CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

RICHARD BLUTE
J.R.B. COCKETT
TIMOTHY PORTER
R.A.G. SEELY

Kähler categories

Cahiers de topologie et géométrie différentielle catégoriques, tome 52, nº 4 (2011), p. 253-268

<http://www.numdam.org/item?id=CTGDC_2011__52_4_253_0>

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
Vol. LII-4 (2011)

KÄHLER CATEGORIES

by Richard BLUTE, J.R.B. COCKETT, Timothy PORTER and R.A.G. SEELY

Résumé. Dans cet article, on établit une relation entre la notion de *catégorie codifférentielle* et la théorie, plus classique, des différentielles de Kähler, qui appartient à l'algèbre commutative. Une catégorie codifférentielle est une catégorie monoïdale additive, ayant une monade T qui est en outre une modalité d'algèbre, c.à.d. une attribution naturelle d'une structure d'algèbre associative à chaque object de la forme $T(C)$. Enfin, une catégorie codifférentielle est équipée d'une transformation dérivante, qui satisfait quelques axiomes typiques de différentiation, exprimés algèbriquement.
La notion classique de différentielle de Kähler définit celle d'un module des formes A-différentielles par rapport à A, où A est une k-algèbre commutative. Ce module est équipé d'une A-dérivation universelle. Une *catégorie Kähler* est une catégorie monoïdale additive, ayant une modalité d'algèbre et un *objet des formes différentielles* associé à chaque objet. Suivant l'hypothèse que la monade algèbre libre existe et que l'application canonique vers T est epimorphique, les catégories codifférentielles sont Kähler.

Abstract. This paper establishes a relation between the notion of a *codifferential category* and the more classic theory of Kähler differentials in commutative algebra. A codifferential category is an additive symmetric monoidal category with a monad T, which is furthermore an algebra modality, *i.e.* a natural assignment of an associative algebra structure to each object of the form $T(C)$. Finally, a codifferential category comes equipped with a deriving transformation satisfying typical differentiation axioms, expressed algebraically.
The traditional notion of Kähler differentials defines the notion of a module of A-differential forms with respect to A, where A is a commutative k-algebra. This module is equipped with a universal A-derivation. A *Kähler category* is an additive monoidal category with an algebra modality and an *object of differential forms* associated to every object. Under the assumption that the free algebra monad exists and that the canonical map to T is epimorphic, codifferential categories are Kähler.

Keywords. Differential categories, Kähler differential, Kähler category
Mathematics Subject Classification (2010). 13N05, 18D10

1. Introduction

Differential categories were introduced in [3] in part to categorify work of Ehrhard and Regnier on differential linear logic and the differential λ-calculus [10, 11]. In the present paper, we shall work with the dual notion of a codifferential category. The notion was also introduced with an eye towards capturing the interaction in certain monoidal categories between an abstract differentiation operator and a (possibly monoidal) monad or comonad. We require our monads to be equipped with *algebra modalities*, *i.e.* each object naturally obtains the structure of an algebra with respect to the monoidal structure. The primary examples of differential and codifferential categories were the categories of vector spaces, relations and sup-lattices, each with some variation of the symmetric algebra monad. Differentiation is formal differentiation of polynomials. The notion of algebra modality is also fundamental in the categorical formulation of linear logic [4]. Thus both the work of Ehrhard and Regnier as well as our work can be seen as an attempt to extend linear logic to include differential structure.

The logical and semantic consequences of this sort of extension of linear logic look to be very promising, likely establishing connections to such areas as functional analysis, as in the *Köthe spaces* or *finiteness spaces* introduced by Ehrhard, [8, 9]. Recent work [5] shows that the category of *convenient vector spaces* [12] is also a differential category. This category is of great interest as it provides underlying linear structure for the category of *smooth spaces* [12], a cartesian closed category in which one can consider infinite-dimensional manifolds.

