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# QUASI-EQUATIONS IN LOCALLY PRESENTABLE CATEGORIES

*by Jiri ADAMEK and Michel HEBERT*

*Dedicated to Francis Borceux on the occasion of his sixtieth birthday*

**Résumé**

Dans la tradition de Hatcher et de Banaschewski-Herrlich, nous définissons une quasi-équation comme étant une paire parallèle de morphismes finitaires. Un objet satisfait une quasi-équation si le foncteur contravariant qui lui correspond égalise la paire de morphismes qui la constitue. Les sous-catégories d'une catégorie localement finiment présentable qui peuvent être présentées par des quasi-équations sont précisément celles qui sont fermées sous les produits, les sous-objets et les colimites filtrées. Nous caractérisons les morphismes de théories correspondants dans le style de Makkai et Pitts, comme étant précisément les morphismes quotient forts. Ces résultats peuvent être vus comme l'analogue du théorème classique de Birkhoff pour les catégories localement finiment présentables. En cours de route, nous démontrons ce résultat plutôt surprenant que dans les catégories localement finiment présentables, tout épimorphisme fort finitaire peut s'écrire comme composé d'un nombre fini d'épimorphismes réguliers.

**Abstract**

Following the tradition of Hatcher and Banaschewski-Herrlich, we introduce quasi-equations as parallel pairs of finitary morphisms. An object satisfies the quasi-equation iff its contravariant hom-functor merges the parallel pair. The subcategories of a locally finitely presentable category which can be presented by quasi-equations are precisely those closed under products, subobjects and filtered colimits. We characterize the corresponding theory morphisms in the style of Makkai and Pitts as precisely the strong quotient morphisms. These results can be seen as an analogue of the classical Birkhoff Theorem for locally finitely presentable categories. On the way, we show the rather surprising fact that in locally finitely presentable categories, every finitary strong epimorphism is a composite of finitely many regular epimorphisms.

# 1 Introduction

Equations in classical (finitary, one-sorted) General Algebra are pairs of terms in $n$ variables, that is, pairs of morphisms

$$u, u' \colon n \rightrightarrows 1$$

in a Lawvere algebraic theory $\mathcal{T}$. An algebra, that is, a functor $A\colon \mathcal{T} \longrightarrow \mathbf{Set}$ preserving finite products, satisfies the equation iff $A(u) = A(u')$. General parallel pairs in $\mathcal{T}$

$$u, u' \colon n \rightrightarrows k$$

are nothing else than $k$-tuples of equations. Analogously, for S-sorted algebras we can form the Lawvere theories, which are categories whose objects are finite words $s_1 s_2 ... s_n$ over S so that the word is a product of the one-letter words $s_i$. Again, parallel pairs

$$u, u' \colon s_1 s_2 ... s_n \rightrightarrows t_1 t_2 ... t_k$$

are just $k$-tuples of properly sorted equations.

In the present paper we apply the same idea to locally finitely presentable categories $\mathcal{K}$ and their Gabriel-Ulmer theories $\mathcal{T}$. Recall that $\mathcal{K}$ can, up to equivalence, be identified with the category $Lex\, \mathcal{T}$ of functors $A\colon \mathcal{T} \longrightarrow \mathbf{Set}$ preserving finite limits. A *quasi-equation* in $\mathcal{K}$ is then a parallel pair of morphisms of $\mathcal{T}$, and an object $A$ satisfies the quasi-equation $(u, u')$ iff $A(u) = A(u')$. The *quasi-equational subcategories*, that is, the full subcategories which can be specified by a set of quasi-equations, are precisely those which are closed under products, subobjects, and filtered colimits in $\mathcal{K}$. This might seem surprising at first sight since in case $\mathcal{K}$ is the category of $\Sigma$-algebras for some signature $\Sigma$, we obtain precisely the concept of a quasi-variety, not that of a variety. However, this simply reflects the fact that whereas the Lawvere theory works with finitely generated free algebras, the Gabriel-Ulmer theory works with all finitely presentable algebras.

It was first observed by Bernhard Banaschewski and Horst Herrlich [5] that quasi-varieties can be presented by orthogonality with respect to finitary regular epimorphisms - and this is, as we demonstrate below, just a variation on presentation by quasi-equations in the Gabriel-Ulmer theory. Thus, the above characterization follows easily from [5], Proposition 2. What is now in our approach is that the existence of regular factorizations is not needed. Considering parallel pairs as a sort of identity was already investigated by Bill Hatcher [10] in a general setting. Actually, since $\mathcal{T}^{op}$ can be seen as a full subcategory of $\mathcal{K}$, our quasi-equations are a special case of what Hatcher calls identities, and we introduce them precisely in this manner.

There is another substantial difference between the case of Lawvere theories and those of Gabriel-Ulmer: in the former one, every equational subcategory $\mathcal{A}$ of $Alg\mathcal{T}$ defines a *congruence* on $\mathcal{T}$; more precisely, it defines a surjective theory morphism

$$Q \colon \mathcal{T} \longrightarrow \mathcal{S}$$

(which means a finite products preserving full functor which is the identity on objects) such that $\mathcal{S}$ is an algebraic theory of $\mathcal{A}$ and the embedding $\mathcal{A} \hookrightarrow Alg\mathcal{T}$ induces the theory morphism $Q$. Conversely, every surjective theory morphism is induced by an equational subcategory of $Alg\mathcal{T}$ (in the sense of the duality of [2]; see [3] for details). In contrast, quasi-equations in a Gabriel-Ulmer theory $\mathcal{T}$ do not, in general, define a congruence on $\mathcal{T}$. Instead, we obtain a *quotient functor* $Q \colon \mathcal{T} \longrightarrow \mathcal{S}$ in the sense of Michael Makkai and Andrew Pitts [16]. This means that

(i) every object of $\mathcal{S}$ is isomorphic to one in $Q[\mathcal{T}]$, and

(ii) every morphism $f\colon QT_1 \longrightarrow QT_2$ of $\mathcal{S}$ has the form

$$\begin{array}{ccc} QT_1 & \xrightarrow{\;f\;} & QT_2 \\ {\scriptstyle s}\downarrow & \nearrow_{Qg} & \\ QT_1' & & \end{array}$$

where $s$ is an isomorphism.

