mathcal{I}_0}, \ldots, \mathcal{S}^{\mathcal{I}_{\mathcal{K}_4}}$ to the sketch of categories, in the same way as in ϵ_λ, together with the induced arrows, i.e. we enlarge the sketch of categories with the diagram:



where the broken arrows represent (the rest of) the restricted (and evaluated) projection. Now, L is the arrow "formally choosing" a limiting cone for each diagram of type $\mathcal{I}$, C is the "choice of comparison" arrow and V is the arrow designed to secure the uniqueness in the universal property of the limiting cones. This works by assuming the following equations to hold:

(1)	$!_1^* L = 1_{\mathcal{S}^{\mathcal{I}_0}}$	(4)	$q_u^2 CL = id q_0^1 L$
(2)	$\alpha^* C = 1_{\mathcal{S}^{\mathcal{I}_1}}$	(5)	$\tau^* V = 1_{\mathcal{S}^{\mathcal{I}_3}}$
(3)	$L !_1^* = \beta^* C$	(6)	$C \beta'^* = t^* V.$

Further explication of the equations:

Ad(1): $!_1$ is just the injection of graphs $\mathcal{I}_0 \hookrightarrow \mathcal{I}_1$ (inherited from the unique arrow $0 \xrightarrow{!_1} 1$ in the category of graphs), so $!_1^* L = 1_{\mathcal{S}^{\mathcal{I}_0}}$ ($!_1^*$ composition with ! from the right) means that for any graph indexed by $\mathcal{I}$, L chooses a projective cone with this very graph as image of the basis.

Ad(2): $\alpha^* C = 1_{\mathcal{S}^{\mathcal{I}_1}}$ makes sure that C takes a cone to a comparison diagram where it itself plays the role as left hand cone.

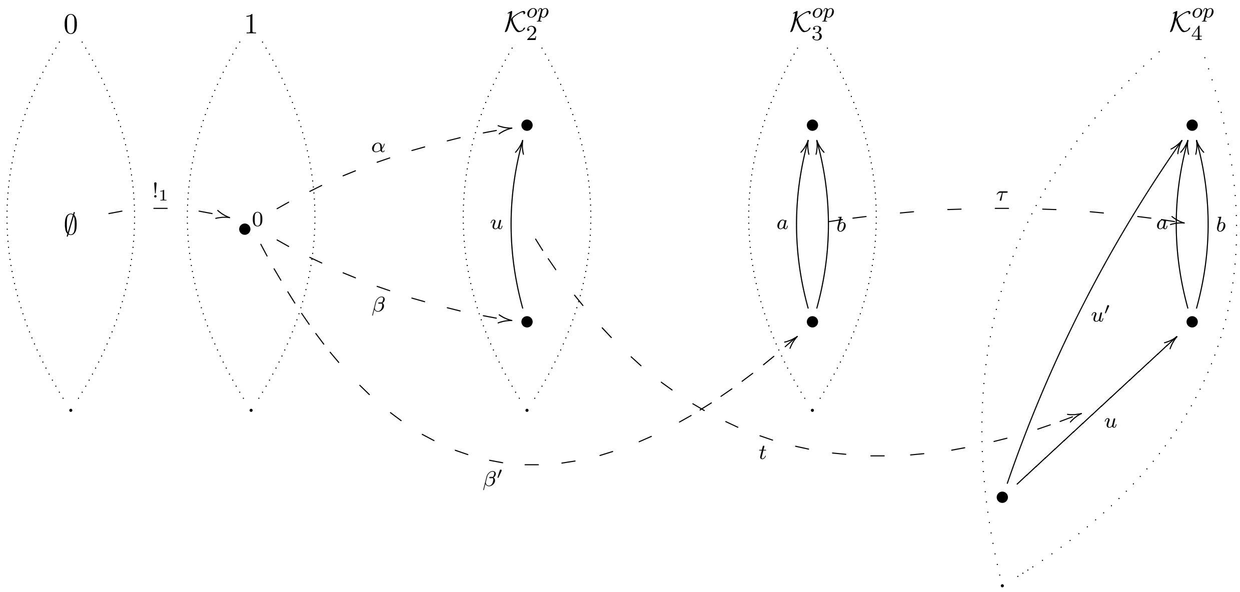
Ad(3): $L!_1^* = \beta^* C$ means that for all cones $\mathcal{I}$ we have that if we forget the vertex (go by arrow $!_1^*$) and then take the "choice of" limiting cone (by arrow L), then this is the same as taking the comparison graph (graph of type $\mathcal{I}_{\mathcal{K}_2}$) and then forget the left hand cone (cone with vertex at the source of the "hook" u), by β^*. That is for any cone with basis indexed by $\mathcal{I}$, taking the limiting cone with same basis is the same as taking the right-hand cone in the comparison graph where it plays the role as left-hand cone, so C compares any cone with the limiting cone of the same basis.

Ad(4): $q_u^2 CL = idq_0^1 L$ makes sure that a chosen limiting cone can only be compared to itself, thus a limiting diagram is (by C) sent to the comparison diagram with hook the identity.

Ad(5): $\tau^* v = 1_{\mathcal{S}^{\mathcal{I}_{\mathcal{K}_3}}}$ means that any graph of type $\mathcal{I}_{\mathcal{K}_3}$ (i.e. a comparison diagram with two hooks; two arrows from one vertex to another in cones with same basis) is the left part of a diagram of type $\mathcal{I}_{\mathcal{K}_4}$, by the arrow V.

Ad(6): $C\beta'^* = t^* V$ means that whenever there are two hooks between vertexes of cones with same basis, then V extends this diagram with a hook to the limiting cone of same basis. So if the right-hand cone in the diagram with two hooks is a limiting cone, then u is the identity, by (4), and hence the two hooks are identical (equal to u', see figures above).

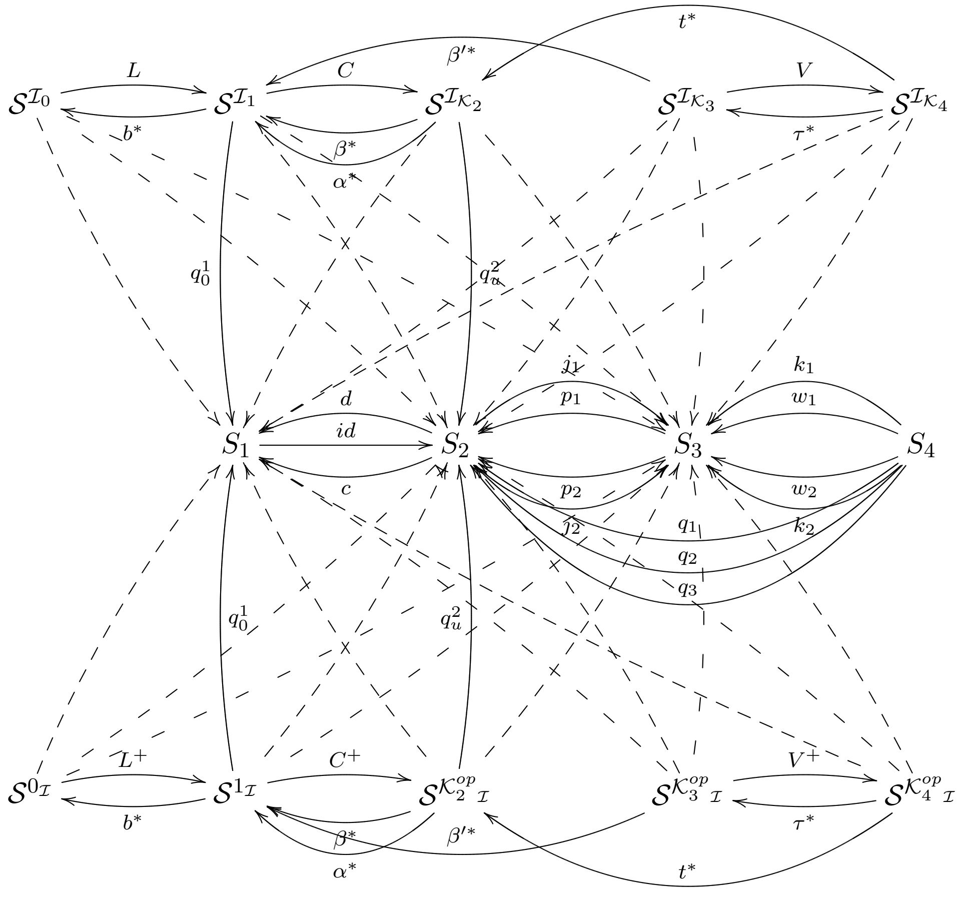
Concerning choice of co-limits, we do the same thing but the arrows reversed in the system of five graphs, i.e. with



and construct the joint graphs $0_{\mathcal{I}}, 1_{\mathcal{I}}, {\mathcal{K}_2^{op}}_{\mathcal{I}}, {\mathcal{K}_3^{op}}_{\mathcal{I}}, {\mathcal{K}_4^{op}}_{\mathcal{I}}$ and then have the same equations except that in the explications left and right should evidently be interchanged.

Remark: Sketching categories with all finite limits can be reduced to sketching categories with finite products (including terminal object) and equalizers, meaning that we only need the above addition to the sketch of categories for the graphs $\{0, \quad 2, \quad \bullet \rightrightarrows \bullet\}$ where 0 is the empty graph and $2 = \{\bullet \quad \bullet\}$ the graph with two objects and only trivial arrows.

So we actually get a finite sketch, with (generating) underlying multiplicative graph of τ_{cl_f}:



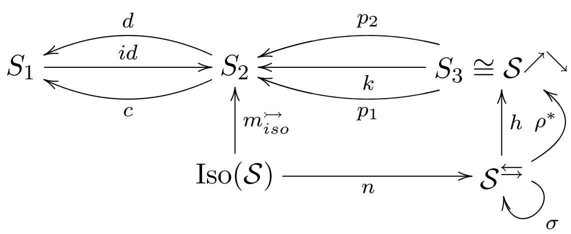
for $\mathcal{I} \in \{0, \quad 2, \quad \bullet \rightrightarrows \bullet\}$ and the six equations as explained above with both L, C, V and L^+, C^+, V^+ plus the equations from the sketch of categories, distinguished cones as in sketch of categories plus all cones with vertexes $\mathcal{S}^{\mathcal{I}_0}, \ldots, \mathcal{S}^{\mathcal{I}_{\mathcal{K}_4}}$ and $\mathcal{S}^{0_\mathcal{I}}, \ldots, \mathcal{S}^{\mathcal{K}_4^{op}}{}_\mathcal{I}$ for the three graph types.

Now for τ_{cl_f} to be a sketch of a theoretical type, there need to be a sketch morphism

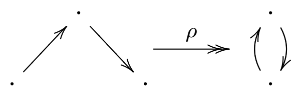
$$\epsilon_\lambda \longrightarrow \tau_{cl_f} .$$

which is evidently not possible to construct directly, since τ_{cl_f} does not have distinguished cones corresponding to all of the distinguished cones in ϵ_λ, we only have the objects of the type $\mathcal{S}^{\mathcal{I}^-}, \mathcal{S}^{\mathcal{I}^+}$ for $\mathcal{I} \in \{0, \quad 2, \quad \bullet \rightrightarrows \bullet\}$. But we can expand the sketch τ_{cl_f} to get a sketch $\bar{\tau}_{cl_f}$ (with same set-theoretical models). We add all missing objects $\mathcal{S}^{\mathcal{I}^-}, \mathcal{S}^{\mathcal{J}^+}$ for $\mathcal{I} \in \mathfrak{J} \setminus \{0, \quad 2, \quad \bullet \rightrightarrows \bullet\}$, $\mathcal{J} \in \mathfrak{J}' \setminus \{0, \quad 2, \quad \bullet \rightrightarrows \bullet\}$. We also add objects of $\mathcal{I}$-limits and $\mathcal{I}$-co-limits for $\mathcal{I} \in \{0, \quad 2, \quad \bullet \rightrightarrows \bullet\}$ in the following way:

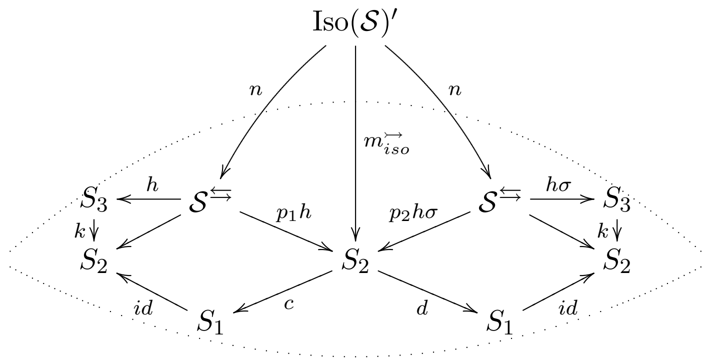
-We introduce the object of isomorphisms, i.e. we add an object $\text{Iso}(\mathcal{S})$, and arrows m_{iso}, n, h to the underlying sketch of graphs, as follows:



where ρ^* is composition from the right by

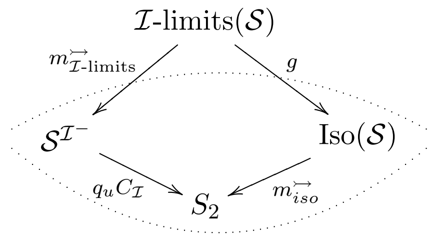


and with m_{iso} distinguished as mono. By the definition of being an isomorphism, we need equations $p_2 h\sigma n = m_{iso}$, $p_1 hn = m_{iso}$ and $khn = idcm_{iso}$, $kh\sigma n = iddm_{iso}$. So we distinguish the cone:



then, when modelled in Set, $\text{Iso}(\mathcal{S})$ will be isomorphic to the image of the mono modelling m_{iso}, which will be the subset of arrows for which there exist an arrow both right and left inverse.

