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DOMINIQUE BOURN
GEORGE JANELIDZE

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CENTRALIZERS IN ACTION ACCESSIBLE CATEGORIES

*by Dominique BOURN and George JANELIDZE**

Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Abstract

Nous introduisons la notion de catégorie accessible qui comprend une grande partie des catégories protomodulaires, dont les catégories des groupes, des anneaux, des algèbres associatives et des algèbres de Lie. Cette notion a l'avantage de permettre de calculer intrinséquement les centralisateurs des sous-objets et des relations d'équivalence. Nous montrons que dans de telles catégories les notions de commutateurs pour les sous-objets et pour les relations d'équivalence coïncident.

We introduce and study action accessible categories. They provide a wide class of protomodular categories, including all varieties of groups, rings, associative and Lie algebras, in which it is possible to calculate centralizers of equivalence relations and subobjects. We show that, in those categories, the equivalence relation and subobject commutators agree with each other.

Key words : Protomodular and semi-abelian categories; centralizers; commutators; split exact sequences.
[2000]primary: 20J05,18G50, 18C15; secondary: 18G30, 18G35.

INTRODUCTION.

When X is a subset of a group A, the centralizer $Z(X)$ of X in A is defined as

$$Z(X) = \{a \in A / x \in X \Rightarrow axa^{-1} = x\}$$

When X is a normal subgroup in A, sending $a \in A$ to the automorphism $c(a)$ of X defined by $c(a)(x) = axa^{-1}$ determines a group homomorphism $c : A \to Aut(X)$, and the centralizer $Z(X)$ can equivalently be defined as: $Z(X) = Ker(c)$.

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When X is a normal subobject in an object A in a semi-abelian category $\mathbb{C}$, the centralizer $Z(X)$ is to be defined as the largest subobject K of A with $[\![X,K]\!] = 0$ (here and below we write $[\![,]\!]$ for the "classical" *subobject commutator* in order to distinguish it from the *equivalence relation commutator* $[,]$). The existence of such a $Z(X)$ can then be proved in the case of a semi-abelian variety, but not in general: a counter-example was constructed by S. A. Huq [12].

Since the group $Aut(X)$ is a particular example of the *split extension classifier* (introduced in [3], and denoted there by $[X]$; see also [2]), it is natural to ask if the equality $Z(X) = Ker(c)$ still holds for the appropriate $c : A \to [X]$ whenever the protomodular category $\mathbb{C}$ is *action representable*, i.e. whenever the split extension classifier $[X]$ exists.

Among others, there are two main results in this paper:

(a) We not just answer positively the question above, but prove a stronger result applicable to a much wider class of categories, which we call *action accessible.* They include e.g. all varieties of groups, rings, associative and Lie algebras.

(b) As an application, we prove that in any action accessible category the equivalence relation and subobject commutators agree in the sense that $[R, S]$ =0 if and only if $[\![I_R, I_S]\!] = 0$, where I_R is the normal subobject associated with R; as we know from [6], this is also true in any strongly protomodular category, but for a very different reason.

The paper is divided into six sections as follows:

Section 1 introduces faithful split extensions and studies their simple properties, especially in the case of rings - which is the most important non-action-representable case. For familiar algebraic categories faithful split extensions

$$0 \longrightarrow X \overset{k}{\rightarrowtail} A \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

correspond to faithful actions of B on X, which is the reason of choosing the term "faithful". Note also that all *generic split extensions*

$$0 \longrightarrow X \rightarrowtail X \ltimes [X] \rightleftarrows [X] \longrightarrow 0$$

(in the sense of [3], with the concept of semidirect product introduced in [9]) are faithful by very definition.

Section 2 defines the *action accessible* categories as those "with access to faithful split extensions", i.e. as those where every split extension admits a morphism into a faithful one. It is shown that the following categories are action accessible:

- any action representable category (trivially);
- the category of rings;
- any Birkhoff subcategory of a *homological* action accessible category;
- the category of split epimorphisms into any object in an action accessible category.