We prove that the theory morphisms corresponding to quasi-equation-al subcategories are precisely the *strong quotient functors*, which means that in (ii) above we can always choose $s = (Qm)^{-1}$ for some strong monomorphism $m\colon T_1' \longrightarrow T_1$ in $\mathcal{T}$. Here we closely follow the results obtained by Jiří Rosický and the authors in [12].

**Acknowledgement.** We are grateful to Enrico Vitale for formulating the problem of characterizing quasi-equational subcategories of locally finitely presentable categories (personal communication).

# 2 Quasi-equations in Finitely Accessible Categories

**2.1. Assumption** Throughout this section $\mathcal{K}$ denotes a finitely accessible category in the sense of [15] or [14]. That is, $\mathcal{K}$ has filtered colimits and a set

$$\mathcal{K}_{fp}$$

representing all finitely presentable objects, and whose closure under filtered colimits is all of $\mathcal{K}$.

**2.2. Conventions** Morphisms with finitely presentable domains and codomains are called *finitary.* By a *finitely presentable morphism* is meant a morphism $f\colon A \longrightarrow B$ which is a finitely presentable object of the slice category $A \downarrow \mathcal{K}$, see [11].

**2.3. Example** A function $f\colon A \longrightarrow B$ in **Set** is

(i) finitary iff the sets $A$ and $B$ are finite

and

(ii) finitely presentable iff the sets $B \setminus f[A]$ and $ker f \setminus \Delta_A$ are finite.

**2.4. Remark** For every finitely presentable object $A$, the finitely presentable morphisms with domain $A$ are precisely the finitary ones, see [11].

**2.5. Definition** By a *quasi-equation* in a category $\mathcal{K}$ is meant a parallel pair of finitary morphisms in $\mathcal{K}$. An object $K$ *satisfies* the quasi-equation

$$u, u' \colon P \rightrightarrows Q$$

provided that the hom-functor $\mathcal{K}(-, K)$ merges that pair. That is,

$$h \cdot u = h \cdot u' \ \text{ for all } \ h \colon Q \longrightarrow K.$$

A full subcategory $\mathcal{A}$ of $\mathcal{K}$ is called *quasi-equational* if there exists a set of quasi-equations in $\mathcal{K}$ satisfied by precisely those objects that lie in $\mathcal{A}$.

**2.6. Example** Let

$$\mathcal{K} = \Sigma\text{-}Alg$$

be the category of algebras of a given signature $\Sigma$ (finitary, one-sorted).

(i) Every equation $v = v'$ (between two terms) can be represented by a parallel pair in the Gabriel-Ulmer theory

$$\mathcal{T} = \mathcal{K}^{op}_{fp}.$$

In fact, let $F_n$ denote a free $\Sigma$-algebra on $n$ generators. If $v, v'$ are elements of $F_n$, consider the homomorphisms

$$v_0, v_0' \colon F_1 \rightrightarrows F_n$$

mapping the generator of $F_1$ to $v$ and $v'$ respectively. This quasi-equation (in the sense of 2.5) is satisfied by precisely those $\Sigma$-algebras which satisfy $v = v'$ in the classical sense.

(ii) More generally, every implication in the classical sense

$$(v_1 = v_1') \wedge \ldots \wedge (v_k = v_k') \quad \Longrightarrow \quad (w = w')$$

can be represented by a parallel pair of morphisms in $\mathcal{T}$. In fact, all the terms in this implication lie in some $F_n$, and then we have a finitely presentable algebra

$$F_n/\sim$$

for the congruence $\sim$ generated by $v_i \sim v_i'$ for $i = 1, ..., n$. The two elements $[w], [w']$ of the algebra $F_n/\sim$ define two homomorphisms

$$u, u' \colon F_1 \rightrightarrows F_n/\sim .$$

The corresponding quasi-equation is satisfied by precisely the $\Sigma$-algebras that satisfy the above implication.

(iii) Conversely, let

$$u, u' \colon P \rightrightarrows Q$$

be any quasi-equation in $\Sigma$-*Alg*. We can represent it by classical implications (as observed already by B. Banaschewski and H. Herrlich [5]). In fact, for $Q$ we have a congruence on a free algebra $F_n$ generated by finitely many pairs, say $(v_1, v_1'), ..., (v_k, v_k') \in F_n \times F_n$, such that $Q \cong F_n/\sim$. For every $x \in P$, choose terms $w_x, w_x' \in F_n$ with $u(x) = [w_x]$ and $(u')(x) = [w_x']$. Now consider all implications

$$(v_1 = v_1') \wedge ... \wedge (v_k = v_k') \quad \Longrightarrow \quad (w_x = w_x')$$

where $x$ ranges through the elements of $P$. A $\Sigma$-algebra satisfies these implications iff it satisfies the given quasi-equation $(u, u')$.

**2.7. Example** Let

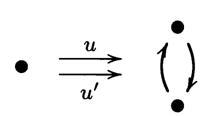
$$\mathcal{K} = Gra$$

be the category of graphs, that is, sets with a binary relation $R$.

(i) Antisymmetry

$$R(x, y) \wedge R(y, x) \;\; \Rightarrow \;\; (x = y)$$

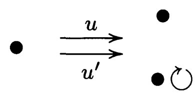
gives rise to a quasi-equation: it is given by the obvious pair



(ii) All graphs without loops, with the terminal object added, form the quasi-equational subcategory given by the formula

$$R(x,x) \;\Rightarrow\; (x=y)$$

or by the obvious parallel pair




(iii) More generally, every formula of the form

$$R(x_{i_1}, x_{i'_1}) \wedge \ldots \wedge R(x_{i_k}, x_{i'_k}) \;\Rightarrow\; (x_j = x_{j'})$$

in variables $x_1, \ldots, x_n$ gives rise to a quasi-equation $u, u' \colon P \rightrightarrows Q$ where the graph of $Q$ has vertices $x_1, \ldots, x_n$ and edges $x_{i_t} \longrightarrow x_{i'_t}$ for $t = 1, 2, \ldots k$.