Two significant areas in which there is a well-established notion of abstract differentiation is algebraic geometry and commutative algebra, where *Kähler differentials* are of great significance. There the Kähler module of differential forms is introduced, for instance see [13, 14]. This is similar in concept to various aspects of the definition of differential category; in particular, the notion of differentiation must satisfy the usual *Leibniz rule*. But, in addition, Kähler differentials have a universal property that the notion of differential category seems to be lacking. Roughly, given a commutative algebra A, the Kähler A-module of differential forms is a module equipped with a derivation satisfying Leibniz, which is universal in the sense that to any other A-module equipped with a derivation, there is a unique A-module map commuting with this differential structure. There is no such (explicit) universal structure in the definition of differential category.

With this in mind, we introduce the new notion of a *Kähler category*. A Kähler category is an additive symmetric monoidal category equipped with a monad T and an algebra modality. We further require that each object be assigned an object of differential forms, *i.e.* an object equipped with a derivation and satisfying a universal property analogous to that arising from the Kähler theory in commutative algebra.

Our main result is that every codifferential category, satisfying a minor structural property, is Kähler. In retrospect, this perhaps should not have been surprising. In any symmetric monoidal category, one can define both the notions of associative algebra and module over an associative algebra. Furthermore if A is any associative algebra in a symmetric monoidal category and C is an arbitrary object, then one can form the free A-module generated by C, as $A \otimes C$. This satisfies the usual universal property of free A-modules. So in a codifferential category, TC is automatically an associative algebra, and thus $TC \otimes C$ is the free TC-module generated by C. This is what we will take to be our object of differential forms.

The difficulty in the proof is in demonstrating that the map of TC-modules arising from the freeness of $TC \otimes C$ also commutes with the differential structure. This is where an additional property, which we call *Property* K, becomes necessary. We assume that our category has sufficient coproducts to construct free associative algebras. As such, there is a canonical morphism of monads between this free algebra monad and the monad giving the differential structure. Property K requires that this morphism be an epimorphism. In many codifferential categories, this is indeed the case. The proof that this condition suffices reveals additional structure in the definition of codifferential category.

A different approach to capturing the universality of Kähler differentials is contained in [7]. There the work is grounded in the notion of Lawvere algebraic theory, as opposed to linear logic in the present framework. A comparison of the two approaches would be interesting.

Acknowledgments Thanks to the University of Ottawa for providing the third author with a *Distinguished Visiting Professorship*. We also want to thank Anders Kock for asking the right question, and the anonymous referee for insightful comments.

2. Codifferential categories

We here review the basic definition in the paper [3]. The emphasis there was on differential categories. We here need the dual definition of codifferential category. We refer the reader to [3] for more details and motivations.

2.1 Basic definitions

Definition 2.1. *1. A symmetric monoidal category C is* additive *if it is enriched over commutative monoids*[1]*. Note that in an additive symmetric monoidal category, the tensor distributes over the sum.*

2. An additive symmetric monoidal category has an algebra modality *if it is equipped with a monad (T, μ, η) such that for every object C in*

[1] In particular, we only need addition on Hom-sets, rather than abelian group structure.

$\mathcal{C}$, the object, $T(C)$, has a commutative associative algebra structure

$$m: T(C) \otimes T(C) \longrightarrow T(C), \qquad e: I \longrightarrow T(C)$$

and this family of associative algebra structures satisfies evident naturality conditions.

3. *An additive symmetric monoidal category with an algebra modality is a* codifferential category *if it is also equipped with a* deriving transformation[2], i.e. *a natural transformation*

$$d: T(C) \longrightarrow T(C) \otimes C$$

satisfying the following four equations[3]*:*

(d1) $e; d = 0$ *(Derivative of a constant is 0.)*

(d2) $m; d = (id \otimes d); (m \otimes id) + (d \otimes id); c; (m \otimes id)$ *(where c is the appropriate symmetry) (Leibniz Rule)*

(d3) $\eta; d = e \otimes id$ *(Derivative of a linear function is constant.)*

(d4) $\mu; d = d; \mu \otimes d; m \otimes id$ *(Chain Rule)*

For a diagrammatic presentation of (the duals of) these equations, see [3].