Section 3 essentially shows that working with split extensions is the same as working with internal groupoids, and therefore allows to apply the constructions with split extensions to (internal) equivalent relations.

Section 4 shows, using the results of Section 3, how to calculate centralizers of equivalence relation as kernel pairs of morphisms into split extensions, and in particular concludes that all action representable and all action accessible (homological) categories admit centralizers.

Section 5 studies centralizers of normal subobjects, compares them with centralizers of equivalence relations and concludes that the equivalence relation and subobject commutators agree in any action accessible category.

Section 6 provides a new characterization of *antiadditivity* (=the property for an object of having trivial centres) via faithfulness of a particular split extension, extending a simple property of groups.

Remark: (a) The authors did their best to adjust the terminology and notation they use with those of the papers they refer to – even though in some cases it almost created disagreements They hope, however, that the choices they made will be most convenient for the readers, especially those who studied the book [1].

(b) The action accessibility defined in this paper has nothing to do with the concept of accessible category – it is only a coincidence of terminology.

1 Faithful split extensions

Let $\mathbb{C}$ be a finitely complete pointed category. Recall it is *protomodular* when for any diagram with $ps = 1_B$ and k the kernel of p:

$$0 \longrightarrow X \overset{k}{\rightarrowtail} A \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

the pair (k, s) is jointly strongly epic.

Now let $\mathbb{C}$ be a fixed pointed protomodular category, and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \overset{k}{\rightarrowtail} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & {\scriptstyle 1_X}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & X & \underset{l}{\rightarrowtail} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D & \longrightarrow & 0 \end{array} \tag{1.1}$$

a diagram in $\mathbb{C}$, which has the following properties:

- it reasonably commutes, i.e. has $l = fk$, $qf = gp$, and $fs = tg$;
- it has $ps = 1_B$ and $qt = 1_D$;
- k and l are kernels of p and q respectively.

We will consider such a diagram as a morphism (g, f) of split extensions (with fixed X), write

$$(g, f) : (B, A, p, s, k) \to (D, C, q, t, l) \tag{1.2}$$

and denote the category of such split extensions by

$$SplExt(X) = SplExt_{\mathbb{C}}(X)$$

The functor

$$SplExt(X) \to \mathbb{C} \tag{1.3}$$

sending (B, A, p, s, k) to B is a faithful fibration in which every *vertical* morphism is an isomorphism (since $g = 1_D \Rightarrow f$ is an isomorphism) and therefore every morphism is *cartesian.* This follows from the protomodularity of $\mathbb{C}$, and, moreover, when $\mathbb{C}$ is just required to be pointed and to have finite limits, this is equivalent to protomodularity.

Observation 1.1. *Using protomodularity we observe:*
(a) to say that the functor (1.3) is faithful is of course the same as to say that the morphism f *in (1.1) (provided it exists) is determined by other morphisms, which follows from the fact that the pair* (k, s) *is jointly (strongly) epic;*
(b) to say that every morphism in $SplExt(X)$ *is cartesian (with respect to the functor (1.3)) is the same as to say that in every diagram in* $\mathbb{C}$ *of the form (1.1) the square* $qf = gp$ *is a pullback;*
(c) the category $SplExt(X)$ *obviously has connected finite limits preserved by the functor (1.3), and, since this functor (1.3) is faithful, it not only preserves, but also reflects monomorphisms.*

Definition 1.2. *An object in* $SplExt(X)$ *is said to be faithful, if any object in* $SplExt(X)$ *admits at most one morphism into it.*

Observation 1.3. *For arbitrary two objects* X *and* Y *in* $\mathbb{C}$, *consider the split extension* $(Y, Y \times X, p_Y, i_Y, i_X)$, *in which* $p_Y : Y \times X \to Y$ *is the product projection, and* $i_Y = \langle 1, 0\rangle : Y \to Y \times X$ *and* $i_X = \langle 0, 1\rangle : X \to Y \times X$ *are the "product injections". This split extension belongs to* $SplExt(X)$, *and it becomes its initial object if and only if* $Y = 0$. *Furthermore, it always admits a (unique) morphism into the initial object, and therefore it is faithful if and only if it is initial. In particular this implies that whenever the category* $\mathbb{C}$ *has at least one object* X *for which every object in* $SplExt(X)$ *is faithful, the category* $\mathbb{C}$ *has no non-zero objects, in other words* $\mathbb{C}$ *is indiscrete.*