(iv) Conversely, every quasi-equation can be expressed by implications of the above form. Note that properties such as reflexivity

$$(x=x) \;\Rightarrow\; R(x,x)$$

do not correspond to any quasi-equation . The fact that this subcategory is not quasi-equational will be clear from the Corollary 2.17, since the morphism $\bullet \longrightarrow \bullet\circlearrowright$ is a monomorphism in *Gra*. In general, universal Horn sentences as above with relation symbols on the right of the connector "$\Rightarrow$" do not define quasi-identities.

**2.8. Example** The smallest quasi-equational subcategory of $\mathcal{K}$ consists of precisely all subterminal objects, i.e., those $A$'s for which there is at most one morphism $X \longrightarrow A$ for each object $X$. In fact, these objects will satisfy all quasi-equations. Conversely, if $A$ is not subterminal, we can find a quasi-equation that $A$ does not satisfy: choose distinct morphisms $u, u' \colon K \rightrightarrows A$. The functor category $\mathcal{K}^{\rightrightarrows}$ is finitely accessible and we can express $(u, u')$ as a filtered colimit of finitary parallel pairs $(u_i, u'_i)$. Then $A$ does not satisfy the quasi-equation $(u_i, u'_i)$ for some $i$ since $u \neq u'$.

**2.9. Remark** As mentioned in the introduction, equations in General Algebra are precisely the parallel pairs of morphisms in Lawvere theories. How does this relate to our concept of quasi-equations ?

By the Gabriel-Ulmer duality (see Section 3 below) every locally finitely presentable category $\mathcal{K}$ has a theory, that is, a small category $\mathcal{T}$ with finite limits, such that $\mathcal{K}$ is equivalent to the category of models:

$Lex\,\mathcal{T}$ = all lex (i.e., finite limit preserving) functors from $\mathcal{T}^{op}$ to **Set**.

The same is true for finitely accessible categories: just drop the requirement of finite limits in $\mathcal{T}$ and instead of lex functors use flat ones (i.e., filtered colimits of representables). Analogously to General Algebra, we can now consider parallel pairs in $\mathcal{T}$ as quasi-equations and say that a model $M: \mathcal{T}^{op} \longrightarrow \mathbf{Set}$ satisfies the quasi-equation $(u, u')$ iff $Mu = Mu'$.

The Gabriel-Ulmer theory $\mathcal{T}$ of a locally finitely presentable category $\mathcal{K}$ is unique up to equivalence, and it is dual to the above $\mathcal{K}_{fp}$ (considered as a full subcategory of $\mathcal{K}$). Thus, parallel pairs in $\mathcal{K}_{fp}$, as in Definition 2.5, are just parallel pairs in the theory - with the arrows reverted. Every object $A$ of $\mathcal{A}$ is represented by the model

$$\mathcal{K}(-, A): \mathcal{T} \longrightarrow \mathbf{Set} \quad \text{for} \quad \mathcal{T} = \mathcal{K}_{fp}^{op}$$

and then the definition of satisfaction in 2.5 is precisely $Mu = Mu'$ for $M = \mathcal{K}(-, A)$.

**2.10. Remarks**

(1) Recall that an object $K$ is *orthogonal* to a morphism $c$ if $\mathcal{K}(-, K)$ turns $c$ into an isomorphism. If $\mathcal{K}$ has coequalizers, then satisfaction of a quasi-equation $u, u': P \rightrightarrows Q$ is equivalent to orthogonality to the coequalizer $c: Q \longrightarrow R$ of $u$ and $u'$.

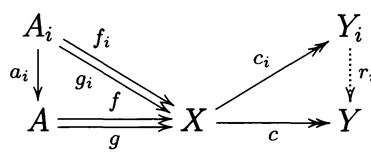
(2) Observe that the coequalizer of a finitary pair is a finitary regular epimorphism. The converse is less obvious, but is true in our context, as was shown in [6] (Theorem 1.3). Actually, their proof can be used to show more:

**2.11. Proposition** *If $\mathcal{K}$ has coequalizers, then the finitely presentable regular epimorphisms are precisely the coequalizers of pairs of morphisms with finitely presentable domain. In particular, finitary regular epimorphisms are precisely the coequalizers of pairs of finitary morphisms.*

*Proof.* The second statement follows from the first by 2.4.

That the coequalizer of a pair of morphisms with finitely presentable domain is necessarily a finitely presentable morphism is a straightforward verification.

The proof of the converse follows the line of the proof for the finitary case in [6]. Given a finitely presentable morphism $c\colon X \twoheadrightarrow Y$ which is a coequalizer of $f, g\colon A \rightrightarrows X$, consider a colimit $(a_i\colon A_i \longrightarrow A)_{i\in I}$ of a filtered diagram, with the $A_i$'s finitely presentable. Let $f_i = fa_i$ and $g_i = ga_i$, and let $c_i\colon X \twoheadrightarrow Y_i$ be the coequalizer of $f_i, g_i$ for each $i$. This induces, in the slice category $X \downarrow \mathcal{K}$, morphisms $r_i\colon c_i \longrightarrow c$:




as well as a filtered diagram $(r_{ij}\colon c_i \longrightarrow c_j)_{i\le j}$. It is straightforward to show that $(r_i)_{i\in I}$ is a colimit of $(r_{ij})_{i\le j}$ in $X \downarrow \mathcal{K}$. But the fact that $c$ is finitely presentable implies that there exist $i \in I$ and $s\colon c \longrightarrow c_i$ such that $r_i \cdot s = 1_c$. Since $c$ is epi, this implies that $r_i$ is an isomorphism. Consequently, $c$ is a coequalizer of $f_i$ and $g_i$, as required. □

**2.12. Corollary** *If $\mathcal{K}$ has coequalizers, then its quasi-equational subcategories are precisely the orthogonality classes $\mathcal{H}^\perp$ of sets $\mathcal{H}$ of finitary regular epimorphisms. Here $\mathcal{H}^\perp$ denotes the full subcategory of all objects orthogonal to members of $\mathcal{H}$.* □

**2.13. Theorem** *If $\mathcal{K}$ has coequalizers and is cowellpowered, then a full subcategory of $\mathcal{K}$ is quasi-equational iff it is closed under filtered colimits and monocones.*

**Remark.** That a subcategory is closed under monocones means that for every collectively monic cone with all codomains in the subcategory, the domain also lies there. The theorem follows, whenever $\mathcal{K}$ has regular factorizations of sources, from Proposition 2 in [5].