We will need an iterated version of the Leibniz rule, which we state now. (The proof is straightforward.)

Lemma 2.2. *In any codifferential category, the composite:*

$$TC^{\otimes n} \xrightarrow{m} TC \xrightarrow{d} TC \otimes C$$

is equal to the sum over i of the composites:

$$\begin{array}{rl} TC^{\otimes n} \xrightarrow{id \otimes id \cdots d \cdots \otimes id} & TC \otimes \cdots TC \otimes C \otimes \cdots TC \\ \xrightarrow{c} & TC \otimes \cdots TC \otimes \cdots TC \otimes C \\ \xrightarrow{m \otimes id} & TC \otimes C \end{array}$$

In this composite the d occurs in the i-th position. The c is the appropriate symmetry to move the C to the final position without changing the order of the TC terms.

[2]We use the terminology of a *deriving transformation* in both differential and codifferential categories.

[3]For simplicity, we assume the monoidal structure is strict

2.2 The polynomial example

We review the canonical example of a codifferential category, as this construction will be generalized in a number of different ways. Let k be a field, and Vec the category of k-vector spaces. It is well-established that Vec is an additive, symmetric monoidal category, and further that the free symmetric algebra construction determines an algebra modality. Specifically, if V is a vector space, set

$$T(V) = k + V + (V \otimes_s V) + (V \otimes_s V \otimes_s V) \ldots,$$

where $\otimes_s$ denotes the usual symmetrized tensor product.

An equivalent, basis-dependent description is obtained as follows. Let J be a basis for V, then

$$T(V) \cong k[x_j \mid j \in J],$$

in other words, $T(V)$ is the polynomial ring generated by the basis J. We have that $T(V)$ provides the free commutative k-algebra generated by the vector space V, and as such provides an adjoint to the forgetful functor from the category of commutative k-algebras to Vec. The adjunction determines a monad on Vec, and the usual polynomial multiplication makes $T(V)$ an associative commutative algebra, and endows T with an algebra modality.

Furthermore Vec is a codifferential category [3]. It is probably easiest to see using the basis-dependent definition. Noting that, even if V is infinite-dimensional, any polynomial only has finitely many variables appearing, the coderiving transformation is defined by

$$f(x_{j_1}, x_{j_2}, \ldots, x_{j_n}) \mapsto \sum_{i=1}^{n} \frac{\partial f}{\partial x_{j_i}}(x_{j_1}, x_{j_2}, \ldots, x_{j_n}) \otimes j_i$$

where $\frac{\partial f}{\partial x_{j_i}}$ is defined in the usual way for polynomial functions.

Theorem 2.3. *(See [3]) The above construction makes* Vec *a codifferential category.*

By similar arguments, we can state:

Theorem 2.4.

1. *The category* Rel *of sets and relations is a differential and codifferential category*[4].

2. *The category* Sup *of sup-semi lattices and homomorphisms is a codifferential category.*

Further details can be found in [3].

3. Review of Kähler differentials

To see the origins of our theory of Kähler categories and introduce our main example, we now consider the classical case of Kähler differentials; see [13, 14] and many other sources, for details.

Let k be a field, A a commutative k-algebra, and M an A-module[5].

Definition 3.1. *An* A-derivation *from A to M is a k-linear map $\partial\colon A \longrightarrow M$ such that $\partial(aa') = a\partial(a') + a'\partial(a)$.*

One can readily verify under this definition that $\partial(1) = 0$ and hence $\partial(r) = 0$ for any $r \in k$.