Using Observation 1.1(c) we obtain:

Proposition 1.4. *For a morphism* $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ *with faithful codomain* (D, C, q, t, l), *the following conditions are equivalent:*
(a) (B, A, p, s, k) *is faithful;*
(b) (g, f) *is a monomorphism in* $SplExt(X)$;
(c) g *is a monomorphism in* $\mathbb{C}$.

In particular, the category $SplExt(X)$ might have a terminal object, which is to be called the *generic split extension with kernel* X (according to [3]), or the *universal split extension of* X (according to [2]); this is the case when $\mathbb{C}$ has *representable object actions* in the sense of [3] or is *action representative* in the sense of [2] (these two concepts coincide, except that the categories considered in [3] were required to be semi-abelian). The image of the generic split extension with kernel X under the functor (1.3) was called the split extension classifier for X and denoted by $[X]$ in [3], and by $D(X)$ in [2]. From Proposition 1.4 we obtain:

Corollary 1.5. *When $\mathbb{C}$ is action representative, the following conditions on an object (B, A, p, s, k) in $SplExt(X)$ are equivalent:*
(a) (B, A, p, s, k) is faithful;
(b) the corresponding classifying morphism $B \to [X]$ is a monomorphism.

Note that in the case of groups the morphism $B \to [X]$ becomes $B \to Aut(X)$, which justifies the term faithful. However there are other justification results beyond the action representative cases, such as Proposition 1.6 below or a similar result for commutative rings. When $\mathbb{C}$ is semi-abelian [13], the category $SplExt(X)$ is equivalent to the category of pairs (B, ξ), where ξ is an action of B on X in the sense of [9] (see [3] and [4] for details). Therefore Definition 1.2 in fact gives a definition of a *faithful object action.*

Proposition 1.6. *Let $\mathbb{C}$ be the variety* **Rg** *of (not-necessarily-unitary) rings, and X an object in $SplExt(X)$. Then the following conditions on an object (D, C, q, t, l) in $SplExt(X)$ are equivalent:*
(a) (D, C, q, t, l) is faithful;
(b) if d and d' are elements in D with $t(d)l(x) = t(d')l(x)$ and $l(x)t(d) = l(x)t(d')$ for all x in X, then $d = d'$;
(c) if d is an element in D with $t(d)l(x) = 0 = l(x)t(d)$ for all x in X, then $d = 0$.

Proof. $(b) \Leftrightarrow (c)$ is obvious.

$(a) \Rightarrow (b)$: for an object (D, C, q, t, l) in $SplExt(X)$ and an element d in D we can construct an object (B, A, p, s, k) in $SplExt(X)$ and a morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ as follows:
- we take B to be the free algebra in $\mathbb{C}$ on a one-element set $\{z\}$;

• we define $g : B \to D$ as the unique ring homomorphism from B to D with $g(z) = d$;
• A is $B \times X$ as an abelian group, with the multiplication defined by

$$(b, x)(b', x') = (bb', bx' + xb' + xx'), \tag{1.4}$$

where bb' and xx' are defined as in B and in X respectively, and bx' and xb' are defined by

$$l(bx') = tg(b)l(x') \text{ and } l(xb') = l(x)tg(b') \tag{1.5}$$

respectively (using the fact that l is injective);
• we define p, s, k, and f by

$$p(b, x) = b,\ s(b) = (b, 0),\ k(x) = (0, x), \text{ and } f(b, x) = tg(b) + l(x) \tag{1.6}$$

respectively.