We can now introduce the object of $\mathcal{I}$-limits "$\mathcal{I}$-limits($\mathcal{S}$)", as the (potential) subobject of $\mathcal{S}^{\mathcal{I}^-}$ such that the comparison map is an isomorphism, i.e adding object and arrows to distinguish the following cone:



Where $C_{\mathcal{I}}$ is the comparison map, taking a $\mathcal{I}$-cone to the diagram comparing it with the chosen $\mathcal{I}$-limit of the same basis and q_u is the projection of the hook between vertexes in the comparison diagram.

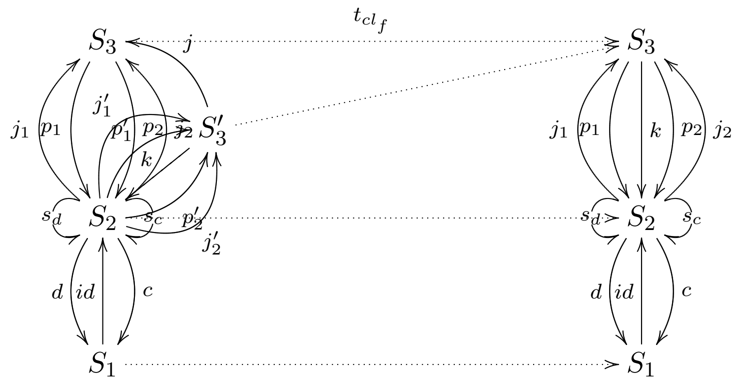
So we get the sketch $\bar{\tau}_{cl_f}$ from τ_{cl_f} by adding some cones (or a whole bunch, though still a set) to be distinguished.

In this way we get an inclusion $\tau_{cl_f} \hookrightarrow \bar{\tau}_{cl_f}$ introducing an equivalence of model categories $\mathrm{Set}^{\tau_{cl_f}} \sim \mathrm{Set}^{\bar{\tau}_{cl_f}}$, and a sketch morphism from ϵ_λ to $\bar{\tau}_{cl_f}$

$$t_{cl_f} : \epsilon_\lambda \longrightarrow \bar{\tau}_{cl_f}$$

to serve as our type of categories with finite limits, defined as follows:

- the "sketch-of-multiplicative-graph" part of ϵ_λ is mapped to the "sketch-of-categories" part of $\bar{\tau}_{cl_f}$ in the evident way, indicated thus:



making all consecutive arrows composable (when modelled).

- for $\mathcal{I} \in \{0, \quad 2, \quad \bullet \rightrightarrows \bullet\}$ we send

$$C_{\mathcal{I}^-} \xrightarrow{m^{\rightarrow}_{\mathcal{I}^-}} \mathcal{S}^{\mathcal{I}^-} \quad \text{to} \quad \mathcal{I}\text{-limits}(\mathcal{S}) \xrightarrow{m^{\rightarrow}_{\mathcal{I}\text{-limits}}} \mathcal{S}^{\mathcal{I}^-}$$

$$\text{and} \quad C_{\mathcal{I}^+} \xrightarrow{m^{\rightarrow}_{\mathcal{I}^+}} \mathcal{S}^{\mathcal{I}^+} \quad \text{to} \quad \mathcal{I}\text{co-limits}(\mathcal{S}) \xrightarrow{m^{\rightarrow}_{\mathcal{I}\text{-co-limits}}} \mathcal{S}^{\mathcal{I}^+}$$

- for $\mathcal{I} \in \mathfrak{I} \setminus \{0, \quad 2, \quad \bullet \rightrightarrows \bullet\}$ we send

$$C_{\mathcal{I}^-} \xrightarrow{m^{\rightarrow}_{\mathcal{I}^-}} \mathcal{S}^{\mathcal{I}^-} \quad \text{to} \quad 0-\text{limits}(\mathcal{S}) \xrightarrow{m^{\rightarrow}_{0-limits}} \mathcal{S}^{0^-} \cong \mathcal{S}^1 \xrightarrow{!^*} \mathcal{S}^{\mathcal{I}^-}$$

and for $\mathcal{I} \in \mathfrak{I}' \setminus \{0, \quad 2, \quad \bullet \rightrightarrows \bullet\}$ we send

$$C_{\mathcal{I}^+} \xrightarrow{m^{\rightarrow}_{\mathcal{I}^+}} \mathcal{S}^{\mathcal{I}^+} \quad \text{to} \quad 0-\text{co-limits}(\mathcal{S}) \xrightarrow{m^{\rightarrow}_{0-co-limits}} \mathcal{S}^{0^+} \cong \mathcal{S}^1 \xrightarrow{!^*} \mathcal{S}^{\mathcal{I}^+}$$

where $!^*$ is composition from the right with the unique arrow $\mathcal{I}^-, \mathcal{I}^+ \longrightarrow 1$ to the graph 1 in the category of λ-small graphs and $\mathcal{S}^{0^-}, \mathcal{S}^{0^+}$ will both be isomorphic to S_1 when modelled, since $0^-, 0^+$ are the two ways of joining the empty graph with the graph of one element, so diagrams in $\mathcal{S}$ of this sort is just all objects (the model of S_1).

When modelled in Set the set of distinguished 0-cones is the set of 0-limits, i.e. the set of terminal elements or, said differently the set of all singletons. The set of 0-co-cones is the set of initial elements $\{\emptyset\}$. So the map $m_{\mathcal{I}}$ in ϵ_λ for $\mathcal{I} \in \mathfrak{J} \setminus \{0, \quad 2, \quad \bullet \rightrightarrows \bullet\}$ destined to choose cones [co-cones] to be distinguished in a model of ϵ_λ, is sent to the map in τ_{cl_f} modelled as the inclusion of terminal [initial] elements into the object of functors $\mathcal{S}^{\mathcal{I}^-}$ $[\mathcal{S}^{\mathcal{I}^+}]$, meaning that in a model of $\bar{\tau}_{cl_f}$ the only limits [co-limits] distinguished by such functors is the constant functors on terminal [initial] elements.

Now that we have a sketch morphism, we just need to assure that it is a strict type, i.e. has the property that a supposed sketch of type t_{cl_f} will, viewed as a model of ϵ_λ, factor uniquely through t_{cl_f}. To show this we will take a more general look at what just happened, by naming concepts treated and state some properties. Furthermore, treating this question more generally will be of much help afterwards in our two examples.

•*Expansion by cones:* for a sketch σ, I will call, an *expansion by distinguished cones* (or just cone-expansion) the sketch $\bar{\sigma}$ obtained by, only, adding objects to σ as vertexes of cones to be distinguished (with all images of the bases in σ), and then only extra arrows and equations as part of the cones added.

> **Proposition 1**: *If σ is a mixed sketch the sketch inclusion $\sigma \overset{\varsigma}{\hookrightarrow} \bar{\sigma}$ induces an equivalence of model categories*
>
> $$\mathrm{Set}^{\bar{\sigma}} \xrightarrow[\varsigma^*]{\sim} \mathrm{Set}^{\sigma} .$$

We remind the reader of the following basic theorem (MacLane, S. [1997]):

> **Theorem** *The following properties of a functor $S : \mathcal{A} \longrightarrow \mathcal{C}$ are logically equivalent:*
> *(i) S is an equivalence of categories,*
> *(ii) S is part of an adjoint equivalence,*
> *(iii) S is full and faithful, and each object $C \in Ob_{\mathcal{C}}$ is isomorphic to SA for some object $A \in Ob_{\mathcal{A}}$.*

■

Proof of proposition 1: ς^* is evidently full and faithful. And each model $R \in \mathrm{Set}^{\sigma}$ can be extended (uniquely up to iso) to a model $\bar{R}$ of $\bar{\sigma}$ such that $\varsigma^* \bar{R} = R$, by taking the added cones to the chosen limits of the respective bases. Meaning if $S' \in Ob_{\bar{\sigma}} \setminus Ob_{\sigma}$ is the vertex of a distinguished cone with basis $B : \mathcal{I} \longrightarrow \sigma$, then $\bar{R}$ takes S' to the vertex of the chosen limit in Set with basis $R \circ B$. Since Set has "chosen" limits for any (λ-small) index graph, we have indeed $\varsigma^*(\bar{R}) = R$.

■

•*Set-model-epi:* We will call *set-model-epi* a sketch morphism $s : \sigma \longrightarrow \sigma'$ in Esq_λ with the property that for all models (sketch morphisms) $\varphi, \psi : \sigma' \longrightarrow \mathrm{Set}$ satisfying $\varphi \circ s = \psi \circ s$ we have $\varphi \cong \psi$.
Notice that a strict theoretical type then is a set-model-epi with source ϵ_λ.

Corollary 1:*For a sketch σ the sketch inclusion $\sigma \overset{\varsigma}{\hookrightarrow} \bar{\sigma}$ is a set-model-epi*

■

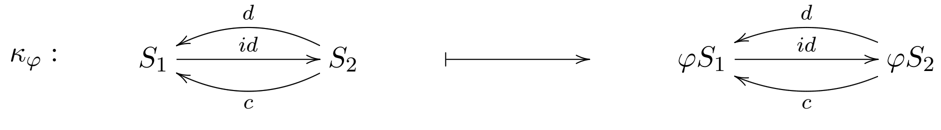
Corollary 2: *if the sketch morphism $t : \epsilon \longrightarrow \sigma$ is a set-model-epi, then so is the expansion $\bar{t} : \epsilon \longrightarrow \bar{\sigma}$ (by cone-expansion of the target).*

■

We are now ready to proof that we have constructed a classical strict type

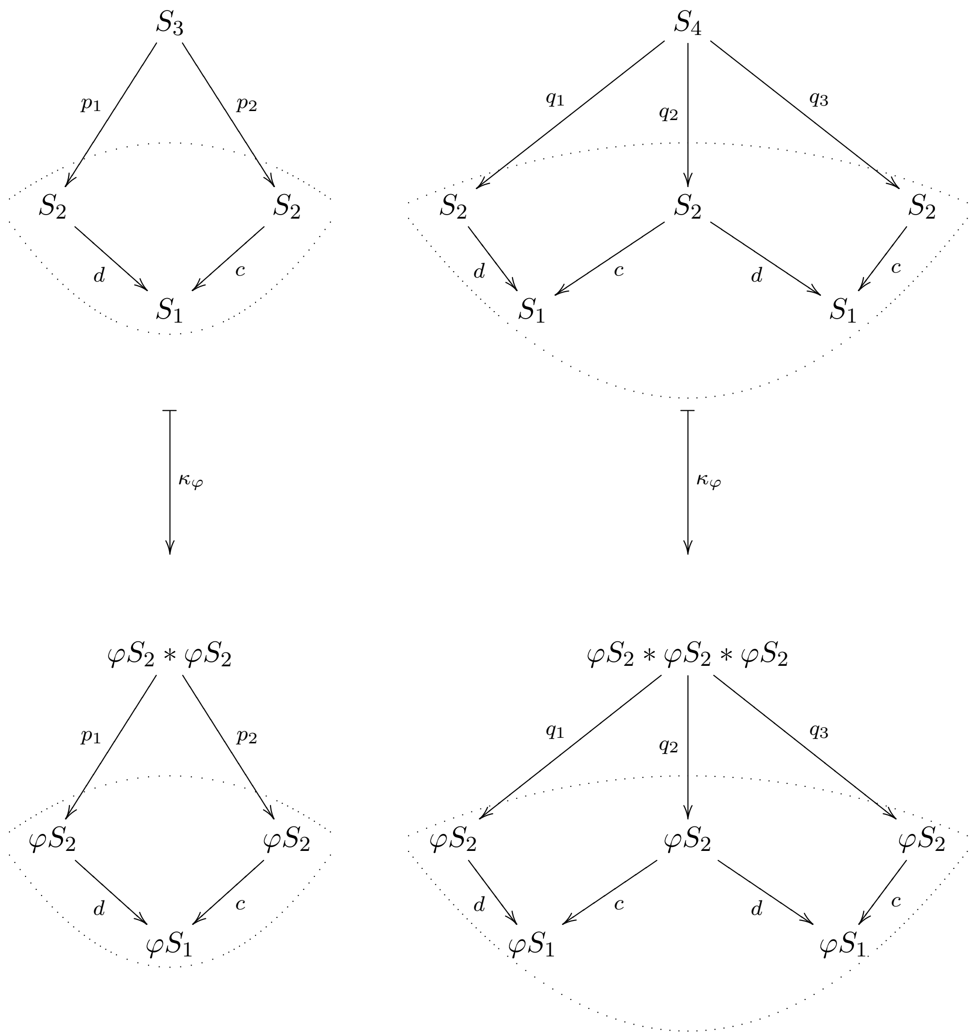
Proposition 3: *$t_{cl_f} : \epsilon_\lambda \longrightarrow \bar{\tau}_{cl_f}$ is a set-model-epi, i.e. t_{cl_f} defines a strict theoretical type.*