*Proof.* It is clear that every quasi-equational subcategory is closed under monocones and filtered colimits. Conversely, let $\mathcal{A}$ be closed under monocones and filtered colimits in $\mathcal{K}$.

(1) $\mathcal{A}$ is a reflective subcategory of $\mathcal{K}$. In order to construct a reflection of an object $K$ we define a transfinite chain $k_{ij}: K_i \longrightarrow K_j$ $(i \leq j \in Ord)$ as follows:

First step: $K_0 = K$.

Isolated step: if $K_i \in \mathcal{A}$ then $K_{i+1} = K_i$ and $k_{i,i+1} = id$. Else choose a parallel pair $u_i, u_i': X_i \rightrightarrows K_i$ with $u_i \neq u_i'$ merged by all morphisms in $K_i \downarrow \mathcal{A}$ (observing that the cone of all these morphisms cannot be a monocone), and let $k_{i,i+1}: K_i \longrightarrow K_{i+1}$ be the coequalizer of $u_i$ and $u_i'$.

Limit step: form the colimit of the previously defined chain.

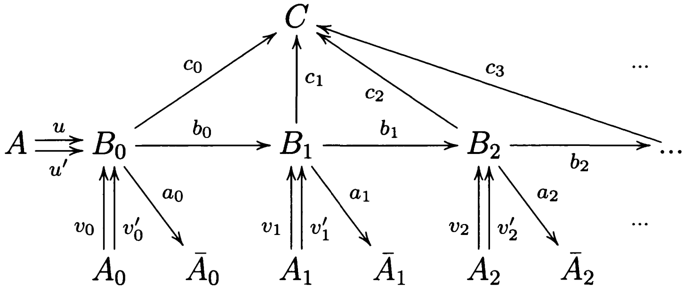
This chain is clearly formed by epimorphisms, and since $\mathcal{K}$ is cowellpowered, there exists $i$ such that $k_{i,i+1}$ is an isomorphism (actually the identity). This implies $K_i \in \mathcal{A}$. We claim that $k_{0,i}: K \longrightarrow K_i$ is a reflection of $K$ in $\mathcal{A}$. In fact, given a morphism $f: K \longrightarrow A$ with $A \in \mathcal{A}$, we get a unique cocone $f_j: K_j \longrightarrow A$ $(j \in Ord)$ with $f_0 = f$: the limit steps are clear, and the isolated steps follow from $f_i \cdot u_i = f_i \cdot u_i'$. In particular, $f = f_i \cdot k_{0,i}$.

(2) The rest of the proof is completely analogous to the proof of Proposition 2 in [5]. □

**2.14. Open Problem** Does 2.13 generalize to the locally finitely multipresentable categories of Y. Diers [7]?

**2.15. Example** of a finitely accessible category and its full subcategory which is closed under filtered colimits and monocones, but is not quasi-equational.

Let $\mathcal{K}$ be the category given by the graph




and the identities

$$c_n = c_{n+1} \cdot b_n, \quad a_n \cdot v_n = a_n \cdot v_n', \quad \text{and} \quad b_n \cdot v_n = b_n \cdot v_n'$$

for all $n \in \mathbb{N}$. This category is finitely accessible with $\mathcal{K}_{fp} = \mathcal{K} - \{C\}$ since the only non-trivial filtered colimit is $C = \mathrm{Colim}_{n\in\mathbb{N}}(b_n)$. The full subcategory $\mathcal{A}$ on the objects

$$\{A\} \cup \{A_n\}_{n\in\mathbb{N}} \cup \{\bar{A}_n\}_{n\in\mathbb{N}}$$

is closed under filtered colimits (trivially) and under monocones: in fact, for $B_n$, the cone $(B_n \longrightarrow \bar{A}_i)_{i\geq n}$ of all objects in $B_n \downarrow \mathcal{A}$ is not a monocone, due to $v_n \neq v_n'$; and for $C$, consider $c_0 \cdot u \neq c_0 \cdot u'$.

However, $\mathcal{A}$ is not quasi-equational: if a quasi-equation in $\mathcal{K}_{fp}$ is satisfied by all objects of $\mathcal{A}$, then it cannot factorize through $(u, u')$, from which it follows that $C$ also satisfies that quasi-equation (from $b_n \cdot v_n = b_n \cdot v_n'$ we have $c_n \cdot v_n = c_n \cdot v_n'$).

**2.16. Example** The category *Pos* of posets has precisely three quasi-equational subcategories: the smallest one (formed by the posets with at most one element), itself, and the subcategory **Set** (represented by the discrete orderings). In fact, observe that if $K$ is not discretely ordered, then for every finite poset $P$ the cone $Pos(P, K)$ is a monocone. Thus, whenever a quasi-equational class contains a non-discrete poset, it is all of *Pos*.

**2.17. Corollary** *Quasi-equational subcategories of locally finitely presentable categories are precisely those closed under*

(i) *products,*

(ii) *subobjects, and*

(iii) *filtered colimits.*

*Proof.* Since a locally finitely presentable category is cocomplete and cowellpowered, we only need to verify that (i) and (ii) imply closedness under monocones. Let $(a_i \colon B \longrightarrow A_i)_{i \in I}$ be a monocone with the $A_i$'s in $\mathcal{A}$. Then there exists a (small) set $I' \subseteq I$ such that $(a_i \colon B \longrightarrow A_i)_{i \in I'}$ is a monocone. In fact, because the finitely presentable objects form a generator, one sees easily that $(a_i \colon B \longrightarrow A_i)_{i \in J}$ with $J \subseteq I$, is a monocone iff for all $f, g \colon B' \rightrightarrows B$ with $B' \in \mathcal{K}_{fp}$ we have that $a_i f = a_i g$ for all $i \in J$ implies $f = g$. Then consider all parallel pairs $f_t, g_t \colon B_t \rightrightarrows B$, $t \in T$, with $B_t \in \mathcal{K}_{fp}$ such that there exists $i = i(t)$ with $a_i f_t \neq a_i g_t$. Choose one such $i(t)$ for each pair $t$, and take $I' = \{i(t) \mid t \in T\}$.