Definition 3.2. *Let A be a k-algebra. A* module of A-differential forms *is an A-module Ω_A together with an A-derivation $\partial\colon A \longrightarrow \Omega_A$ which is universal in the following sense: for any A-module M, for any A-derivation $\partial'\colon A \longrightarrow M$, there exists a unique A-module homomorphism $f\colon \Omega_A \longrightarrow M$ such that $\partial' = \partial f$.*

Lemma 3.3. *For any commutative k-algebra A, a module of A-differential forms exists.*

There are several well-known constructions. The most straightforward, although the resulting description is not that useful, is obtained by constructing the free A-module generated by the symbols $\{\partial a \mid a \in A\}$ divided out by the evident relations, most significantly $\partial(aa') = a\partial(a') + a'\partial(a)$. Of more value is the following description, found, for instance, as Proposition 8.2A of [13].

[4]Noting the self-duality which commutes with the monoidal structure.

[5]All modules throughout the paper will be *left* modules.

Lemma 3.4. *Let A be an k-algebra. Consider the multiplication of A:*

$$\mu: A \otimes A \longrightarrow A.$$

Let I be the kernel of μ and set $\Omega_A = I/I^2$. Define a map $\partial: A \longrightarrow \Omega_A$ by

$$\partial b = [1 \otimes b - b \otimes 1]$$

where we use square brackets to represent the equivalence class. The pair (Ω_A, ∂) acts as a module of differential forms. □

Example 3.5. For the key example, let $A = k[x_1, x_2, \ldots, x_n]$, then Ω_A is the free A-module generated by the symbols $dx_1, dx_2, \ldots, dx_n$, so a typical element of Ω_A looks like

$$f_1(x_1, x_2, \ldots, x_n)dx_1 + f_2(x_1, x_2, \ldots, x_n)dx_2 + f_n(x_1, x_2, \ldots, x_n)dx_n.$$

Note how this compares to our polynomial example of a codifferential category. If V is an n-dimensional space, then there is a canonical isomorphism:

$$\Omega_{T(V)} \cong T(V) \otimes V.$$

This provides the basis for our main theorem on Kähler categories below.

4. Kähler categories

In all of the following, the category $\mathcal{C}$ will be symmetric, monoidal and additive. Unless otherwise stated, all algebras will be assumed to be both associative and commutative for the remainder of the paper.

Definition 4.1. *Let A be an algebra, and $M = \langle M, \cdot_M : A \otimes M \longrightarrow M \rangle$ an A-module. Then an A-*derivation *to M is an arrow $\partial: A \longrightarrow M$ such that*

$$\mu; \partial = c; id \otimes \partial; \cdot_M + id \otimes \partial; \cdot_M \qquad \text{and} \qquad \partial(1) = 0$$

Note that if we are enriched over abelian groups, the second condition may be dropped.

Definition 4.2. *A* Kähler category *is an additive symmetric monoidal category with*

- *a monad T,*
- *a (commutative) algebra modality for T,*
- *for all objects C, a module of $T(C)$-*differential forms $\partial_C: T(C) \longrightarrow \Omega_C$, viz *a $T(C)$-module Ω_C, and a $T(C)$-derivation, $\partial_C: T(C) \longrightarrow \Omega_C$, which is universal in the following sense: for every $T(C)$-module M, and for every $T(C)$-derivation $\partial': T(C) \longrightarrow M$, there exists a unique $T(C)$-module map $h: \Omega_C \longrightarrow M$ such that $\partial; h = \partial'$.*

$$\begin{array}{ccc} T(C) & \xrightarrow{\partial} & \Omega_C \\ & \underset{\partial'}{\searrow} & \vdots\, h \\ & & M \end{array}$$