Proof: Take any pair of models $\psi, \varphi : \bar{\tau}_{cl_f} \longrightarrow \text{Set}$ such that $\varphi \circ t_{cl_f} = \psi \circ t_{cl_f}$, then we need to proof that $\varphi \cong \psi$, i.e. that there exists a natural isomorphism between φ and ψ.
I will show this by showing that every model $\varphi : \bar{\tau}_{cl_f} \longrightarrow \text{Set}$ is isomorphic to, what I will call, its corresponding *reference model* $\kappa_\varphi : \bar{\tau}_{cl_f} \longrightarrow \text{Set}$, which, in word, will be "the model enabling us to understand the model φ", i.e. the set-theoretical model we "thought of" when we constructed the sketch $\bar{\tau}_{cl_f}$. Meaning that κ_φ equals φ on the "oriented-graph-part" of $\bar{\tau}_{cl_f}$ (the set of objects and the set of arrows between objects), i.e.



we won't write $\kappa_\varphi f$ for modelled arrows (f an arrow in the sketch-of-categories-part of $\bar{\tau}_{cl_f}$) since these are in the model κ_φ exactly what they were named to become in a model, when we made the sketch.
Then κ_φ takes the distinguished cones in the "sketch-of-categories-part" to the expected fiber products, i.e. the actual set of consecutive pairs of arrows and consecutive triples of arrows,

which we illustrate:

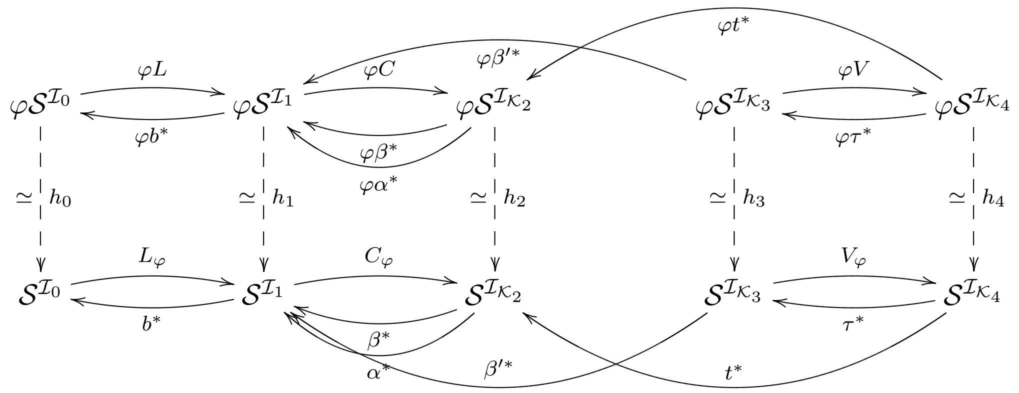


Then in the model κ_φ all the arrows are exactly what they were thought to be:

$$\begin{aligned}
w_1 &: \quad (f_1, f_2, f_3) \longmapsto (f_1, f_2) \\
w_2 &: \quad (f_1, f_2, f_3) \longmapsto (f_2, f_3) \\
k_1 &: \quad (f_1, f_2, f_3) \longmapsto (k(f_1, f_2), f_3) \\
k_2 &: \quad (f_1, f_2, f_3) \longmapsto (f_1, k(f_2, f_3))
\end{aligned}$$

p_1, p_2 the restrictions of the Cartesian projections etc. Only we set the composition $\kappa_\varphi(k)$ to be determined by the composition $\varphi(k)$ composed with the isomorphism of limit vertexes $\varphi S_3 \cong \varphi S_2 * \varphi S_2$.
Now the objects of the sort $\mathcal{S}^{\mathcal{I}}$ will be sent to exactly the set of functors from $\mathcal{I}$ to $\mathcal{S}$ (all diagrams indexed by $\mathcal{I}$) as introduced in the sketching of sketches earlier, one could say that κ_φ is the model justifying the notation, which is why we will write $\kappa_\varphi \mathcal{S}^{\mathcal{I}} = \mathcal{S}^{\mathcal{I}}$. The choice of

limit/choice of co-limit diagram in the sketch $\bar{\tau}_{cl_f}$ will then be sent to the diagram induced by φ via the isomorphisms $\varphi\mathcal{S}^{\mathcal{I}} \xrightarrow[h]{\simeq} \mathcal{S}^{\mathcal{I}}$ meaning that all natural squares in the following diagram commute:



where $h_i : \varphi\mathcal{S}^{\mathcal{I}_i} \longrightarrow \mathcal{S}^{\mathcal{I}_i}; \quad x \longmapsto (I \mapsto f_I(x), i \mapsto f_i(x), (i', i'') \mapsto f_{(i',i'')})$ as mentioned in the introduction of the objects of the sort $\mathcal{S}^{\mathcal{I}}$.

We now claim that the obvious isomorphism of objects between the models φ and κ_φ is natural. This is almost obvious by definition of κ_φ. But let us go through it just once: we name the claimed natural isomorphism $h : \varphi \longrightarrow \kappa_\varphi$, and start by looking at the "sketch-of-categories-part" of $\bar{\tau}_{cl_f}$. The only natural squares that do not totally obviously commute by force of being projections in a limiting cone, are

$$\begin{array}{ccc} \varphi S_4 & \xrightarrow{h_{S_4}} & \varphi S_2 * \varphi S_2 * \varphi S_2 \\ \downarrow{\scriptstyle \varphi f} & & \downarrow{\scriptstyle f} \\ \varphi S_3 & \xrightarrow[h_{S_3}]{} & \varphi S_2 * \varphi S_2 \end{array} \qquad \text{for } f \in \{w_1, w_2, k_1, k_2\}$$

but commutativity follows immediately by the equations in the sketch of categories, since if we take $x \in \varphi S_4$ and suppose $h_{S_4}(x) = (f_1, f_2, f_3)$, then

$$\begin{aligned} \varphi p_1 \varphi w_1(x) &= \varphi q_1(x) = q_1 h_{S_4}(x) = f_1 \\ \varphi p_2 \varphi w_1(x) &= \varphi q_2(x) = q_2 h_{S_4}(x) = f_2 \end{aligned}$$

and h_{S_3} is the unique map from φS_3 to $\varphi S_2 * \varphi S_2$ s.t. $p_1 h_{S_3} = \varphi p_1$ and $p_2 h_{S_3} = \varphi p_2$, giving us that

$$\begin{aligned} p_1 h_{S_3} \varphi w_1(x) &= f_1 \\ p_2 h_{S_3} \varphi w_1(x) &= f_2 \end{aligned}$$

so $h_{S_3}\varphi w_1(x) = (f_1, f_2) = w_1(f_1, f_2, f_3) = w_1 h_{S_4}(x)$, same goes for w_2.
And since, by definition of κ_φ, we have $kh_{S_3} = \varphi k$, we get

$$\begin{aligned} \varphi p_1 \varphi k_1(x) &= \varphi k \varphi w_1(x) = k h_{S_3} \varphi w_1(x) = k(f_1, f_2) \\ \varphi p_2 \varphi k_1(x) &= \varphi q_3(x) = q_3 h_{S_4}(x) = f_3 \\ \text{which gives} \quad p_1 h_{S_3} \varphi k(x) &= k(f_1, f_2), \quad p_2 h_{S_3} \varphi k(x) = f_3 \end{aligned}$$

so $h_{S_3}\varphi k_1(x) = (k(f_1, f_2), f_3) = k_1 h_{S_3}$, same for k_2.
So φ and κ_φ gives isomorphic underlying categories.
Now this natural isomorphism clearly (or by corollary 2.) extends to all the distinguished cones with vertex of the sort $\mathcal{S}^{\mathcal{I}}$, and furthermore, by definition of κ_φ, naturality extends to the "choice-of-limit-diagrams" (p. 26). The only part of $\bar{\tau}_{cl_f}$ not mentioned yet concerns the object of isomorphisms Iso($\mathcal{S}$) and the objects $\mathcal{I}$-limits($\mathcal{S}$) and $\mathcal{I}$-co-limits($\mathcal{S}$), but these are all introduced as vertexes of distinguished cones, hence again by corollary 2. we still get an isomorphism. Thus $h : \varphi \longrightarrow \kappa_\varphi$ is indeed a natural isomorphism.
Now to finish the proof of the proposition we need to justify that φ and ψ (agreeing on $t_{cl_f}(\epsilon_\lambda)$) must have isomorphic *reference models* $\kappa_\varphi \cong \kappa_\psi$. But since φ and ψ are equal on the oriented-graph-part of $\bar{\tau}_{cl_f}$ their reference models will evidently be exactly the same except for the choice of limit maps L_φ and L_ψ, but they are clearly isomorphic since by uniqueness (up to isomorphism) of limits, they have isomorphic images (we are in Set) (same for choice of co-limits). So we can without pain conclude that $\kappa_\varphi \cong \kappa_\psi$, which gives the natural isomorphism of φ and ψ through their reference models.

■

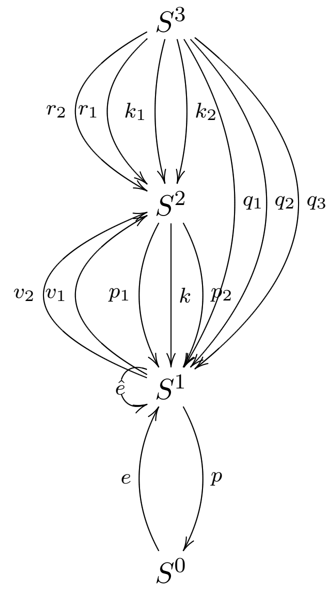
Remark: The essence of this proof is just pointing out how we have been sketching while thinking of certain (canonical) set-theoretical models

7.2 Two sketches of fields

In this section we go through a simple example of the general problem (p. 24) where we compare two sketches of fields that have the same models in Set but not in the category Top of topological spaces with continuous maps.
The construction of the first sketch of fields ξ_1 is following the standard set theoretical definition of a field and will be sketched gradually in the order of its underlying structures.
First the sketch of monoids, following the set theoretical definition

> Def. 1: A monoid (M, k, e) is a set M with a binary operation "k" everywhere defined, associative, and with a distinguished unit element e with respect to the operation.

Underlying multiplicative graph:



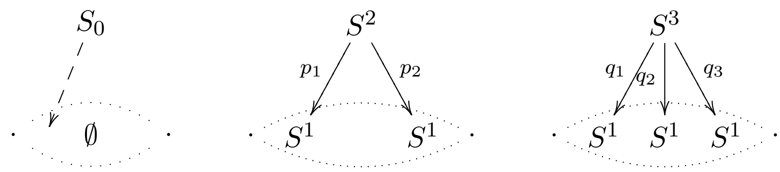
Equations concerning binary operation k:

$$\begin{aligned} p_1 \cdot r_1 &= q_1, \quad p_2 \cdot r_1 = q_2 \\ p_1 \cdot r_2 &= q_2, \quad p_2 \cdot r_2 = q_3 \\ k \cdot r_1 &= l_1, \quad k \cdot r_2 = l_2 \quad \text{just naming the composite} \\ p_1 \cdot k_1 &= l_1, \quad p_2 \cdot k_1 = q_3 \\ p_1 \cdot k_2 &= q_1, \quad p_2 \cdot k_2 = l_2 \\ k \cdot k_1 &= k \cdot k_2 \quad \text{associativity of k} \end{aligned}$$

Equations concerning the unit element, or 0-ary operation e:

$$\begin{aligned} \hat{e} &= e \cdot p \\ p_1 \cdot v_1 &= 1_{S_1}, \quad p_2 \cdot v_1 = \hat{e} \\ p_1 \cdot v_2 &= \hat{e}, \quad p_2 \cdot v_2 = 1_{S_1} \\ k \cdot v_1 &= k \cdot v_2 = 1_{S_1} \quad \text{unitarity of } e \text{ with respect to } k. \end{aligned}$$

Distinguished projective cones:



We can then think of S^0 as 1, S^1 as the "set of elements" (M), S^2 as the "set of pairs of elements" $(M \times M)$ and S^3 as "the set of triples of elements" $(M \times M \times M)$. The arrows p_1, p_2, q_1, q_2, q_3 are thought of as the canonical projections, r_1, r_2 partial projections