But then $< a_i >_{i \in I'} \colon B \longrightarrow \prod_{i \in I'} A_i$ is a monomorphism, and hence $B \in \mathcal{A}$. □

**2.18. Remark** This last corollary and Proposition 2.11 imply that in locally finitely presentable categories there is no difference between orthogonality classes with respect to

(a) finitary strong epimorphisms,

(b) finitary regular epimorphisms, and

(c) coequalizers of finitary parallel pairs.

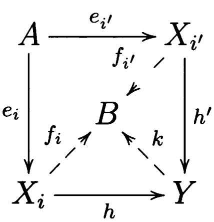
In fact, from (a), the closure properties (i)-(iii) above easily follow, from which we derive (c). Here is an explanation of this phenomenon, which seems to be of independent interest:

**2.19. Proposition** *In a locally finitely presentable category every finitely presentable strong epimorphism is a composite of finitely many finitely presentable regular epimorphisms..*

*Proof.* The class $\mathcal{E}$ of all composites of finitely many finitely presentable regular epimorphisms in a locally finitely presentable category $\mathcal{K}$ is closed under composition and under pushouts (see [11]). Given a finitely presentable strong epimorphism $f\colon A \longrightarrow B$, we prove $f \in \mathcal{E}$.

Let $D$ be the full subcategory of $A \downarrow \mathcal{K}$ with objects all the morphisms $e_i\colon A \longrightarrow X_i$ $(i \in I)$ in $\mathcal{E}$ through which $f$ factors: $f = f_i \cdot e_i$ for some (necessarily unique) $f_i\colon X_i \longrightarrow B$. Note that $D$ is small, since $A \downarrow \mathcal{K}$ is locally finitely presentable. As a diagram in $\mathcal{K}$, $D$ is filtered:

(a) Given objects $X_i$ and $X_{i'}$ of $D$, there is a cospan in $D$. In fact, form the pushout:




Then $h, h' \in \mathcal{E}$. Since $f_i e_i = f = f_{i'} e_{i'}$, there exists $k\colon Y \longrightarrow B$ with $f_i = kh$ and $f_{i'} = kh'$. Consequently, $he_i$ is a member of $\mathcal{E}$ through which $f$ factorizes: $f = khe_i$. Then there exists $j \in I$ with $e_j = he_i$ and $X_j = Y$, and we have the connecting morphisms

$$h\colon X_i \longrightarrow X_j \quad \text{and} \quad h'\colon X_{i'} \longrightarrow X_j$$

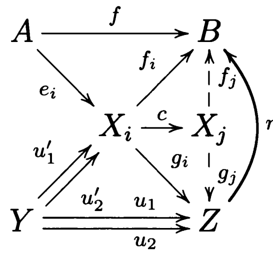
of $D$.

(b) There is no parallel pair of (distinct) connecting morphisms $h, k\colon X_i \rightrightarrows X_{i'}$ in $D$, as those necessarily satisfy $he_i = e_{i'} = ke_i$, hence $h = k$.

Moreover, the morphisms $f_i\colon X_i \longrightarrow B$ $(i \in I)$ form a colimit of $D$. In fact, consider the colimit $(g_i\colon X_i \longrightarrow Z)_{i \in I}$ of $D$. The cocone of the $f_i$'s is compatible with $D$: from $he_i = e_j$, it follows that $f_j h = f_i$, since $e_i$ is epi. The factorizing morphism $m\colon Z \longrightarrow B$ with $mg_i = f_i$ $(i \in I)$ is a monomorphism: given

$$mu_1 = mu_2 \text{ for } u_1, u_2\colon Y \rightrightarrows Z,$$

we prove $u_1 = u_2$; without loss of generality, we may assume that $Y$ is finitely presentable (since $\mathcal{K}_{fp}$ is a generator). Since $Z = \mathrm{Colim} X_i$ is a filtered colimit, the pair $u_1, u_2$ factorizes through the colimit morphism $g_i$ for some $i \in I$:




From Proposition 2.11, the coequalizer $c$ of $u'_1, u'_2$ is a finitely presentable regular epimorphism, therefore,

$$ce_i \in \mathcal{E}.$$

From

$$f_i \cdot u'_1 = m \cdot g_i \cdot u'_1 = m \cdot u_1 = m \cdot u_2 = m \cdot g_i \cdot u'_2 = f_i \cdot u'_2$$

we conclude that $f_i$ factorizes through $c$, thus, $f$ factorizes through $ce_i$. This implies that

$$ce_i = e_j \text{ for some } j \in I.$$

We get, from $g_i = g_j c$, that

$$u_1 = g_i \cdot u'_1 = g_j \cdot c \cdot u'_1 = g_j \cdot c \cdot u'_2 = g_i \cdot u'_2 = u_2,$$

as requested.

Choose any $i \in I$ and observe that since $m$ is a monomorphism and $f$ is a strong epimorphism with

$$f = f_i \cdot e_i = m \cdot (g_i \cdot e_i),$$

$m$ is an isomorphism. This proves

$$B = \mathrm{Colim}_{i \in I} X_i$$

as claimed.