Remark 4.3. We remark that Ω is functorial, indeed, is left adjoint to a forgetful functor, in the following sense. Consider the category $\mathcal{D}er(T)$ of "T-derivations": its objects are tuples (C, M, ∂), for C an object of $\mathcal{C}$, M a $T(C)$-module, and $\partial: T(C) \longrightarrow M$ a derivation. A morphism $(C, M, \partial) \longrightarrow (C', M', \partial')$ is a pair $(f: C \longrightarrow C', g: M \longrightarrow M')$, where f is a morphism in $\mathcal{C}$ and g is a $T(C)$-module morphism, satisfying $\partial; g = T(f); \partial': T(C) \longrightarrow M'$. The universal property of Ω allows us to regard it as a functor $\mathcal{C} \longrightarrow \mathcal{D}er(T)$, since given $f: C \longrightarrow C'$, $T(f); \partial': T(C) \longrightarrow \Omega_{C'}$ is a derivation if ∂' is, and hence f induces $\Omega_f: \Omega_C \longrightarrow \Omega_{C'}$. Moreover Ω is easily seen to be left adjoint to the forgetful functor $\mathcal{D}er(T) \longrightarrow \mathcal{C}$ given by the first projection.

Theorem 4.4. *The category of vector spaces over an arbitrary field is a Kähler category, with structure as described in the previous section.*

We would like to show that codifferential categories are Kähler, but are not in a position to do so at the moment, although we do not have a counterexample. The difficulty in getting a general result lies in the fact that in the definition of differential or codifferential category, there is no *a priori* universal property; evidently universality is fundamental in Kähler theory.

However there is a universal property at our disposal: since our monad is equipped with an algebra modality, we can use the fact that $T(C) \otimes C$ is the free $T(C)$ module generated by C.

Now suppose that $\mathcal{C}$ is a Kähler category. For each object C, we wish to construct an object Ω_C, with a universal derivation. As already suggested, we will define $\Omega_C = T(C) \otimes C$.

So suppose we have a $T(C)$-derivation $\partial: T(C) \longrightarrow M$. We must construct the unique $T(C)$-module map $h : T(C) \otimes C \longrightarrow M$ with the required property. But because of the universal property of the free left $T(C)$-module generated by C, we already know there is a unique $T(C)$-module map $h: T(C) \otimes C \longrightarrow M$.

It remains to verify that $d; h = \partial$, which is the focus of the remainder of the paper. The key to our approach is that there must be an interaction between the T-algebra structure and the associative algebra structure.

4.1 Free associative algebras *vs.* algebra modalities

We assume we have a symmetric monoidal additive category with an algebra modality and with finite biproducts and countable coproducts. We will also need to consider the tensor algebra, *i.e.*

$$F(C) = I + C + C \otimes C + C \otimes C \otimes C \ldots$$

As always, this is the free (not-necessarily-commutative) associative algebra generated by C. As such, the functor induces a monad $(F, \bar{\mu}, \bar{\eta})$ on our category, and that monad has its own (noncommutative) algebra modality.

Because of the existence of biproducts, we are able to establish close connections between the tensor algebra monad and the associative algebras arising from our algebra modality. These are expressed as a collection of natural transformations.

By the universality of F, we have the following natural transformations: $\alpha: FT \longrightarrow T$ (given by the lifting of the identity $T \longrightarrow T$), and $\varphi: F \longrightarrow T$ (given by the lifting of the unit $\eta: I \longrightarrow T$). More explicitly, these are given by the following constructions.

For any object C, $\alpha_C: FT(C) \longrightarrow T(C)$ can be built out of each component (since its domain is a coproduct). So we want a map $\alpha_n: T(C)^{\otimes n} \longrightarrow T(C)$, but this is just the n-fold multiplication on $T(C)$. In the case where $n = 0$, there is the canonical map $\eta: I \longrightarrow T(C)$. The map α_C is the usual quotient of the free associative algebra generated by (the underlying object of) $T(C)$ onto $T(C)$.

Also we observe that $\varphi_C: FC \longrightarrow TC$ is simply $F\eta_C; \alpha_C: FC \longrightarrow FTC \longrightarrow TC$.