$((x_1, x_2, x_3) \longmapsto (x_1, x_2), (x_2, x_3))$, k the "addition of elements", k_1, k_2 can be thought of as:

$$k_1 : (x_1, x_2, x_3) \longmapsto (x_1 + x_2, x_3)$$

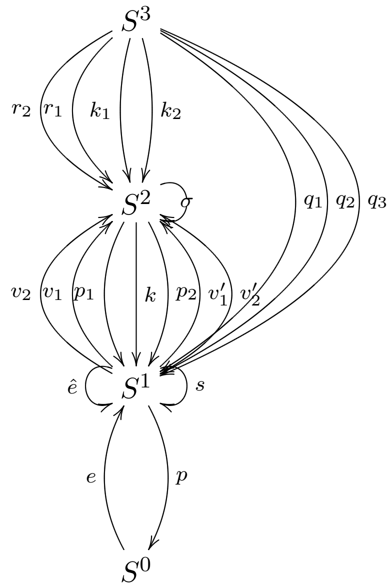
$$k_2 : (x_1, x_2, x_3) \longmapsto (x_1, x_2 + x_3)$$

e is the unit element (0) with respect to k (addition) and this is assured by v_1, v_2 thought of as

$$v_1 : x \longmapsto (x, 0)$$

$$v_2 : x \longmapsto (0, x)$$

Now we add what is missing to obtain the sketch of abelian groups, i.e. arrows and equations representing the fact fact that all elements should have a left and right inverse, with respect to the composition "addition", and furthermore that the composition is additive. To this we add arrows: s ("taking an element to its inverse"), v'_1, v'_2 ("taking an element x resp. to the pairs $(-x, x)$ and $(x, -x)$"), and σ (a permutation). Giving us the underlying multiplicative graph:



Equations as in sketch of monoids plus:
-new equations concerning binary operation k:

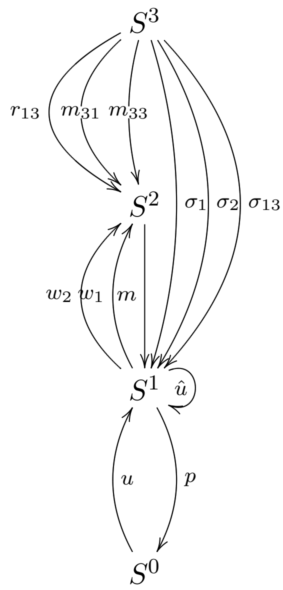
$$\begin{aligned} p_1 \cdot \sigma &= p_2, \quad p_2 \cdot \sigma = p_1 \quad \sigma \text{ a permutation} \\ k \cdot \sigma &= k \quad \text{k is additive.} \end{aligned}$$

-new equations concerning the inverses of elements:

$$\begin{aligned} p_1 \cdot v'_1 &= p_2 \cdot v'_2 \\ p_1 \cdot v'_2 &= 1_{S^1} = p_2 \cdot v'_1 \\ k \cdot v'_1 &= k \cdot v'_2 = e. \end{aligned}$$

Distinguished projective cones as above.

We continue to get the sketch of a unitary ring, i.e. we add what is needed to describe a unitary multiplication, which is



and further new equations:
-concerning partial projections:

$$\begin{aligned} p_1 \cdot r_{13} &= q_1, \quad p_2 \cdot r_{13} = q_3 \\ \sigma_{13} &= m \cdot r_{13} \end{aligned}$$

-concerning multiplication m and its distributivity (from left and right) on the "addition" (composition k):

$$\begin{aligned} p_1 \cdot m_{31} &= \sigma_1, \quad p_1 \cdot m_{33} = \sigma_{13} \\ p_2 \cdot m_{31} &= m \cdot k_2, \quad p_2 \cdot m_{33} = \sigma_2 \\ k \cdot m_{31} &= m \cdot k_2, \quad k \cdot m_{33} = m \cdot k_1 \end{aligned}$$

-concerning 0-ary operation u:

$$\begin{aligned} u \cdot p &= \hat{u} \\ p_1 \cdot w_1 &= \hat{u}, \quad p_2 \cdot w_1 = 1_{S^1} \\ p_1 \cdot w_2 &= 1_{S^1}, \quad p_2 \cdot w_2 = \hat{u} \\ m \cdot w_1 &= m \cdot w_2 = 1_{S^1} \end{aligned}$$

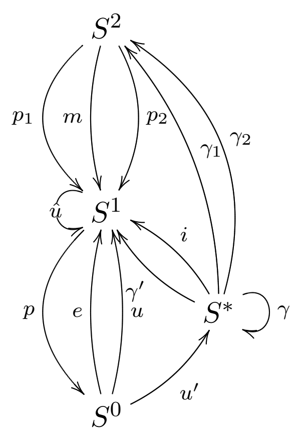
Where we can think of these arrows as:

$$\begin{aligned}
w_1 &: \quad x \longmapsto (1,x)\\
w_2 &: \quad x \longmapsto (x,1)\\
r_{13} &: \quad (x_1,x_2,x_3) \longmapsto (x_1,x_3)\\
\sigma_1 &: \quad (x_1,x_2,x_3) \longmapsto x_1x_2\\
\sigma_2 &: \quad (x_1,x_2,x_3) \longmapsto x_2x_3\\
\sigma_{13} &: \quad (x_1,x_2,x_3) \longmapsto x_1x_3\\
m_{13} &: \quad (x_1,x_2,x_3) \longmapsto (x_1x_2, x_1x_3)\\
m_{33} &: \quad (x_1,x_2,x_3) \longmapsto (x_1x_3, x_2x_3)
\end{aligned}$$

Finally we can talk about sketches of fields:
First take the sketch of unitary rings, that we just constructed above. We need all symbols and their signification, essentially:

$k : S^2 \longrightarrow S^1$ formal law of addition
$m : S^2 \longrightarrow S^1$ formal law of multiplication
$e : S^0 \longrightarrow S^1$ formal law selecting the "0" of the field
$\hat{e} : S^1 \longrightarrow S^1$ formal law "constant on 0" of the field
$u : S^0 \longrightarrow S^1$ formal law selecting the "1" of the field
$\hat{u} : S^1 \longrightarrow S^1$ formal law "constant on 1".

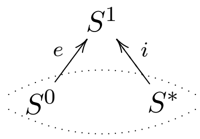
We then add the following piece of graph, wherein the new symbols are $S^*, i, u', \gamma, \gamma', \gamma_1, \gamma_2$:



with further new equations:

$$\begin{aligned}
\gamma' &= i\cdot\gamma, \quad i\cdot u' = u\\
p_1\cdot\gamma_1 &= i, \quad p_1\cdot\gamma_2 = \gamma'\\
p_2\cdot\gamma_1 &= \gamma', \quad p_2\cdot\gamma_2 = i\\
m\cdot\gamma_1 &= m\cdot\gamma_2 = \hat{u}
\end{aligned}$$

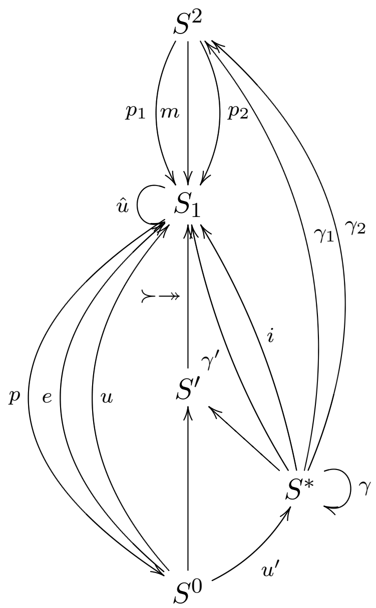
and a single new distinguished co-cone:



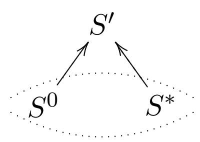
S^* thought of as all "non-zero elements" and the arrows thougt of as:

$$\begin{array}{rcl}
\gamma & : & x \longmapsto x^{-1} \\
\gamma_1 & : & x \longmapsto (x, x^{-1}) \\
\gamma_2 & : & x \longmapsto (x^{-1}, x) \\
i & : & \longmapsto x
\end{array}$$

This sketch is indeed set-theoretical, because if we realize this sketch in the category of topological spaces we will not get the usual topological fields, i.e a field with a topology such that all the field operation maps are continuous. The problem is that a non-discrete topological space K is in general not the sum of its subspaces $\{0\}$ and K^* (elements non-null), An example is the space $\mathbb{R}$ with the standard topology, then $\mathbb{R}$ is not the the direct sum of its subspaces $\{0\}, \mathbb{R}^*$, since 0 is not an isolated point in the standard topology.
To get a sketch ξ_2 giving us the topological fields when realized in topological spaces we need to modify the above sketch a bit (Guitart, R. [1988]), by distinguishing S^1 not as the sum $S^0 + S^*$, but as the target of an epi-mono with source this sum giving us instead:



and instead, the distinguished co-cone:



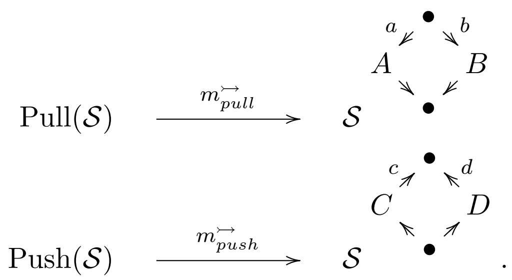
Then ξ_2 gives us topological fields, when we model in Top (exercise!)

We thus have two sketches of fields ξ_1 and ξ_2 with an obvious sketch morphism $\xi_2 \longrightarrow \xi_1$ (just collapsing S^1 and S') which clearly gives rise two an equivalence of model categories whenever we are in the classical type $\bar{\tau}_{cl}$ with the extra property that all epi-monos are isos.

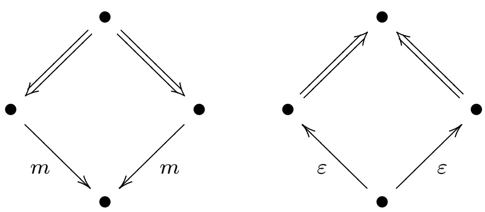
7.2.1 A type confusing the two sketches of fields

In order to get a strict type confusing ξ_1 and ξ_2, we now sketch the extra property that all arrows that are both epi and mono, are isomorphisms. We will call this property $*_{fields}$ and the extended sketch we then denote $\bar{\tau}_{{cl_f}^{*fields}}$.

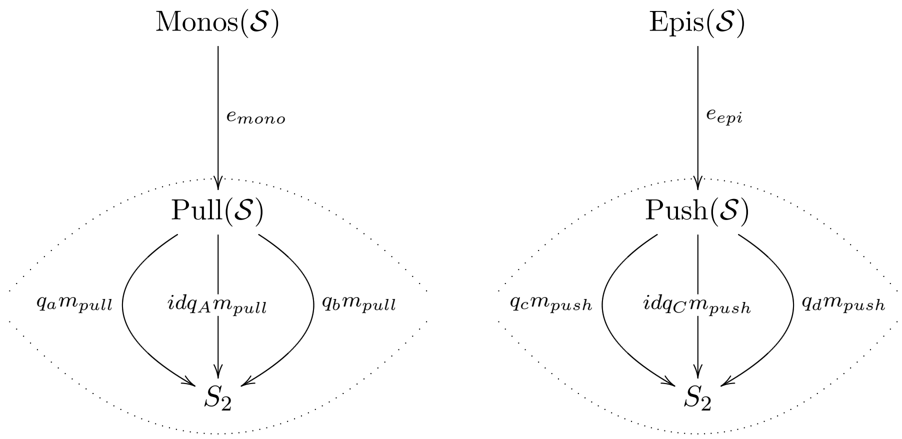
We can introduce objects Pull($\mathcal{S}$) of pullback diagrams in $\mathcal{S}$ and Push($\mathcal{S}$) of pushout diagrams in $\mathcal{S}$ in the same way as the objects $\mathcal{I}$-limits($\mathcal{S}$), $\mathcal{I}$-co-limits($\mathcal{S}$) were introduced previously, i.e as certain distinguished subobjects:

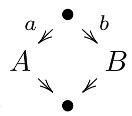


This enable us to introduce objects "Epis($\mathcal{S}$)" and "Monos($\mathcal{S}$)" as the subobjects of Push($\mathcal{S}$) and Pull($\mathcal{S}$) where the "pushed out" or the "pulled back" part of the diagrams are the identity maps, recalling the definition of epis and monos by certain diagrams being pushouts resp. pullbacks: ε epi and m mono if and only if we have pushout and pullback:

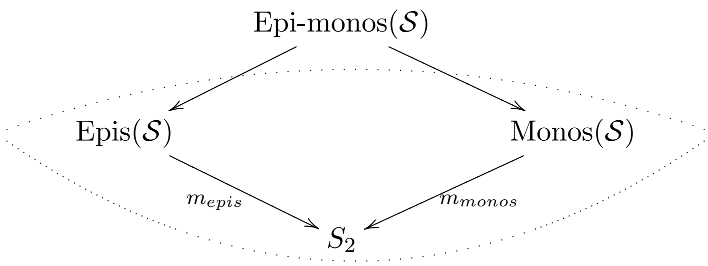


We can then sketch the object of monos and epis by adding objects and arrows to distinguish two cones:

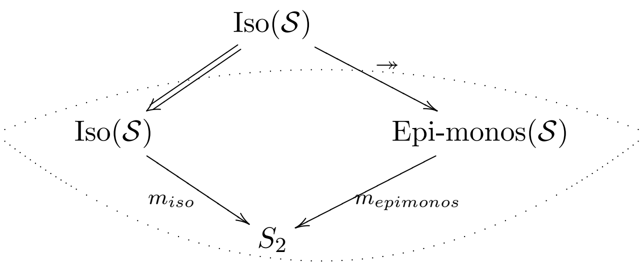


where q_a, q_b, q_A are projections in the distinguished cone with $\mathcal{S}$ 

as vertex, likewise for q_c, q_d, q_C. Meaning that when modelled in Set we get that Monos($\mathcal{S}$) is the unique subset of Pull($\mathcal{S}$) (pullbacks in the underlying graph) equalizing the "pulled-back" part (model of maps a, b) with the identity on A. This gives $A = B$ and the map $A \longrightarrow \bullet$ a monomorphism by the universal property of the pullback.
Then of course the object "Epi-monos($\mathcal{S}$)" will be sketched by distinguishing the cone:



hence Epi-monos($\mathcal{S}$) will, when modelled in Set, be the subobject of $Ar(\mathcal{S})$ of maps both epi and mono.
At last to get the property we set out to sketch, namely that all epi-monos are isos, we distinguish cone



"Epi-monos($\mathcal{S}$)" will then in Set be realised as isomorphic to the object "Iso($\mathcal{S}$)", because we distinguished the pullback of m_{iso} along $m_{epimonos}$ to be an epi and it will evidently be a

mono, so since we are modelling in set it is an iso, which sketchs our property.
Remark: One should here be careful not to confuse the fact that epi-monos are isos in Set, with the fact that we are sketching this property for models in Set.