Now, seeing $D = (e_i \longrightarrow e_j)_I$ as a diagram in $A \downarrow \mathcal{K}$, it is also filtered, and it is easily seen that $(f_i \colon e_i \longrightarrow f)_I$ is its colimit. Since $f$ is finitely presentable, the morphism $id_f \colon f \longrightarrow \mathrm{Colim}_{i\in I} e_i$ factorizes through the colimit morphism $f_i$ for some $i \in I$. That is, there exists

$$r \colon f \longrightarrow e_i \text{ with } f_i r = id_f.$$

In particular, $rf = e_i$, so $r$ is epi. But also, $f_i r = id_B$, so $r$ is a split mono, hence an isomorphism. Therefore, $f = r^{-1} e_i$ belongs to $\mathcal{E}$, since $e_i$ does. □

**2.20. Corollary** *In a locally finitely presentable category, every finitary strong epimorphism is a composite of finitely many finitary regular epimorphisms.*

In fact, this follows from Proposition 2.19 and Remark 2.4.

**2.21. Remark**

(i) John Isbell proved in [13] that in a suitably complete category, every strong epimorphism is a chain-composite of regular epimorphisms. Later, John MacDonald and Arthur Stone [17] demonstrated that the minimum length of this chain can be an arbitrary cardinal. Thus, the main message of Proposition 2.19 is that this cardinal is finite in case of finitary (or even finitely presentable) strong epimorphisms in locally finitely presentable categories.

(ii) Note that if a composite of finitely many regular epimorphisms is finitary, this does not imply that each one is; however the proposition says that there must be a finite path for the composite, made of finitary regular epimorphisms.

**2.22. Corollary** *For a full subcategory $\mathcal{A}$ of a locally finitely presentable category the following conditions are equivalent:*

*(i) $\mathcal{A}$ is quasi-equational*

*(ii) $\mathcal{A}$ is strongly epireflective and closed under filtered colimits*

*(iii) $\mathcal{A}$ is closed under products, subobjects and filtered colimits*

*(iv) $\mathcal{A}$ is the orthogonality class with respect to a set of finitary regular epimorphisms*

*(v)* *$\mathcal{A}$ is the orthogonality class with respect to a set of finitary strong epimorphisms.* □

**2.23. Remark** Following W. Hatcher [10], call *identity* a parallel pair of (arbitrary) morphisms, and *quasiprimitive* a full subcategory $\mathcal{A}$ of $\mathcal{K}$ defined by any class of identities: i.e., there exists a family E of identities such that the objects of $\mathcal{A}$ are precisely the objects of $\mathcal{K}$ which satisfy (in the sense of Definition 2.5) all identities in E. In order to compare with Corollary 2.22, we mention:

*For a full subcategory $\mathcal{A}$ of a locally finitely presentable category the following conditions are equivalent:*

*(i)′* *$\mathcal{A}$ is quasiprimitive*

*(ii)′* *$\mathcal{A}$ is strongly epireflective*

*(iii)′* *$\mathcal{A}$ is closed under products and subobjects*

*(iv)′* *$\mathcal{A}$ is the orthogonality class with respect to a class of finitely presentable regular epimorphisms*

*(v)′* *$\mathcal{A}$ is the orthogonality class with respect to a class of finitely presentable strong epimorphisms*

*Moreover, "finitely presentable" can be left out in (iv)′ and (v)′.*

In fact, the possibility of deleting "finitely presentable" is clear from the fact that every strong epimorphism is a filtered colimit of finitely presentable strong epimorphisms (proved as Corollary 2.10(1) of [1]). See [10] for the equivalence of (i)′ and (iii)′. The rest follows easily.

# 3 Strong Quotient Functors

In the present section we give the corresponding characterization of quasi-equations on the level of theories.

**3.1. Assumption** Throughout this section $\mathcal{K}$ denotes a locally finitely presentable category.

**3.2. Remark** Recall from [8] the Gabriel-Ulmer duality between the 2-category

$$\mathrm{LFP}$$

of locally finitely presentable categories with

1-cells: right-adjoints preserving filtered colimits, and

2-cells: natural transformations,

and the 2-category

$$\mathrm{LEX}$$

of (Gabriel-Ulmer) theories, that is, small categories with finite limits, with

1-cells: lex-functors, and

2-cells: natural transformations.

We have a biequivalence

$$Lex\colon\ \mathrm{LEX}^{op} \longrightarrow \mathrm{LFP}$$

assigning to every theory $\mathcal{T}$ the category $Lex\,\mathcal{T}$ of all lex functors from $\mathcal{T}$ to **Set**. To every 1-cell $Q\colon \mathcal{T} \longrightarrow \mathcal{S}$ it assigns the functor

$$Lex\,Q\colon Lex\,\mathcal{S} \longrightarrow Lex\,\mathcal{T},\quad H \longmapsto H \cdot Q.$$

In the opposite direction the biequivalence

$$GU\colon \mathrm{LFP} \longrightarrow \mathrm{LEX}^{op}$$

assigns to every locally finitely presentable category $\mathcal{K}$ its Gabriel-Ulmer theory $GU(\mathcal{K}) \cong \mathcal{K}_{fp}^{op}$.

**3.3. Example** Every quasi-equational subcategory $\mathcal{A}$ of the locally finitely presentable category $\mathcal{K} = Lex\,\mathcal{T}$ is strongly epireflective and closed under filtered colimits in $\mathcal{K}$ (see 2.22). Consequently, $\mathcal{A}$ is locally finitely presentable by 1.46 in [4] and the embedding $\mathcal{A} \hookrightarrow \mathcal{K}$ is a morphism of LFP, and as such has, up to natural isomorphism, the form $Lex\,Q$ for a lex functor

$$Q\colon GU(\mathcal{K}) \longrightarrow GU(\mathcal{A}).$$

**3.4. Convention** We call $Q\colon GU(\mathcal{K}) \longrightarrow GU(\mathcal{A})$ the theory morphism *induced* by the quasi-equational subcategory $\mathcal{A}$. It is determined uniquely up to equivalence in the sense that given equivalence functors $E\colon \mathcal{T} \longrightarrow$
$GU(\mathcal{K})$ and $E'\colon GU(\mathcal{A}) \longrightarrow \mathcal{S}$, then also the composite

$$\mathcal{T} \xrightarrow{E} GU(\mathcal{K}) \xrightarrow{Q} GU(\mathcal{A}) \xrightarrow{E'} \mathcal{S}$$

is induced by $\mathcal{A}$ (and conversely, every induced theory morphism is of the form $E' \cdot Q \cdot E$).