Checking that this determines a morphism in $SplExt(X)$ requires a long but straightforward calculation, which we omit. Let us now compare the morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ with the morphism $(g', f') : (B', A', p', s', k') \to (D, C, q, t, l)$ constructed in exactly the same way but with an element d' instead of d. We claim that if

$$t(d)l(x) = t(d')l(x) \text{ and } l(x)t(d) = l(x)t(d')$$

for all x in X, then $(B', A', p', s', k') = (B, A, p, s, k)$. Indeed, we observe:
• $B' = B$, $A' = A$ as abelian groups, and p', s', k' are the same maps as p, s, k respectively in any case. Therefore we only need to show that for all b, b' in B and x, x' in X, $(b, x)(b', x')$ in A' is the same as $(b, x)(b', x')$ in A.
• According to (1.4) and (1.5), to show that $(b, x)(b', x')$ in A' is the same as $(b, x)(b', x')$ in A for all b, b' in B and x, x' in X, it suffices to show that:

$$tg(b)l(x) = tg'(b)l(x) \text{ and } l(x)tg(b) = l(x)tg'(b) \tag{1.7}$$

for all b in B and x in X.
• Since, by the assumption on d and d', the equalities (1.7) hold for

$b = z$, it suffices to show that the set of elements b in B for which the equalities (1.7) hold form a subring in B. Moreover, since that set is obviously a subgroup of the additive group of B, we only need to show that it is closed under the multiplication in B. This, however, easily follows from the fact that tg and tg' are ring homomorphisms and the multiplication in D is associative.

Next, since (D, C, q, t, l) is faithful, $(B', A', p', s', k') = (B, A, p, s, k)$ implies $g = g'$, and so $d = d'$.

(*b*) $\Rightarrow$ (*a*): let (g, f) and (g', f') be morphisms from (B, A, p, s, k) to (D, C, q, t, l) and b and x be elements in B and X respectively. Since $p(s(b)k(x)) = ps(b)pk(x) = 0$, there exists y in X with $k(y) = s(b)k(x)$, and we have:

$$tg(b)l(x) = fs(b)fk(x) = f(s(b)k(x)) = fk(y) = l(y) = f'k(y)$$

$$= f'(s(b)k(x)) = f's(b)f'k(x) = tg'(b)l(x)$$

and similarly $l(x)tg(b) = l(x)tg'(b)$. Condition (b) then tells us that $g(b) = g'(b)$ for all b in B. That is, $g = g'$, and since k and s are jointly epic this also gives $f = f'$, as desired. □

2 Action accessibility

Definition 2.1. *Let $\mathbb{C}$ be a pointed protomodular category. An object in SplExt(X) is said to be accessible, if it admits a morphism into a faithful object. If every object in SplExt(X) is accessible, we will say that $\mathbb{C}$ is action accessible.*

As immediately follows from this definition, *every action representative category* is *action accessible.* So this is in particular the case for the categories **Gp** of groups and R-**Lie** of Lie R-algebras. The following example of action accessible category will show that the converse is not true:

Proposition 2.2. *The variety* **Rg** *of (not-necessarily-unitary) rings is action accessible.*

Proof. For an object (B, A, p, s, k) in $SplExt(X)$ we construct the desired morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ into a faithful object as follows:

• The set $I = \{b \in B \mid \forall_{x \in X}\ s(b)k(x) = 0 = k(x)s(b)\}$ is an ideal in B. Indeed, for b in I and b' in B we have $s(b'b)k(x) = s(b')s(b)k(x) = 0 = s(b)s(b')k(x) = s(bb')k(x)$, where the third equality follows from the fact that $s(b')k(x) = k(y)$ for some y in X; similarly $k(x)s(b'b) = 0 = k(x)s(bb')$. We take $D = B/I$.

• The image $s(I)$ of I under s is an ideal in A. In order to prove this, it suffices to show that for b in I, b' in B, and x in X, the elements $s(b')s(b)$, $s(b)s(b')$, $k(x)s(b)$, and $s(b)k(x)$ are in $s(I)$. For the elements $s(b')s(b) = s(b'b)$ and $s(b)s(b') = s(bb')$ this follows from the fact that I is an ideal in B. The elements $k(x)s(b)$ and $s(b)k(x)$ are simply equal to 0 by definition of I. We take $C = A/s(I)$.