The theoretical type $t_{cl_f^{*fields}}$ and its corresponding free generator

$\bar{\tau}_{cl_f^{*fields}}$ is constructed by a finite sequence of cone-expansions of $\bar{\tau}_{cl_f}$, so the sketch morphism

$$t_{cl_f^{*field}} : \epsilon_\lambda \longrightarrow \bar{\tau}_{cl_f^{*fields}}$$

will indeed by propositions 3 and corollary 2. (in section 4.1.1) be a strict type morphism (a "set-model-epi" from ϵ_λ)
So we have constructed a strict type confusing the two sketches of fields ξ_1, ξ_2. We already saw that Top discriminate between ξ_1 and ξ_2 so, following the general problem, the only thing left for us to do is to make some conjecture about the free $t_{cl_f^{*fields}}$-type generated by Top.
The free $t_{cl_f^{*fields}}$-type of Top: Lets see what happen if we force all epi-monos in Top to be isos. For any topological space X (object of Top) the identity on X (as a set) gives, for any topology $\mathcal{T}$ on X rise to a continuous epi-mono

$$(X, \mathcal{P}(X)) \rightarrowtail\!\!\twoheadrightarrow (X, \mathcal{T}) .$$

where $\mathcal{P}(X)$ is the discrete topology on X. Hence if we force epi-monos in Top to be isomorphisms, we get that all topologies are the discrete topology. We thus conjecture $T_{t_{cl_f^{*fields}}}(Top) \cong$ Set.

Since $t_{cl_f^{*fields}}$ is strict we can profit from the application of *generalized associated sheaf theorem*, in the ways discussed in section 6. We conclude:
-The sketch morphism $\xi_2 \longrightarrow \xi_1$ indicate the theorem that fields in the sense of ξ_2 are co-products of their 0-subspace and non-0-subspace. valid in all categories with finite limits and co-limits where all epi-monos are isos. This is trivial since ξ_1 is itself the general proof, it evidently fulfill this property.

-If our favorite semantics is of type $t_{cl_f^{*fields}}$ then there is no need for the more general sketch ξ_2, when interpreting field-structures, we can stick to the simpler ξ_1.

The other way around, we should not believe that the simplest (set theoretical) way of sketching a structure always meet our purpose, since we saw that the first sketch of fields ξ_1 does not give us topological fields when modelled in Top. We should free our minds of set-theory and describe concepts by sketching them according to where they are needed.

7.3 Two sketches of monoids

We now look at another example a little more complicated than the previous, concerning two sketches of monoids following two different set theoretical definitions of monoids. We will establish a strict theoretical type confusing these two sketches, as well as look at one example

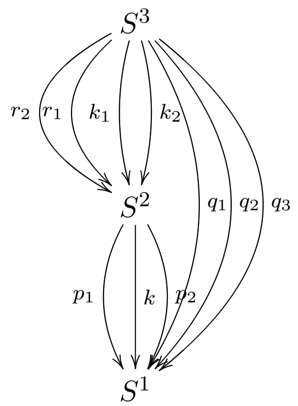
of a category that discriminate between them.
Two ways to define a monoid in the classical set-theoretical setting:

Def. 1: A monoid (M, k, e) is a set M with a binary operation "k" everywhere defined, associative, and with a distinguished unit element e with respect to the operation.

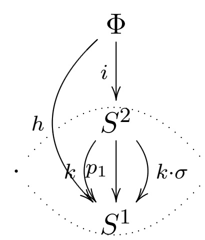
Def. 2: A monoid (M, k) is a set M with a binary operation "k" everywhere defined, associative, and for which there exist a unit element.

Sketch of monoids following definition 1: This is exactly the sketch of monoids as constructed as first step towards the sketch of fields ξ_1 in the previous section (p. 36), we will call it μ_1.

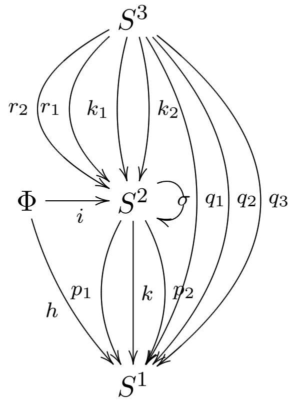
Sketch of monoids following definition 2:
Sketch of semi-groups:



which is just the first sketch of monoids μ_1, deprived of everything concerning the 0-ary operation e, the given unit element.
In stead of the 0-ary operation we add the object Φ representing the formula (conjunction) $x \cdot e = e \cdot x = x$ as the vertex of a distinguished cone:

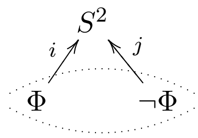


where $\sigma : S^2 \longrightarrow S^2$ is to be thought of as a permutation (i.e. $p_1 \cdot \sigma = p_2, \quad p_2 \cdot \sigma = p_1$). This gives us the sketch S_Φ:

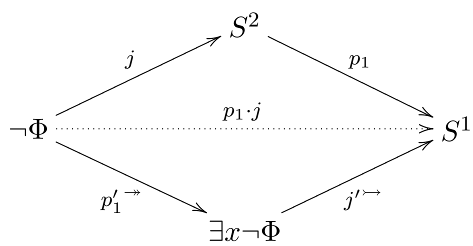


Thus we have added the equations: $h = p_1 \cdot i = k \cdot \sigma \cdot i = k \cdot i$ (h just naming the composite). When modelled in Set this will exactly mean that Φ is the subset of pairs of elements that satisfy the unity-condition, i.e. the subset $\{(x, e) \in M \times M | x \cdot e = e \cdot x = x\}$.

But since we need the formula $\exists e \neg(\exists x \neg(x \cdot e = e \cdot x = x))$, we introduce $\neg\Phi$ by distinguishing the discrete co-cone

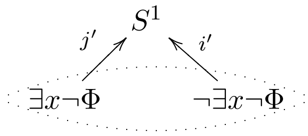


following the set theoretical meaning of the complement of Φ in S^2 (i.e. in terms of formulas all pairs not fulfilling Φ).
Remark: One should here keep in mind that the uniqueness of $\neg\Phi$ given Φ, is in no way guaranteed in an arbitrary model, since it is a special property of Set. This means that we are already envisaging certain types of models having the property that one co-projection in a binary co-product completely determines the other co-projection as subobject of the co-product (sum).
To express the central part of the definition 2. (i.e. the existence of a unit element as a property of the binary operation; not itself introduced as a 0-ary operation as in μ_1) we need to distinguish certain factorizations of certain arrows: for a formula Θ (figuring in a sketch as a vertex of a distinguished cone, like Φ) we can express $\exists y \Theta$ as the "image" of the projection forgetting the variable y. So to get $\exists x \neg\Phi$ we add a factorization of the map $p_1 \cdot j$:

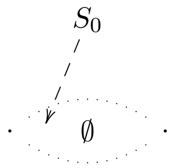


and distinguish p_1' as a potential epi and j' as a potential mono.
As before (with Φ) we distinguish the "complement" of $\exists x \neg\Phi$ as "complementary subobject"

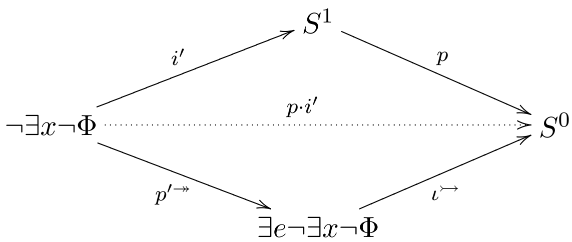
in S^1, by distinguishing the co-cone



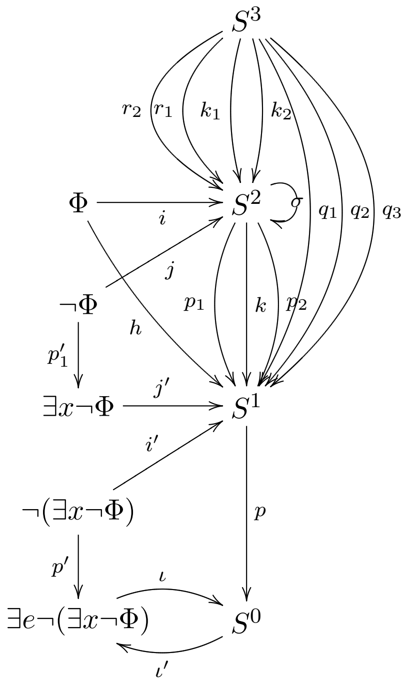
To be able to talk about the truth of our formula we add the object S^0 $(=1)$ a vertex of the empty cone (terminal object in the model)



and projection $p : S^1 \longrightarrow S^0$. Thus we can as before get the formula $\exists e \neg (\exists x \neg \Phi)$, as "sub-object" of 1, by adding the factorization:



and again p' should be realized as epi and ι as mono.
Our formula $\exists e \neg \exists x \neg \Phi$ is to be true in any model, so we need distinguish ι as invertible, i.e. add an arrow $\iota' : 1 \longrightarrow \exists e \neg \exists x \neg \Phi$ and equation $\iota \cdot \iota' = 1_1$ and $\iota' \cdot \iota = 1_{\exists e \neg \exists x \neg \Phi}$.
All in all, we get the underlying multiplicative graph of μ_2:



Cones and equations are the ones gradually introduced in the above.

Following the general problem we wish to find an equivalence $\varphi : \mathrm{Set}^{\mu_2} \sim \mathrm{Set}^{\mu_1}$, derived from a sketch-morphism $\upsilon : \mu_2 \longrightarrow \mu_1$, i.e. such that $\varphi = \mathrm{Set}^{\upsilon}$ but obviously it is not possible to get a sketch morphism $\mu_2 \longrightarrow \mu_1$ in the strict sense, by lack of distinguished cones and co-cones in μ_1. So we add to μ_1 what is needed to get a sketch-morphism, in a way such that the new larger sketch $\bar{\mu_1}$ has the same models in Set as both μ_1 and μ_2.
We thus get two sketch morphisms to $\bar{\mu_1}$ in the following way

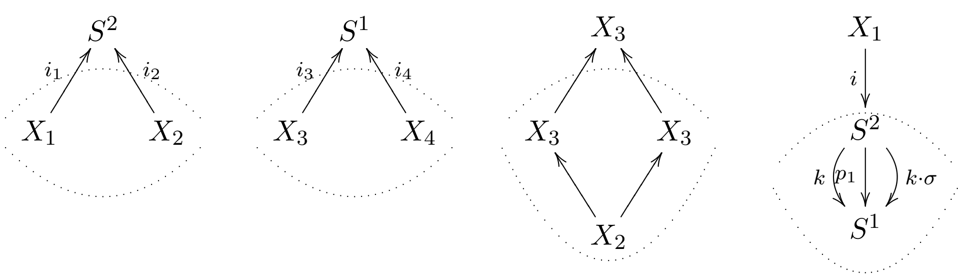
$$\begin{array}{ccc} \mu_2 & \xrightarrow{\upsilon} & \bar{\mu}_1 \\ & & \uparrow{\scriptstyle incl} \\ & & \mu_1 \end{array}$$

and we are then interested in determining a general type in which, both of these two sketch-morphisms induce an equivalence of model categories and hence an equivalence:

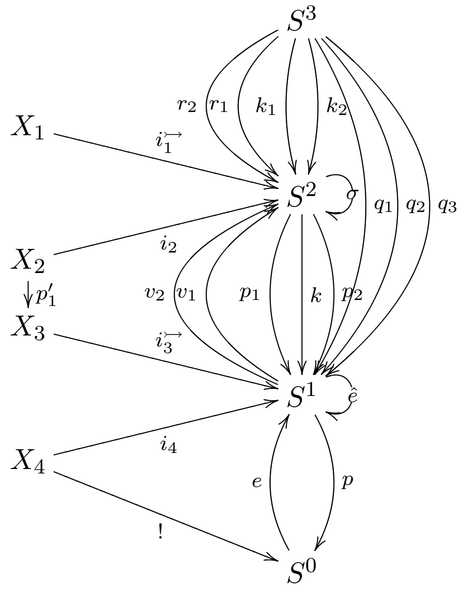
$$\mathrm{Mod}_{\mathcal{C}}(\mu_1) \sim \mathrm{Mod}_{\mathcal{C}}(\mu_2)$$

for all categories $\mathcal{C}$ of this type.