**3.5. Remarks** As mentioned in the Introduction, in case of Lawvere algebraic theories, the theory morphisms

$$Q\colon L(\mathcal{K}) \longrightarrow L(\mathcal{A})$$

corresponding to equational subcategories (varieties) $\mathcal{A}$ are precisely the surjective functors which are the identity on objects. This does not work for Gabriel-Ulmer theories:

**3.6. Example** Consider the trivial signature $\Sigma$ of two nullary symbols $u, u'$ and let $\mathcal{A}$ be the quasi-equational class of all algebras $A$ with $u_A = u'_A$. Here, $GU(\mathcal{A})$ is the dual of the category of finite pointed sets, and $GU(\mathcal{K})$ is the dual of the category of finite bipointed sets. The induced functor $Q$ just merges the two distinguished points to one.

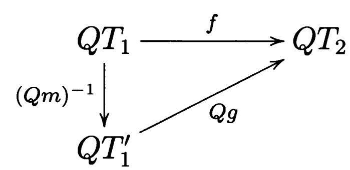
Observe that $Q$ is not surjective on hom-sets: if 1 denotes the terminal object of $GU(\mathcal{K})$ and 2 the initial one, then $id\colon Q(1) \longrightarrow Q(2)$ has no preimage in $GU(\mathcal{K})$.

**3.7. Definition** A lex functor $Q\colon \mathcal{T} \longrightarrow \mathcal{S}$ is called a *strong quotient* provided that

(i) every object of $\mathcal{S}$ is isomorphic to $QT$ for an object $T$ of $\mathcal{T}$,

and

(ii) every morphism $f\colon QT_1 \longrightarrow QT_2$ of $\mathcal{S}$ has the form



for some strong monomorphism $m\colon T_1' \longrightarrow T_1$ and some morphism $g\colon T_1' \longrightarrow T_2$ of $\mathcal{T}$.

**3.8. Remark** This definition is just a variation on the concept of *quotient functor* introduced by M. Makkai [16], see Introduction.

**3.9. Theorem** *The theory morphisms induced by quasi-equational subcategories are precisely the strong quotient functors.*

**3.10. Remark** The proof of the theorem will be a variation of the analogous result concerning orthogonality in [12]. Let us recall this result first:

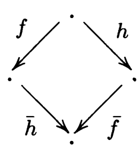
(1) Given a set $\mathcal{H}$ of finitary morphisms in $\mathcal{K}$, the full subcategory $\mathcal{A} = \mathcal{H}^{\perp}$ is called an $\omega$-orthogonality class. It is locally finitely presentable and the theory morphisms induced by the embeddings $\mathcal{A} \hookrightarrow \mathcal{K}$ of $\omega$-orthogonality classes are precisely the quotient functors $Q\colon GU(\mathcal{K}) \longrightarrow \mathcal{S}$.

(2) The following connection to the categories of fractions of Gabriel and Zisman [9] was made explicit:

Recall that a set $\mathcal{H}$ of morphisms in a category $\mathcal{K}$ is said to *admit a left calculus of fractions* provided that

(i) $\mathcal{H}$ contains all isomorphisms and is closed under composition,

(ii) for every span $\cdot \xleftarrow{f} \cdot \xrightarrow{h} \cdot$ with $h \in \mathcal{H}$ there exists a commutative square




with $\bar{h} \in \mathcal{H}$,

and

(iii) for every parallel pair equalized by a member of $\mathcal{H}$ there exists a member of $\mathcal{H}$ coequalizing this pair.

Recall further that given a set $\mathcal{H}$ of morphisms in a category $\mathcal{C}$, the *category of fractions of* $\mathcal{H}$ is a category $\mathcal{C}[\mathcal{H}^{-1}]$ together with a functor

$$Q_{\mathcal{H}}\colon \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{H}^{-1}]$$

which takes the members of $\mathcal{H}$ to isomorphisms, and is universal for this property: every functor $\mathcal{C} \longrightarrow \mathcal{C}'$ taking members of $\mathcal{H}$ to isomorphisms factors uniquely through $Q_{\mathcal{H}}$.

The connection to $\omega$-orthogonality classes established in [12] is this: Let $\mathcal{H} \subseteq mor\ \mathcal{K}_{fp}$ admit a calculus of left fractions in $\mathcal{K}_{fp}$. Then

$$Q_{\mathcal{H}}\colon \mathcal{K}_{fp} \longrightarrow \mathcal{K}_{fp}[\mathcal{H}^{-1}]$$

is the dual of the theory morphism induced by the embedding of the $\omega$-orthogonality class $\mathcal{H}^{\perp} \hookrightarrow \mathcal{K}$.

*Proof of Theorem 3.9*

(I) We first prove that every strong quotient $Q\colon GU(\mathcal{K}) \longrightarrow \mathcal{S}$ is induced by a quasi-equational subcategory of $\mathcal{K}$ (with theory $\mathcal{S}$).

Let then

$$Q\colon GU(\mathcal{K}) \longrightarrow \mathcal{S}$$

be a strong quotient. Due to [12], for the category $\mathcal{A} = Lex\,\mathcal{S}$, the functor $Lex\,Q\colon \mathcal{A} \longrightarrow \mathcal{K}$ is the embedding of the $\omega$-orthogonality class $\mathcal{A}$. It is easy to verify that the set

$$\mathcal{H} = \{h \in \mathcal{K}_{fp} \mid Q(h) \text{ is an isomorphism}\}$$

admits a left calculus of fractions in $\mathcal{K}_{fp}$ and that it fulfills

$$\mathcal{A} = \mathcal{H}^{\perp} \text{ and } \mathcal{S}^{op} \cong \mathcal{K}_{fp}[\mathcal{H}^{-1}].$$

If $\mathcal{H}_0$ denotes the set of all strong epimorphisms (of $\mathcal{K}$) in $\mathcal{H}$, then the fact that $Q$ is a *strong* quotient implies that every morphism $h\colon QT_1 \longrightarrow QT_2$ of $\mathcal{S}$ has the form $h = Qg \cdot (Qm)^{-1}$ for $m \in \mathcal{H}_0$ (in $\mathcal{S}$, thus in $\mathcal{K}_{fp}$). Hence every morphism in $\mathcal{K}_{fp}[\mathcal{H}^{-1}]$ is actually a morphism in $\mathcal{K}_{fp}[\mathcal{H}_0^{-1}]$, so that

$$\mathcal{K}_{fp}[\mathcal{H}_0^{-1}] = \mathcal{K}_{fp}[\mathcal{H}^{-1}].$$

By (2) in Remark 3.8, we have

$$\mathcal{A} = \mathcal{H}_0^{\perp},$$

and therefore $\mathcal{A}$ is the orthogonality class of strong epimorphisms. It follows from Corollary 2.22 that $\mathcal{A}$ is a quasi-equational subcategory.