• We define q, t, and l as the morphisms induced by p, s, and k respectively, and take f and g to be the canonical morphisms $A \to A/s(I)$ and $B \to B/I$. This obviously determines a morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$, since all the maps involved are ring homomorphisms and the resulting diagram considered as a diagram in the category of abelian groups becomes isomorphic to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \rightarrowtail & X \oplus B & \rightleftarrows & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \rightarrowtail & X \oplus B/I & \rightleftarrows & B/I & \longrightarrow & 0
\end{array}$$

of canonical morphisms.

It remains to prove that (D, C, q, t, l) is faithful. According to Proposition 1.6 it suffices to prove that if d is an element in D with $t(d)l(x) = 0 = l(x)t(d)$ for all x in X, then $d = 0$. We have $d = b + I$ for some b in B, and then $t(d)l(x) = 0 = l(x)t(d)$ in D means that the elements $s(b)k(x)$ and $k(x)s(b)$ are in $s(I)$. On the other hand, $d = 0$ in D means that b is in I, i.e. that $s(b)k(x) = 0 = k(x)s(b)$ for all x in X. That is, we have to prove the implication

$$\forall_{x \in X}\ s(b)k(x), k(x)s(b) \in s(I) \Rightarrow \forall_{x \in X}\ s(b)k(x) = 0 = k(x)s(b)$$

However, it follows from the much stronger and obvious implication

$$s(b)k(x), k(x)s(b) \in s(B) \Rightarrow s(b)k(x) = 0 = k(x)s(b)$$

□

Many other examples of action accessible categories can be obtained from

Proposition 2.3. *If $\mathbb{C}$ is an action accessible homological (i.e. pointed protomodular regular) category and $\mathbb{D}$ is a Birkhoff subcategory in $\mathbb{C}$, then $\mathbb{D}$ also is action accessible.*

Proof. For X in $\mathbb{D}$ and a morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ with (B, A, p, s, k) in $SplExt_{\mathbb{D}}(X)$ and a faithful object (D, C, q, t, l) in $SplExt_{\mathbb{C}}(X)$ just take ($\mathbb{C}$ being regular) $(g', f') : (B, A, p, s, k) \twoheadrightarrow (D', C', q', t', l')$, where (D', C', q', t', l') is the suitably constructed image of (g, f), and (g', f') is induced by (g, f). □

Now let $Pt_{\mathbb{C}}(Y)$ denote the category whose objects are the split epimorphisms above Y and morphims are the commutative triangles between those split epimorphims. When $\mathbb{C}$ is protomodular, then $Pt_{\mathbb{C}}(Y)$ is pointed protomodular.

Proposition 2.4. *Let $\mathbb{C}$ be a pointed protomodular category.*
(a) Given a morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ in the category $SplExt(X)$, the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\langle p,f\rangle} & B \times C & \underset{B\times t}{\overset{B\times q}{\rightleftarrows}} & B \times D \\ {\scriptstyle p}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle p_B}\downarrow\uparrow{\scriptstyle \langle 1,tg\rangle} & & {\scriptstyle p_B}\downarrow\uparrow{\scriptstyle \langle 1,g\rangle} \\ B & = & B & = & B \end{array} \qquad (2.1)$$

is a split extension in $Pt_{\mathbb{C}}(B)$ that is faithful whenever so is (D, C, q, t, l).
(b) If $\mathbb{C}$ is action accessible, then, for any object B, the category $Pt_{\mathbb{C}}(B)$ is action accessible.