<u>Construction of $\bar{\mu_1}$:</u> we just add and distinguish the cones and co-cones missing to get a morphism $\upsilon : \mu_2 \longrightarrow \bar{\mu_1}$. We thus add to μ_1 objects X_1, X_2, X_3, X_4 and distinguish co-cones and cone:



hence $\bar{\mu_1}$ has underlying multiplicative graph:

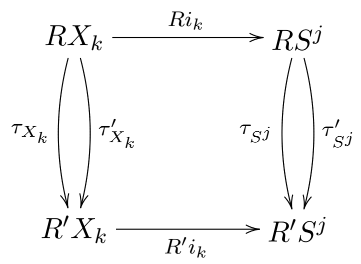


with equations and cones as μ_1 plus $u \circ i_4 =!$, $p_1 \circ i_2 = i_3 \circ p_1'$, the three distinguished co-cones and the distinguished cone, above.
$\bar{\mu_1}$ has same models as μ_1 in Set, since any model $R : \mu_1 \longrightarrow \text{Set}$ can be expanded to a model $\bar{R} : \bar{\mu_1} \longrightarrow \text{Set}$. Because $\bar{R}X_1$ is determined as the vertex of a limit, then X_2 can (up to iso) only be mapped to the complement of $\bar{R}X_1$ in $\bar{R}S_1 \times \bar{R}S_1$ and $\bar{R}X_3$ will be determined by epi-mono factorization and at last $\bar{R}X_4$ will be determined in the same way as $\bar{R}X_2$. So we in fact get an isomorphism of set model categories.
Now this makes use of some properties particular for the category Set: the existence of complements and unique epi-mono factorization. But we are interested in identifying which general properties a category need to satisfy for μ_1 and $\bar{\mu}_1$ to have the same models therein. We want to arrive at a general strict type of categories that has the properties needed.
We can systematize this procedure by checking the properties of *(iii)* in the standard theorem stated earlier concerning equivalences of categories (theorem 7.1).
We thus investigate the properties of a category $\mathcal{C}$ needed for $\text{Mod}(\bar{\mu_1}, \mathcal{C}) \xrightarrow{incl^*} \text{Mod}(\mu_1, \mathcal{C})$ to have the properties of *(iii)*:
-*incl* is an inclusion so $incl^*$ is evidently full.
-$incl^*$ is faithful, i.e. for any R, R' models of $\bar{\mu_1}$, the map

$$\text{Hom}_{\text{Mod}(\bar{\mu_1})}(R, R') \longrightarrow \text{Hom}_{\text{Mod}(\mu_1)}(incl^* R, incl^* R')$$

is injective: take $\tau, \tau' : R \longrightarrow R'$ two natural transformations such that $\tau \neq \tau'$, then there is at least one object S in $Ob_{\bar{\mu_1}}$ such that $\tau_S \neq \tau'_S$. If S is in μ_1 we evidently have $incl^* \tau \neq$

$incl^*\tau'$. If not we must have $S = X_k$ for some $k \in \{1, \dots, 4\}$ and by naturality



$$R'i_k\tau_{X_k} = \tau_{S^j}Ri_k$$

and

$$R'i_k\tau_k = \tau'_{S^j}Ri_k$$

Now if all $R'i_k$'s are monomorphism, we have $\tau_{X_k} \neq \tau'_{X_k}$ implies $R'i_k\tau_{X_k} \neq R'i_k\tau'_{X_k}$ and hence $\tau_{S^j}Ri_k \neq \tau'_{S^j}Ri_k$ which gives us $\tau_{S^j} \neq \tau'_{S^j}$ and hence $incl^*\tau \neq incl^*\tau'$, i.e. $incl^*$ is faithful. We have that $R'i_1$ and $R'i_3$ are monomorphisms for $\mathcal{C}$ any category, and if $\mathcal{C}$ has the property that *a co-projection in a sum is a monomorphism* then $R'i_2$ and $R'i_4$ will be monomorphisms too.

To show the last property of (iii) take a model of μ_1 $R : \mu_1 \longrightarrow \mathcal{C}$, we then wish to find an expansion of R to a model $\bar{R}$ of $\bar{\mu_1}$ such that $incl^*\bar{R} \cong R$.

First we suppose the category $\mathcal{C}$ to be at least of type t_{cl_f}, i.e. it has finite limits and co-limits, then $\bar{R}X_1$ is determined up to isomorphism (it is the vertex of a equalizer cone), so we need $\mathcal{C}$ to have properties assuring the existence of objects $\bar{R}X_2, \bar{R}X_3, \bar{R}X_4$ defining a model $\bar{R}$ of $\bar{\mu_1}$.

As already pointed out, the properties that *all monomorphisms are co-projections* in a binary sum and that all such co-projection determines the other up to isomorphism are desired. We will call the latter the *sum-property*. Now:

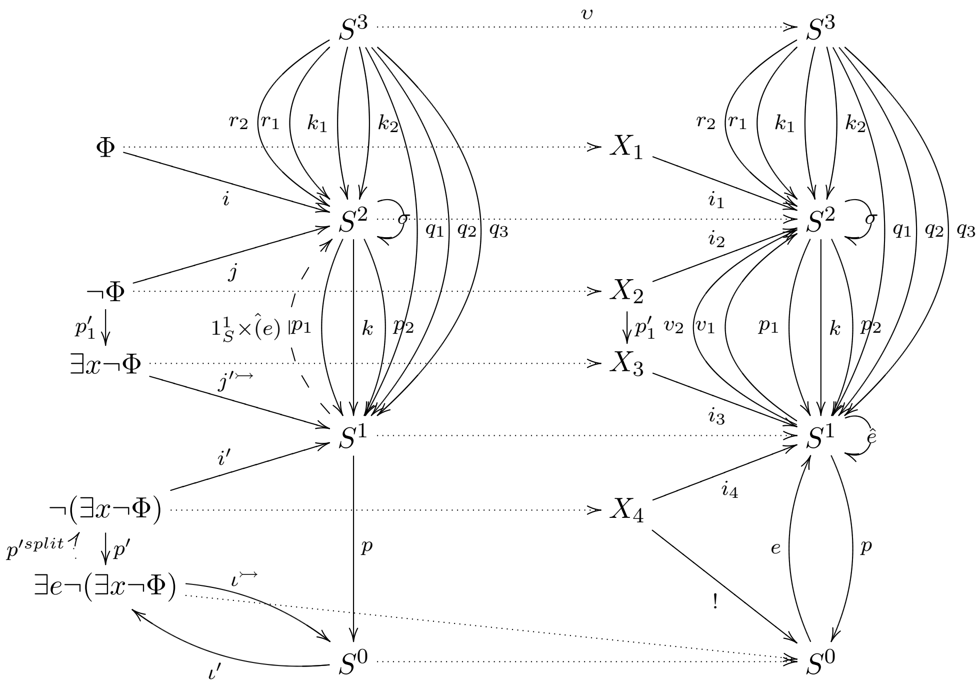
-We can expand R to model X_2: if monomorphisms are co-projections in some sum (in $\mathcal{C}$) $\bar{R}X_2$ will be determined by $\bar{R}X_1$ up op isomorphism by the *sum-property*.

-We can expand R to model X_3: because of the way we (in μ_2) sketched the notion of existence following the second definition of monoids any category serving as target for models will have some epi-mono-factorization property. So if we demand that $\mathcal{C}$ has *epi-mono-factorization for all arrows* (not necessarily unique) then the existence of $\bar{R}X_3$ will follow.

-We can expand R to model X_4: again if all monomorphisms are co-projections in some sum, $\bar{R}X_4$ will be determined by the sum-property.

We conclude that $incl^*$ is an equivalence of model categories if $\mathcal{C}$ is of the type t_{cl_f} with the extra properties: the sum-property, all arrows have epi-mono-factorization, all monomorphisms are co-projections in a binary sum and all co-projections are monomorphisms.

We now look at $\upsilon : \mu_2 \longrightarrow \bar{\mu_1}$



We are interested in when $\upsilon^* : \mathrm{Mod}_{\mathcal{C}}(\bar{\mu_1}) \longrightarrow \mathrm{Mod}_{\mathcal{C}}(\mu_2)$ is an equivalence, so again we check the properties of (iii):
We start with the last property of (iii) (assuming that $\mathcal{C}$ has the properties arrived at for $incl^*$ to be an equivalence): Take $R : \mu_2 \longrightarrow \mathcal{C}$ a model of μ_2. We then again wish the existence of an extended model $\bar{R}$ of $\bar{\mu}_1$ such that $\upsilon^*\bar{R} \cong R$. If $\mathcal{C}$ has properties that permit, from $R(\mu_2)$ the existence of arrows modelling the arrows $e, \hat{e}, v_1, v_2$ (in $\bar{\mu_1}$), we will have the extension searched for. If C has the property that *all epimorphisms are split* and we note ε^{split} the split morphism of an epimorphism ε in $\mathcal{C}$, then $\bar{R}$ can be defined as equal to R on everything hit by υ (meaning that we get $\upsilon^*\bar{R} = R$) and then we define $\bar{R}(e) = R(i')R(p')^{split}R(\iota')$ and $\bar{R}(v_1) = 1_{RS^1} \times \bar{R}(\hat{e})$, $\bar{R}(v_2) = \bar{R}(\hat{e}) \times 1_{RS^1}$. Then $\bar{R}$ is obviously a functor extending R to $\bar{\mu}_1$, but it is not obvious that it defines a model. We need to proof that $\bar{R}(p_1)\bar{R}(v_1) = \bar{R}(k)\bar{R}(v_1) = \bar{R}(\sigma k)\bar{R}(v_1)$.

I had overlooked this in the first version of this paper, so I have not yet proven that we only need the properties arrived at so far in order to prove that $\bar{R}$ is a model of $\bar{\sigma_1}$. But it is obviously true for set theoretical models, since sums are disjoint unions in Set, the question is wether the sum property assures the connection between the two ways of sketching unitarity, I am not sure, but if we add the property that all arrows from the terminal object to a sum will facter through one and only one of its co-projections, then we get the connection searched for. We shall leave this at that since we have no particular use of this example other than to illustrate how the "the mashinery" works.

υ^* is faithful: same procedure as before (with $incl^*$) and it follows from the fact that υ is surjective on objects.

υ^* is full: take any pair of models $R, R' \in Ob_{\mathrm{Mod}(\bar{\mu_1},\mathcal{C})}$ then

$$\mathrm{Hom}_{\mathrm{Mod}(\bar{\mu_1},\mathcal{C})}(R,R') \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(\mu_2,\mathcal{C})}(\upsilon^* R, \upsilon^* R')$$

is surjective: if we take a natural transformation

$$\tau : \upsilon^* R \longrightarrow \upsilon^* R'$$

then we need to find a natural transformation $\bar{\tau} : R \longrightarrow R'$ such that $\upsilon^* \bar{\tau} = \tau$. υ is surjective on objects, so it is enough to show that the naturality of τ extends to the arrows $v_1, v_2, e, \hat{e}$. As long as we are talking about models in a category $\mathcal{C}$ with all epis split there is no harm in adding the pair of potential split maps p'^{split}, $1^1_S \times \hat{e}$ of p', p_1 to the sketch μ_2 (as above indicated by dotted arrows).
Then we can set $\upsilon(1^1_S \times \hat{e}) = v_1$ and the naturality of τ will follow easily by the equations $v_2 = \sigma v_1$, $\hat{e} = p_2 v_1$, $ep = \hat{e}$.

7.3.1 A type confusing the two sketches of monoids

By the above systematic request for equivalent model categories, we can conclude[1] that a sufficient[2] type of category, in which the two sketches μ_1 and μ_2 have the same models, is a category with finite limits and co-limits and furthermore satisfying: the sum-property, all epimorphisms are split, all arrows have some epi-mono factorization, all monomorphisms are co-projections in some sum and all co-projections in a sum are monic.
In fact the last property follows from the others:

> **Lemma**: *A sufficient type for which μ_1 and μ_2 have equivalent model categries, is the type: categories with finite limits and co-limits, with the sum-property, where all epis split, where all arrows have epi-mono factorization and where all monos are co-projection in some sum.*

Proof:
In a category of the type in the Lemma, all co-projections in a binary sum are monomorphisms: take a sum

$$A \xrightarrow{a} A+B \xleftarrow{b} B$$

and take an epi-mono factorization of, lets say, a

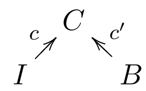
$$\begin{array}{ccc} A & \xrightarrow{a} & A+B \\ & {\scriptstyle\varepsilon}\searrow \quad \nearrow{\scriptstyle m} & \\ & I & \end{array}$$

then the diagram

$$I \xrightarrow{m} A+B \xleftarrow{b} B$$

[1] up to the missing verification.
[2] remember that we nowhere proved necessity of all of these properties

is a sum. Because, take any diagram



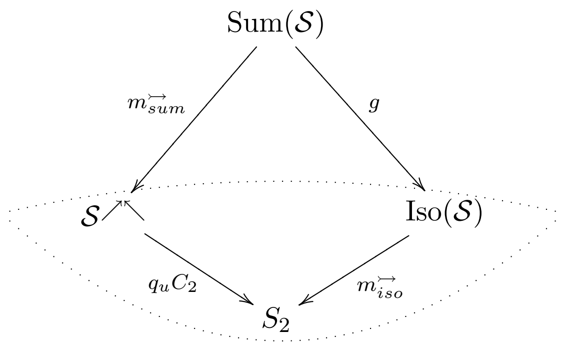
then, since $(A+B,(a,b))$ is a sum, there exists a unique arrow $h : A+B \longrightarrow C$ such that $ha = c\varepsilon$ and $hb = c'$. Now since $a = m\varepsilon$ we have that $ha = hm\varepsilon = c\varepsilon$ which, by ε epimorphic, gives $hm = c$. So $(A+B,(m,b))$ is a sum, and hence by sum-property $A \cong I$, thus a is a monomorphism (same goes for b).