Lemma 4.5. *φ is a morphism of monads*

Proof. This follows immediately from Proposition 6.1, Chapter 3 of [1] (where the reader can also find the definition of a morphism of monads). That proposition states that φ will be a morphism of monads if the following diagrams commute:

$$
\begin{array}{ccc}
T(C) & \xrightarrow{\eta} & FT(C) \\
& \searrow^{1} & \downarrow{\scriptstyle \alpha} \\
& & T(C)
\end{array}
\qquad
\begin{array}{ccccc}
FFT(C) & \xrightarrow{\mu} & FT(C) & \xleftarrow{F\mu} & FTT(C) \\
\downarrow{\scriptstyle F\alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
FT(C) & \xrightarrow[\alpha]{} & T(C) & \xleftarrow[\mu]{} & TT(C)
\end{array}
$$

These are straightforward, and in fact are an immediate consequence of the universal property of F, since the individual morphisms in these diagrams are all associative algebra maps (and so each composite is the unique lifting of the obvious map). More concretely, since objects of the form $F(C)$ are all coproducts, it suffices to check the equations componentwise, which is a simple exercise. □

Definition 4.6. *The monad T satisfies* Property K *if the natural transformation $\varphi: F \longrightarrow T$ is a componentwise epimorphism.*

If we are working in a category in which there is an evident monad, we will say that the category satisfies Property K, rather than the monad.

Proposition 4.7. *The categories of vector spaces, relations and sup-lattices, as described in Theorems 2.3, 2.4, satisfy Property* K.

Proof. (Sketch) For vector spaces, for example, this is the usual quotient by symmetrizing. The other two examples are similar. □

4.2 Codifferential categories satisfying K are Kähler

We now present the main result of the paper. In fact, we offer two proofs to illustrate different aspects of the notions involved.

Theorem 4.8. *If $\mathcal{C}$ is a codifferential category, whose monad satisfies Property* K*, then $\mathcal{C}$ is a Kähler category, with $\Omega_C = T(C) \otimes C$.*

Proof. We consider the "inclusion" map $\eta; d: C \longrightarrow T(C) \otimes C$. By equation 1 in the definition of codifferential category, we have $\eta; d = u; e \otimes id_C$.

Hence by the freeness of $T(C) \otimes C$, for any $T(C)$-module M and for any morphism $h: C \longrightarrow M$, there exists a unique *map of $T(C)$-modules*, $\hat{h}: T(C) \otimes C \longrightarrow M$ such that $\eta; d; \hat{h} = u; e \otimes id_C; \hat{h} = h$. Suppose as in the definition of Kähler category that we have a $T(C)$-module M and a $T(C)$-derivation $\partial: T(C) \longrightarrow M$. Taking $h = \eta; \partial$, we thus have a unique $T(C)$-module map $\hat{h}: T(C) \otimes C \longrightarrow M$ such that $\eta; d; \hat{h} = h = \eta; \partial$

So our goal is to show that we can cancel the η's in the previous equation.

Proof #1 The first proof is a straight calculation. We consider the morphisms:

$$\Phi = F\eta; \alpha; d; \hat{h} \quad \text{and} \quad \Psi = F\eta; \alpha; \partial$$

If we can show these two maps are equal, we are done given that Property K gives that $F\eta; \alpha$ is surjective and thus $d; \hat{h} = \partial$.

Since the domain of Φ and Ψ is a coproduct, it suffices to show that the maps are equal on each component.

For the I component, both composites are 0, by definition.

For the C component, we have $\eta; d; \hat{h} = \eta; \partial$, which has already been shown.