Proof. (a): Omitting straightforward verification of the first assertion, consider another split extension

$$\begin{array}{ccccc} A & \xrightarrow{h} & A' & \underset{s'}{\overset{p'}{\rightleftarrows}} & B' \\ {\scriptstyle p}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle u}\downarrow\uparrow{\scriptstyle x} & & {\scriptstyle v}\downarrow\uparrow{\scriptstyle y} \\ B & = & B & = & B \end{array} \qquad (2.2)$$

with the same A, p, and s, and two morphisms (m, n) and (m', n') from it to the split extension (2.1). We have to prove that $n = n'$. Since $p_B n = v = p_B n'$, it suffices to prove that the composites $p_D n$ and $p_D n'$ of n and n' with the projection $p_D : B \times D \to D$ are equal to each other. This, however, follows from the fact that $(p_C m, p_D n)$ and $(p_C m', p_D n')$ can be presented as two parallel morphisms into (D, C, q, t, l) in $SplExt_{\mathbb{C}}(X)$ and (D, C, q, t, l) is faithful.

(b): Let us now begin with an arbitrary split extension (2.2) in $Pt_{\mathbb{C}}(B)$, and let (X, k') be the kernel of p'. Since $\mathbb{C}$ is action accessible, there is a faithful split extension (D, C, q, t, l) and a morphism

$$(g', f') : (B', A', p', s', k') \to (D, C, q, t, l)$$

in $SplExt_{\mathbb{C}}(X)$. After that all we need is to observe that the morphism (g', f') induces a morphism from the split extension (2.2) to the split extension

$$\begin{array}{ccccc} A & \xrightarrow{\langle p,f\rangle} & B \times C & \underset{B \times t}{\overset{B \times q}{\rightleftarrows}} & B \times D \\ {\scriptstyle p}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle p_B}\downarrow\uparrow{\scriptstyle \langle 1,tg\rangle} & & {\scriptstyle p_B}\downarrow\uparrow{\scriptstyle \langle 1,g\rangle} \\ B & = & B & = & B \end{array}$$

constructed as follows:

• putting $g = g'y$ makes $B \times q$ a morphism $(B \times C, p_B, \langle 1, tg\rangle) \to (B \times D, p_B, \langle 1, g\rangle)$ in the category $Pt_{\mathbb{C}}(B)$, and we define $k : (A, p, s) \to (B \times C, p_B, \langle 1, tg\rangle)$ as the kernel of that morphism;

• we then define $f : A \to C$ as the composite of k with the product projection $B \times C \to C$, which makes $k = \langle p, f\rangle$;

- and that this split extension is faithful by (a). □

3 The fibration of X-groupoids

In this section we extend the previous observations to internal reflexive graphs and groupoids, which we shall need to introduce centralizers. For an object X in $\mathbb{C}$, by a reflexive graph structure on an object (B, A, p, s, k) in $SplExt(X)$ we will mean a morphism $u : A \to B$ with $us = 1_B$; we will then also say that (B, A, p, s, u) is the underlying reflexive graph of $((B, A, p, s, k), u)$. Conversely, given any reflexive graph

(B, A, d_0, s_0, d_1), the morphism d_1 gives a reflexive graph structure on the object (B, A, d_0, s_0, k) in $SplExt(X)$, where (X, k) is any kernel of d_0.

Since every protomodular category is a Maltsev category, being an internal groupoid in $\mathbb{C}$ is the same as being an internal reflexive graph in $\mathbb{C}$ satisfying certain property (not having an additional structure). Specifically, a reflexive graph (B, A, d_0, s_0, d_1) is a groupoid if and only if the commutator

$$[R[d_0], R[d_1]] \tag{3.1}$$

is trivial, where $R[f]$ denotes (as in [1]) the equivalence relation determined by the kernel pairs of any morphism f:

$$R[f] \underset{p_1}{\overset{p_0}{\underset{\longleftarrow s_0 \longrightarrow}{\longrightarrow}}} X \xrightarrow{f} Y$$

and where, given any pair (R, S) of equivalence relations on an object X, we say that the commutator $[R, S]$ is trivial and write $[R, S] = 0$ when the pair (R, S) has a connector [7], i.e. it admits a morphism:

$$p : R \times_X S \to X,$$

which, written with generalized elements as $(xRySz) \mapsto p(x, y, z)$, satisfies the identities $p(x, y, y) = x$ and $p(y, y, z) = z$.