■

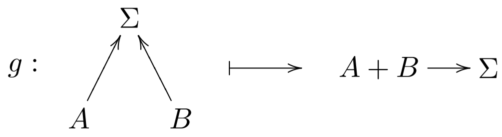
We will call the properties arrived at, the $*_{monoids}$-properties. i.e. the properties: sum-property, all epimorphisms split, all arrows have some epi-mono factorization and all monos are co-projections. The corresponding sketch of the type confusing the two sketches of monoids, will hence be noted $\bar{\tau}_{cl_f^{*monoids}}$, and this sketch will be constructed in the following, by adding the $*_{monoids}$-properties to the sketch $\bar{\tau}_{cl_f}$ one by one:

The sum-property: given a binary co-product (co-limit indexed by the discrete graph 2), if one co-projection is known then the other is known up to isomorphism. So in the sketch $\bar{\tau}_{cl_f}$ we need make the restriction, required by this sum-property, on the object destined to be the set of binary co-products:
For $\mathcal{I} = 2 = \{\bullet \quad \bullet\}$ we have our choice of co-limits, by the arrow L_2 destined to choose a 2-co-limit of all pairs of objects in the underlying graph of the model. We will call sums all co-limits indexed by 2, so "the object of sums" will be the object 2-co-limits($\mathcal{S}$) as introduced in the construction of $\bar{\tau}_{cl_f}$. This is practical since the sum-property concerns all sums and not just the chosen ones, though the chosen ones are representative up to isomorphism.

We have (in the model) the object of sums Sum($\mathcal{S}$), as the subobject of $\mathcal{S}^{\nearrow\nwarrow}$ such that the hook by the comparison map (the arrow u between the vertexes of the two co-cones compared) is an isomorphism, i.e the following cone is distinguished:

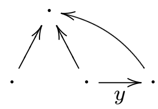


Where C_2 is the comparison map, taking a 2-co-cone to the diagram comparing it with the chosen sum of the same basis and q_u is the projection of the hook between vertexes in the comparison diagram. So g will (up to isomorphism) be the map taking a sum to the isomorphism from the chosen sum of the same basis:

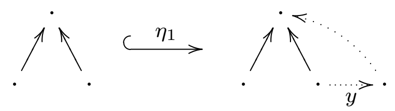


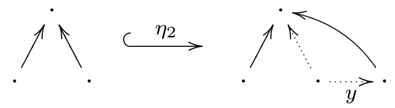
where we (abusively) let $A + B$ note our choice of co-limit by (the model of) L_2 for the pair A, B.

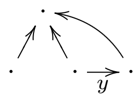
We introduce the graph $\mathcal{K}_5 = \cdot \qquad \cdot \longrightarrow \cdot$ and then the joint graph $1_{\mathcal{K}_5}$:

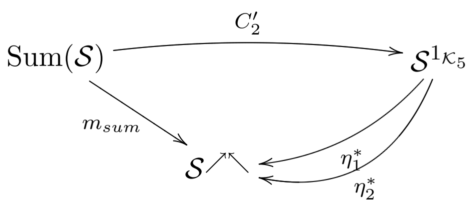


and inclusions:

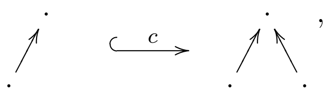




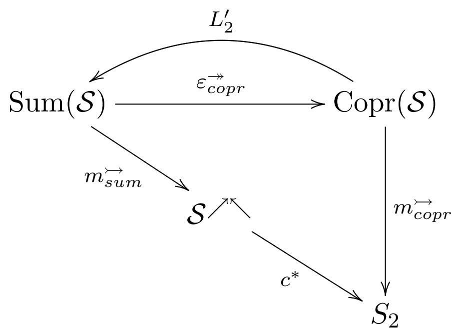
the object $\mathcal{S}$  $= \mathcal{S}^{1_{\mathcal{K}_5}}$ and a new comparison map C_2':



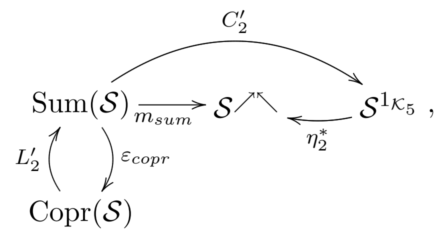
such that $\eta_1^* C_2' = m_{sum}$, to make sure that any sum is embedded into a comparison graph; it is actually being compared. We then continue to consider what we are comparing a sum to, by the map C_2'.
If we note c^* the composition from the right by graph morphism



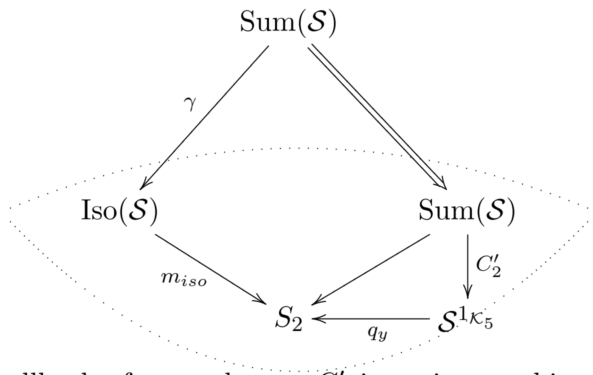
we can introduce (up to isomorphism) the "object of co-projections", as "Copr($\mathcal{S}$)" by adding the following factorization of the map $c^* m_{sum}$ that forgets the right leg in a sum:



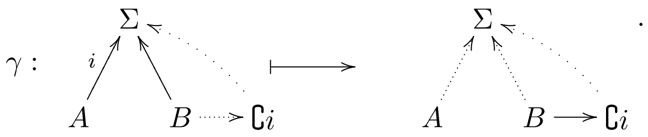
where L'_2 is added destined two be a right inverse to the epi ε_{copr}, i.e. we add equation $\varepsilon_{copr} L'_2 = 1_{\text{Copr}(\mathcal{S})}$. Since we sketch theoretical types set-theoretically, i.e. to be modelled in Set, there is nothing "wild" in adding L'_2 to the sketch, because in Set all epimorphisms (surjections) have a right inverse (given of course that we believe in the axiom of choice).
Then in a model R, RL'_2 will be a new choice map, from the object of co-projections to the object of sums, taking a co-projection to a sum where it is the left leg, we will call this sum the "complement sum".
Now we add equation $\eta_2^* C'_2 = m_{sum} L'_2 \varepsilon_{copr}$



meaning that C'_2 compares (by "new hook" y) any sum to its corresponding sum obtained by forgetting one co-projection and then choosing a sum by L'_2 and that all sums, in this way, will be compared to their corresponding "complement sum" by a (model of) the "new hook" map y.
Now the sum-property comes down to adding a map $\gamma : \text{Sum}(\mathcal{S}) \longrightarrow \text{Iso}(\mathcal{S})$ and distinguish the cone

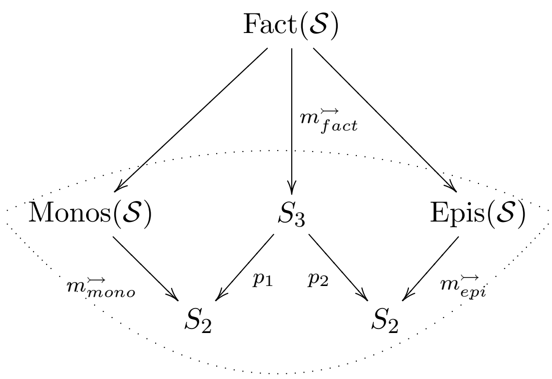


Then in a model the pullback of m_{iso} along $q_y C'_2$ is an isomorphism, meaning that for any sum, we have that the "new hook" in the "new comparison graph" (comparing the sum to the "complement sum" of one of its co-projections) is an isomorphism, and hence γ is the arrow modelled as



We have now sketched the sum-property: for a co-projection q in a sum, any sum with q as one co-projection ("left leg"), will compare two $L'_2(q)$ by an isomorphism, so q does determine the other co-projection up to isomorphism.

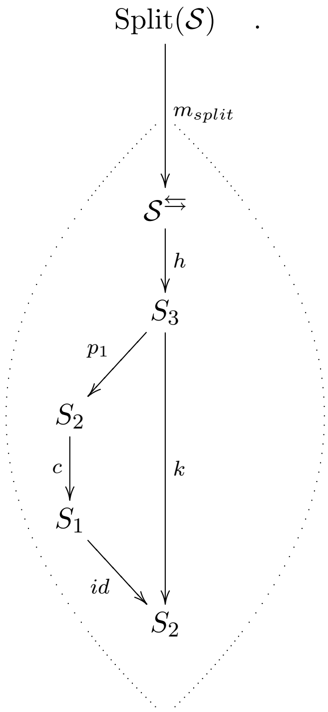
All arrows have an epi-mono factorization: we add an object, that we call "Fact($\mathcal{S}$)", to be modelled as a subobject of composable pairs of arrows consisting of an epimorphism followed by a monomorphism. This object can be added to the sketch as vertex of the following distinguished cone:



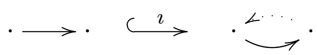
i.e m_{fact} is modelled as, at the same time, the pullback of the subobject of monomorphisms along the first projection of pairs of composable arrows, and the pullback of the subobject of epimorphisms along the second projection of composable arrows, so clearly m_{fact} is a potential monomorphism.
Then distinguish $km^{\twoheadrightarrow}_{fact}$ as a potential epimorphism. Since epis have right inverse in Set we get that there are choices of epi-mono factorizations for all arrows (in any given model in Set), by the possible right inverses (sections) of the model of km_{fact}.

All epimorphisms are split: first we add the object "of all split diagrams", "Split($\mathcal{S}$)", to be modelled as the subobject of the object $\mathcal{S}^{\leftrightarrows}$, of diagrams of the sort $\bullet \underset{a}{\overset{a'}{\leftrightarrows}} A$ such that

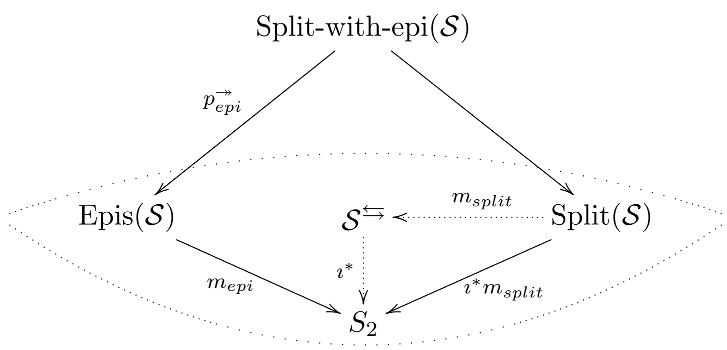
$aa' = 1_A$. By distinguishing the cone



Notice how this is just the left part of the diagram we saw earlier (p.30) distinguishing the object "Iso($\mathcal{S}$)".
Now if we denote $\imath^* : \mathcal{S}^{\leftrightarrows} \longrightarrow S_2$ the composition from the right with the graph inclusion



Then we distinguish the object "Split-with-epi($\mathcal{S}$)", to be modelled as subobject of split diagrams where the co-section is an epimorphism, by pulling back m_{epi} along $\imath^* m_{split}$. We thus distinguish the cone:



and furthermore that the pullback p_{epi} of $\imath^* m_{split}$ along m_{epi} is a potential epimorphism, which, for a model R, in set gives the existence of a right inverse

$$R(p_{epi})' : \text{Epis}(\mathcal{S}) \longrightarrow \text{split-with-epi}(\mathcal{S}) \ ,$$

choosing, for each epi, a split diagram where it itself is the left inverse.
<u>Monos are coprojections:</u> The lemma gives a potential mono $Copr(\mathcal{S}) \rightarrowtail Monos(\mathcal{S})$ so we just need to distinguish this map to sketch an epi as well and our sketch is final.

The sketch $\bar{\tau}_{cl_f^{*monoids}}$ was here constructed in a manner taking very much advantage of the fact that it is to be modelled in set. As well as viewing the fact that we wish for a strict type, i.e. we wish to take advantage of the proposition 3 and corollary 2 and hence want, as much as possible, to restrict our expansion of t_{cl_f} to a finite series of expansions by cones.

The theoretical type $t_{cl_f^{*monoids}}$ and its corresponding free generator $T_{t_{cl_f^{*monoids}}}$ of $t_{cl_f^{*monoids}}$-types.