We next argue the binary $C \otimes C$ component, to demonstrate the techniques for the n-ary case. We wish to show that the composite

$$\Phi_2 = C \otimes C \xrightarrow{\eta\otimes\eta} TC \otimes TC \xrightarrow{m} TC \xrightarrow{d} TC \otimes C \xrightarrow{\hat{h}} M$$

is equal to:

$$\Psi_2 = C \otimes C \xrightarrow{\eta\otimes\eta} TC \otimes TC \xrightarrow{m} TC \xrightarrow{d'} M$$

Proceed as follows. Throughout the proof, we assume strict associativity. Any unit isomorphism is denoted by u and c always denotes a symmetry. It will always be clear from the context what the relevant symmetry is.

$$\begin{aligned}\Phi_2 &= \eta\otimes\eta; id\otimes d; m\otimes id; \hat{h} + \eta\otimes\eta; d\otimes id; c; m\otimes id; \hat{h} \\ &= \eta\otimes u; id\otimes e\otimes id; m\otimes id; \hat{h} + u\otimes\eta; id\otimes e\otimes id; c; m\otimes id; \hat{h} \\ &= \eta\otimes id; \hat{h} + id\otimes\eta; c; \hat{h}\end{aligned}$$

Now note that

$$\begin{aligned}\Psi_2 &= \eta\otimes\eta; id\otimes\partial; \cdot_M + \eta\otimes\eta; \partial\otimes id; \cdot_M \\ &= \eta\otimes h; \cdot_M + h\otimes\eta; c; \cdot_M\end{aligned}$$

The result then follows from the universal property of $(\hat{-})$. In particular, $id_{TC}\otimes h; \cdot_M = \hat{h}$.

This calculation shows the structure for the general n-ary case, which requires the n-ary Leibniz rule of Section 2. The n-ary versions of Φ and Ψ are

$$\Phi_n = \eta^{\otimes n}; m^{\otimes n-1}; d; \hat{h} \qquad \Psi_n = \eta^{\otimes n}; m^{\otimes n-1}; \partial$$

Expanding, we obtain

$$\Phi_n = \sum_{i=1}^{n} \eta^{\otimes i-1}\otimes id\otimes\eta^{\otimes n-i}; c; m^{\otimes n-2}; \hat{h}$$

and

$$\Psi_n = \sum_{i=1}^{n} \eta^{\otimes i-1}\otimes h\otimes\eta^{\otimes n-i}; c; m^{\otimes n-2}; \cdot_M$$

The result again follows from the definition of $\hat{h}$. □

We now give a more conceptual proof, using the universality of F (as the free associative algebra functor), rather than its explicit construction.

Suppose that A is a (commutative) algebra, and M an A-module. Then in fact $A+M$ has the structure of an algebra, in the following way. The unit is $I \xrightarrow{\langle e,0\rangle} A+M$, and the multiplication $(A+M)\otimes(A+M) \longrightarrow A+M$

is induced by the following three maps:

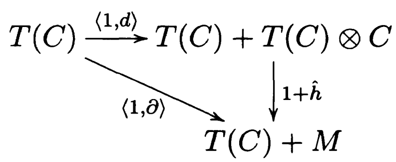
$$\begin{array}{ccccc} A \otimes A & \xrightarrow{m} & A & \longrightarrow & A + M \\ A \otimes M & \xrightarrow{\bullet} & M & \longrightarrow & A + M \\ M \otimes M & \xrightarrow{0} & M & \longrightarrow & A + M \end{array}$$

Moreover, this construction is functorial in M, so given a module morphism $M \longrightarrow N$, the map $A + M \longrightarrow A + N$ is an algebra morphism.

The following well-known observation [6] was used in the early work of Beck [2].

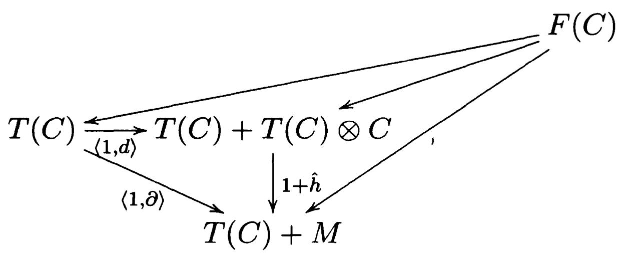
Lemma 4.9. *If A is a (commutative) algebra, M an A-module, then $A \xrightarrow{\partial} M$ is a derivation iff $A \xrightarrow{\langle 1, \partial \rangle} A + M$ is an algebra morphism.*