(II) We now show that for every quasi-equational subcategory $\mathcal{A} \hookrightarrow \mathcal{K}$, the induced theory morphism $Q\colon GU(\mathcal{K}) \longrightarrow GU(\mathcal{A})$ is a strong quotient.

Let $\mathcal{A}$ be a quasi-equational subcategory of $\mathcal{K}$. By Remark 2.10, we have a set $\mathcal{H}$ of finitary regular epimorphisms in $\mathcal{K}$ with

$$\mathcal{A} = \mathcal{H}^{\perp}.$$

The closure $\bar{\mathcal{H}}$ of $\mathcal{H}$ under isomorphism, composition and pushout in $\mathcal{K}_{fp}$ is a set of finitary strong epimorphisms with

$$\mathcal{A} = \bar{\mathcal{H}}^{\perp}.$$

Moreover, $\bar{\mathcal{H}}$ clearly admits a left calculus of fractions in $\mathcal{K}_{fp}$. By (1) in Remark 3.10 the induced theory morphism

$$Q\colon GU(\mathcal{K}) \longrightarrow GU(\mathcal{A})$$

is a quotient functor. More detailed: the following was shown in the last part of the proof of V.2 in [12]: (i) Let

$$R\colon \mathcal{K} \longrightarrow \mathcal{A}$$

be a reflector of $\mathcal{A}$ with reflection morphisms $\eta_K\colon K \longrightarrow RK$ chosen so that $R\eta_K = id_{RK}$ for all $K \in \mathcal{K}$. We have a domain-codomain restriction

$$R_0\colon \mathcal{K}_{fp} \longrightarrow \mathcal{A}_{fp}$$

and we can assume $R_0 = Q^{op}$. (ii) Given a morphism

$$f\colon R_0 L \longrightarrow R_0 \bar{L} \quad (L, \bar{L} \in \mathcal{K}_{fp})$$

there exist morphisms $g\colon L \longrightarrow C_h$, $h\colon \bar{L} \longrightarrow C_h$ in $\bar{\mathcal{H}}$ and $c_h\colon C_h \longrightarrow R_0\bar{L}$ ($C_h \in \mathcal{K}_{fp}$) such that $f = R(c_h \cdot g)$ and $c_h \cdot h = \eta_{\bar{L}}$. This last equation yields $Rc_h \cdot Rh = id_{R_0\bar{L}}$, thus, $Rh = R_0h$ is invertible and

$$f = (R_0h)^{-1} \cdot (R_0g).$$

This proves that $Q = R_0^{op}$ is a strong quotient: for every morphism $f\colon Q\bar{L} \longrightarrow QL$ we have $f = Qg \cdot (Qh)^{-1}$ and $h$ is a strong monomorphism in $GU(\mathcal{K}) = \mathcal{K}_{fp}^{op}$. □

**3.11. Remark** Theorem 3.9 characterizes theory morphisms induced by strongly epireflective subcategories closed under filtered colimits (see 2.22). Let us mention a related result of M. Makkai and A. Pitts [16] characterizing theory morphisms induced by all full reflective subcategories of $\mathcal{K}$ closed under filtered colimits. These are precisely the lex functors $Q\colon \mathcal{S} \longrightarrow \mathcal{T}$ such that

(i) every object of $\mathcal{S}$ is isomorphic to $QT$ for an object $T$ of $\mathcal{T}$,

and

(ii) every morphism $f\colon QT_1 \longrightarrow QT_2$ of $\mathcal{S}$ has the form

$$\begin{array}{ccc} QT_1 & \xrightarrow{\;f\;} & QT_2 \\ {\scriptstyle s}\downarrow & \nearrow_{Qg} & \\ QT_1' & & \end{array}$$

for some morphism $g\colon T_1' \longrightarrow T_2$ of $\mathcal{T}$ and some morphism $s\colon QT_1 \longrightarrow$
$QT_1'$ having a splitting in $\mathcal{S}$:

$$r \cdot s = id_{QT_1}$$

with $Qg = f \cdot r$.

**3.12. Conclusions** For locally finitely presentable categories the concept of equation which naturally corresponds to the classical equations of General Algebra is that of a parallel pair of morphisms in the Gabriel-Ulmer theory. This was studied by W. Hatcher [10] and B.Banascheski and H.Herrlich [5] more than 30 years ago. (In the latter work the more general case of locally $\lambda$-presentable categories was considered, where the equations are parallel pairs of $\lambda$-presentable morphisms.) We call such parallel pairs quasi-equations.

In our paper we derived from the above earlier work that the quasi-equational subcategories of a locally finitely presentable category are precisely those closed under products, subobjects, and filtered colimits. We just used slightly less restrictive assumptions. And we characterized the theory morphisms between the Gabriel-Ulmer theories that precisely correspond to the quasi-equational classes. A generalization to locally $\lambda$-presentable categories is straightforward: the quasi-equational classes are those full subcategories that are closed under products, subobjects, and $\lambda$-filtered colimits. The concept of a quotient functor of M.Makkai and A. Pitts in [16] is also clearly definable in this infinitary setting; again, the theory morphisms corresponding to the quasi-equational classes are precisely the strong quotients. The proof is completely analogous to the proof of Theorem 3.9, one just works with the theory given by the dual of the category of all $\lambda$-presentable objects.

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