<u>proposition 4:</u>

$$t_{cl_f^{*monoids}} : \epsilon_\lambda \longrightarrow \bar{\tau}_{cl_f^{*monoids}}$$

is a set-model-epi, i.e a strict theoretical type.

<u>Proof:</u> The only part of $\bar{\tau}_{cl_f^{*monoids}}$ not achieved by a finite sequence of expansions by cones of $\bar{\tau}_{cl_f}$ are the maps $L_2' : \text{Copr}(\mathcal{S}) \longrightarrow \text{Sums}(\mathcal{S})$ "choice of complement sum" and $C_2' : \text{Sums}(\mathcal{S}) \longrightarrow \mathcal{S}^{1\kappa_5}$ "comparison (by y)of a sum to the complement sum of its left leg".
Take $\varphi \xrightarrow{s} \psi$ the natural isomorphism between models of $\bar{\tau}_{cl_f^{*monoids}}$ induced by corollary 2. (i.e not yet natural on L_2', C_2') and take q an element of $\varphi(\text{Copr}(\mathcal{S}))$, then

$$\begin{array}{rcl} \varphi L_2'(q) & = & \cdot \xrightarrow{q} \cdot \xleftarrow{\complement_\varphi q} \cdot \\ \psi L_2'(s_{copr} q) & = & \cdot \xrightarrow{s_{copr} q} \cdot \xleftarrow{\complement_\psi s_{copr} q} \cdot \end{array}$$

and since $s_{Copr(\mathcal{S})} \varphi \varepsilon_{copr} = \psi \varepsilon_{copr} s_{Sum(\mathcal{S})}$ we get that $s_{Sum(\mathcal{S})}(q, q') = (s_{Corp(\mathcal{S})} q, s_{Copr(\mathcal{S})} q')$ and then $s_{Sum(\mathcal{S})} \varphi L_2'(q) = s_{Sum(\mathcal{S})}(q, \complement_\varphi q) = (s_{Copr(\mathcal{S})}, s_{Copr(\mathcal{S})} \complement_\varphi q)$ and $\psi L_2' s_{Copr(\mathcal{S})}(q) = (s_{Copr(\mathcal{S})} q, \complement_\psi s_{Copr(\mathcal{S})})$. Now by the sum-property we get $\complement_\psi s_{Copr(\mathcal{S})} \cong s_{Copr(\mathcal{S})} \complement_\varphi q$, meaning we only get naturality of the isomorphism s up to isomorphism. This is however not a serious problem, because the model of L_2' is just one of several isomorphic sections of the epimorphism forgetting the "right-leg" co-projection in a given sum. Any model (φ) will naturally posses all possibilities for modelling L_2 and one of these (or rather all of them together) will furnish the pure naturality of $s : \varphi \xrightarrow{\cong} \psi$. Said differently, the model $\psi : \bar{\tau}_{cl_f^{*monoids}} \longrightarrow \text{Set}$ posses (in Set) a map isomorphic to $\psi L_2'$ and naturally isomorphic to $\varphi L_2'$ by the isomorphism s.

■

We have thus found a strict type $t_{cl_f^{*monoids}}$ confusing our two sketches of monoids μ_1 (following "Def. 1") and μ_2 (following "Def. 2").

Now keeping to the general problem, and in good mathematical tradition, we point out at least one category discriminating between μ_1 and μ_2, i.e. a category $\mathcal{C}$ such that $\mathcal{C}^{\mu_1}$ is not equivalent to $\mathcal{C}^{\mu_2}$.

Discrimination of the two sketches of monoids:(Coppey, L. [1992]) If K is a commutative field, we call K-Vect the category of K K-vector spaces with arrows all K-linear maps.
Then if (M, k, e) signifies a monoid in the sense of μ_1 internally of the category K-Vect (i.e. (M, k, e) is the image of a model $R_M : \mu_1 \longrightarrow K - \text{Vect}$ we get:

-M is a K-vector space.
-The unit element $e : M^0 \longrightarrow M$ is a K-linear map, so since M^0 is the zero-space e must be the zero-map. Meaning that the unit element for the (K-linear) composition k is the vector 0. Now since $(x, y) = (x, 0) + (0, y)$ we get by unitarity (in the sense of μ_1):

$$\begin{aligned} k(x,y) &= k((x,0)+(0,y)) \\ &= k(x,0)+k(0,y) \\ &= k(x,e)+k(e,y) \\ &= x+y. \end{aligned}$$

So for a K-Vector space m there is exactly one possibility for a monoid-structure on M (internally of K-Vect), the one obtained by the addition of vectors in M.

Now suppose that (M, k) is a monoid in the sense of μ_2 (internally of K-Vect), then:

-M is a K-vector space.
- $k : M \times M \longrightarrow M$ is a K-linear map.
-We already have $k(0, 0) = 0$.
-Because of associativity of k, we get:

$$k(k(x,0),0) = k(x,k(0,0)) = k(x,0)$$

-Consequently the map $\lambda : M \longrightarrow M$, $\lambda(x) = k(x, 0)$ is an idempotent K-linear map; the same goes for the map $\phi : M \longrightarrow M$, $\phi(y) = k(0, y)$, and we see that:

$$\begin{aligned} k(k(x,y),z) &= \lambda(\lambda(x)+\phi(y))+\phi(z) \\ &= \lambda^2(x)+\lambda\phi(y)+\phi(z) \\ &= \lambda(x)+\lambda\phi(y)+\phi(z) \end{aligned}$$

and

$$\begin{aligned} k(x,k(y,z)) &= \lambda(x)+\phi(\lambda(y)+\phi(z)) \\ &= \lambda(x)+\phi\lambda(y)+\phi^2(z) \\ &= \lambda(x)+\phi\lambda(y)+\phi(z). \end{aligned}$$

Hence by associativity of k we can deduce $\lambda\phi = \phi\lambda$, thus the law of composition must be of the form $k = \lambda p_1 + \phi p_2$ where p_1, p_2 are the two projections $M^2 \longrightarrow M$ and λ, ϕ are two commuting idempotent K-linear maps.
The other way around, if M is a K-vector space and $\lambda, \phi : M \longrightarrow M$ two commuting idempotent K-linear maps, then (M, k) is, at least, a semi-group (internally of K-Vect). But there is nothing in the previous calculations that prevents unitarity (in the sense of μ_2):

> -The "only possible monoid structure" in the sense of μ_1, is one among "the possible" structures in the sense of μ_2, by putting $\lambda, \phi = 1_M$.
> -Better yet, there exist non-isomorphic monoid structures in the sense of μ_2 (internally of K-Vect) having the same underlying semi-group.

To show the last statement, take for example $\lambda = 1_M$ and $\phi = 0$. Then $k = p_1$ and the trivial semi-group (M, p_1) can be equipped with two different monoid structures in the sense of μ_2:

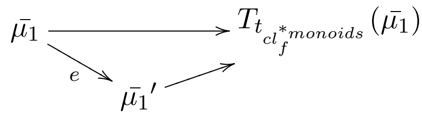
- Φ is the diagonal of M^2.
- $\neg\Phi$ is one "complement" of this diagonal.
- $\exists x \neg\Phi$ is the then either isomorphic to M or to $\{0\}$, according to the choice of "complement" above (either it is different from $p_2(M^2)$ or it is equal to $p_2(M^2)$).
- $\exists e \neg(\exists x \neg\Phi)$ is always true since every K-linear subspace of $\{0\}$ is $\{0\}$.

If we only require $M \neq \{0\}$, we here have two non-isomorphic monoid structures on the same vector space in K-Vect. Hence K-Vect discriminate between μ_1 and μ_2, and hence is not of type $t_{cl_f^{*monoids}}$.
However, we observe that monoids in K-Vect, in the sense of μ_2, perhaps have a unit element as models of μ_2. But at the same time (and in the same sense) they don't have unit element, because all models of μ_2 into K-Vect extend (uniquely) to a model (into K-Vect) of the sketch obtained by adding the object $\neg(\exists e \neg(\exists x \neg\Phi))$ to the sketch μ_2.
There is no more contradiction in this than in the fact that $0 = 1$ in K-Vect. So we are just confirming the fact that our choice of interpretation of the negation, when constructing μ_2, was indeed set-theoretical (or Boolean); we were not thinking about monoids in the sense of Def. 2 internally of additive categories when we sketched μ_2.

The core of the above demonstration of K-Vect not being of type $t_{cl_f^{*monoids}}$ is that K-Vect does not satisfy the sum-property, for example the diagonal of M^2 has complements that define different subobjects of M^2 whenever the vector space M is non-zero.

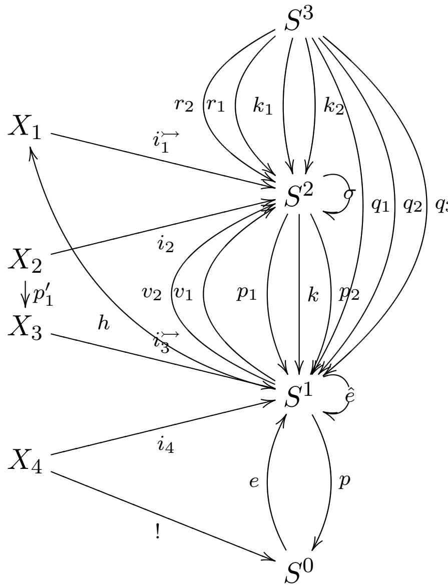
We thus conjecture that $T_{t_{cl_f^{*monoids}}}(K - Vect)$ ("forcing" the sum-property) will give the trivial category of just one object and one arrow (the zero-vector space with zero-map)

Since $t_{cl_f^{*monoids}}$ is strict we can profit from the application of *generalized associated sheaf theorem*, in the ways discussed in section 6. We conclude:

-The two sketch morphisms $incl : \mu_1 \longrightarrow \bar{\mu_1}$, $\upsilon : \mu_2 \longrightarrow \bar{\mu_1}$ indicate the theorem, relative to the two sketches μ_1 and μ_2 for $t_{cl_f^{*monoids}}$-models, that the two sketches describe the same mathematical theory. The proof is obtained by the factorization 

where $\bar{\mu_1}'$ is a sketch evidently modelling both sketches and could be the folllowning sketch:

Underlying multiplicative graph of $\bar{\mu_1}'$:



only adding the arrow $h : S^1 \longrightarrow X_1$ and equations $v_1 = i_1 h$ which in any model is assured by the universal property of the equalizer distinguished by X_1, since we in μ_1 have the equations $kv_1 = k\sigma v_1 = p_1 v_1 = 1_{S^1}$. This makes evident that a model of $\bar{\mu_1}'$ is at the same time a model of μ_1 and μ_2 describing one and same structure, by the equations $v_2 = \sigma v_1, \quad \hat{e} = p_2 v_1, \quad ep = \hat{e}.$
One could then discuss whether it is necessary to add the arrow h, or even whether it is enough to add h to obtain the *evidence.* Personally I think there should be something in the proof-factorization directly showing the connection between the two descriptions, even though anyone with a trained eye will notice the existence of the map h in any model.

-The natural conjecture that μ_1 and μ_2 are confused when modelled in any boolean topos, has been checked and confirmed! (Boolean toposes are clearly of type $t_{cl_f^*monoids}$).

-In the process of showing that K-Vect discriminate between μ_1 and μ_2, we came across another theorem: *that any law k of a semi group internally of K-Vect, is of the form $k = \lambda p_1 + \phi p_2$ where λ, ϕ are commuting idempotent linear maps.* This gives rise to an object proving a theorem relative to the sketch of semi groups, valid in the additive type (exercise).

8 Conclusion

We have clearly been supporting the seven mottos stated in the introduction and we have arrived at a basic understanding of theoretical strict types as subcategories in the category of sketches furnishing a general machinery for proving and discriminating.
A machinery that applies whenever we (within the given size frame) are able to sketch our problem, i.e. whenever we can get a strict type containing the theory in which we want to show a certain property related to a certain concept, and whether we can sketch this concept and formulate the property as a sketch morphism. Then we can examine whether the property gives rise to a theorem in our theory by examining (as we did in the example of monoids) for what types of models the sketch morphism gives rise to an equivalence of model categories, or by directly progressing towards the free types (in our Type) generated by the sketches of our sketch morphism. Then if the sketch morphism gives rise to an isomorphism/equivalence of generated free types we have a general proof that our concept has the considered property in any category of the considered strict type, thus also in the theory we started out with.
The examples supply us with ideas for a manual to this machinery of *checking conjectures* since they give samples of how to proceed in praxis, when we want two confuse/discriminate sketches or prove theorems in the frame Esq_λ.
We also see from the examples that many usual (λ-small) categories will be of some strict type so the above machinery can be supposed to apply to many mathematical problems and perhaps fruitfully.

Concerning further work, there are two main questions now urged upon us:
Firstly, how do we optimize the search for the *confusing type*, meaning on the one hand that it is wishful to minimize the properties needed in a category in order for two sketches to have the same models therein and on the other hand assuring this by a systematization of the progression/mounting towards a sketch that evidently shows the confusion in models of a necessary and sufficient type. Secondly, the question of application, of finding interesting examples showing that this machinery does indeed bare fruit in mathematical practice.

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