Proof #2 We note that $d; \hat{h} = \partial$ if and only if

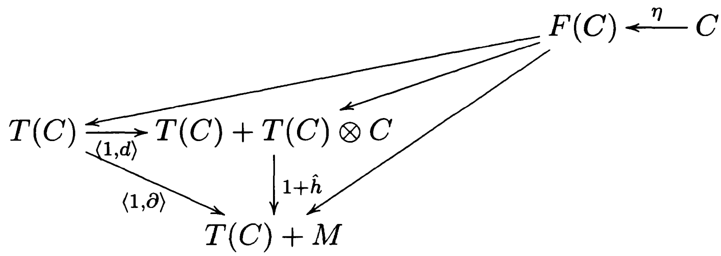


(*)

Now, given property K, this previous diagram commutes if and only if



Note that a $T(C)$-derivation followed by a $T(C)$-module map is a derivation. So in the diagram above, every morphism is a morphism of algebras. Since $F(C)$ is the free algebra generated by C, this diagram commutes if and only if it commutes on the image of C.



But this amounts to the equation $\eta; d; \hat{h} = \eta; \partial$, which is already established. □

References

[1] [M. Barr, C. Wells, 1985] *Toposes, Triples and Theories*, Springer-Verlag. Reprinted in Reprints in Theory and Applications of Categories, (2005).

[2] [J. Beck, 1967] *Triples, Algebras and Cohomology*, thesis. Reprinted in Reprints in Theory and Applications of Categories, (2003).

[3] [R. Blute, J.R.B. Cockett, R.A.G. Seely, 2006] Differential categories, Mathematical Structures in Computer Science 16, pp. 1049-1083.

[4] [R. Blute, J.R.B. Cockett, R.A.G. Seely, 1996] ! and ?: Storage as Tensorial Strength, Mathematical Structures in Computer Science 6, pp. 313-351.

[5] [R. Blute, T. Ehrhard, C. Tasson, 2011] A convenient differential category. To appear in Cahiers de Topologie et Géométrie Différentielle Catégorique.

[6] [N. Bourbaki, 1985] *Algebra*, Ch. III, section 10.10-10.12, Hermann.

[7] [E. Dubuc, A. Kock, 1984] On 1-form classifiers, Communications in Algebra 12, pp. 1471-1531.

[8] [T. Ehrhard, 2002] On Köthe sequence spaces and linear logic. Mathematical Structures in Computer Science 12, pp. 579-623.

[9] [T. Ehrhard, 2005] Finiteness spaces. Mathematical Structures in Computer Science 15, pp. 615-646.

[10] [T. Ehrhard, L. Regnier, 2003] The differential lambda-calculus. Theoretical Computer Science 309, pp. 1-41.

[11] [T. Ehrhard, L. Regnier, 2006] Differential interaction nets. Theoretical Computer Science 364, pp 166-195.

[12] [A. Frölicher, A. Kriegl, 1988] *Linear Spaces and Differentiation Theory*, Wiley.

[13] [R. Hartshorne, 1977] *Algebraic Geometry*. Springer-Verlag.

[14] [H. Matsumura, 1970] *Commutative Algebra*, W.A. Benjamin Co.

Richard Blute[6]
Dept of Mathematics and Statistics
University of Ottawa
Ottawa, Ontario, Canada
rblute@uottawa.ca

J.R.B. Cockett[6]
Dept of Computer Science
University of Calgary
Calgary, Alberta, Canada
robin@cpsc.ucalgary.ca

Timothy Porter
WIMCS
School of Computer Science
University of Bangor
Bangor, Wales, United Kingdom
t.porter@bangor.ac.uk

R.A.G. Seely[7]
Dept of Mathematics and Statistics
McGill University
Montreal, Quebec, Canada
rags@math.mcgill.ca

[6]Research supported in part by NSERC.
[7]Research supported in part by NSERC & FQRNT.