Accordingly, by a groupoid structure on an object (B, A, p, s, k) in $SplExt(X)$ we will mean a morphism $u : A \to B$ for which $us = 1_B$ and $[R[p], R[u]] = 0$; the system (B, A, p, s, k, u) will then be called an X-groupoid. X-groupoids form a category $Grpd(X) = Grpd_{\mathbb{C}}(X)$, in which a morphism

$$(g, f) : (B, A, p, s, k, u) \to (D, C, q, t, l, v) \tag{3.2}$$

is a morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ in $SplExt(X)$ with $vf = gu$. Similarly to the functor (1.3), the functor

$$Grpd(X) \to \mathbb{C} \tag{3.3}$$

sending (B, A, p, s, k, u) to B is a faithful fibration in which every vertical morphism is an isomorphism and therefore every morphism is cartesian. This last point means that every morphism in $Grpd(X)$ determines a discrete fibration of (internal) groupoids.

Similarly to Definitions 1.2 and 2.1, we introduce

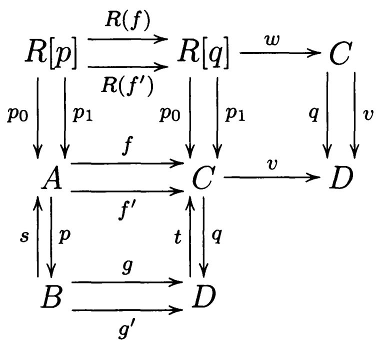
Definition 3.1. *(a) An X-groupoid is said to be faithful, if any X-groupoid admits at most one morphism into it.*
(b) An X-groupoid is said to be accessible, if it admits a morphism into a faithful X-groupoid. If every X-groupoid is accessible, we will say that $\mathbb{C}$ is groupoid accessible.

Lemma 3.2. *An X-groupoid is faithful if and only if its underlying object of $SplExt(X)$ is faithful.*

Proof. Let (D, C, q, t, l, v) be a faithful X-groupoid, and

$$(g, f), (g', f') : (B, A, p, s, k) \rightrightarrows (D, C, q, t, l)$$

a pair of morphisms in $SplExt(X)$. Consider the diagram



where the top parts of the first two columns the kernel equivalence relations of p and q, the top morphisms between them are induced by (g, f) and (g', f'), and w is the "division map" (with generalized elements it would be written as $w(\phi, \psi) = \psi\phi^{-1}$) of the kernel equivalence relation of q considered as a groupoid. Since the X-groupoid (D, C, q, t, l, v) is faithful, we get $vf = vf'$. Thus we have $g = vtg = vfs = vf's' = g'$.

Conversely, let (D, C, q, t, l, v) be a X-groupoid with a faithful underlying action. Then any pair

$$(g, f), (g', f') : (B, A, p, s, k, u) \rightrightarrows (D, C, q, t, l, v)$$

of morphisms in $Grpd(X)$ determines an underlying pair of morphisms in $SplExt(X)$, and consequently $g = g'$. □

By this lemma and Proposition 5.1 in [2] which shows that any split extension classifier underlies an internal groupoid, *any action representative category* $\mathbb{C}$ *is groupoid accessible.* We then get the following:

Proposition 3.3. *Suppose the category* $\mathbb{C}$ *is pointed protomodular and groupoid accessible; then it is action accessible.*

Proof. Just observe that any object (B, A, p, s, k) in $SplExt(X)$ admits a morphism into the underlying object of an X-groupoid, e.g. of the kernel equivalence relation of p, and use the previous lemma. □

On the other hand we have:

Proposition 3.4. *Suppose* $\mathbb{C}$ *is homological. Let*

$$(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$$

be a morphism in $SplExt(X)$ *in which* g *(and therefore also* f*) is a normal epimorphism. When* (B, A, p, s, k) *has a reflexive graph structure or a groupoid structure* u*, the object* (D, C, q, t, l) *also has such a structure* v *with* $vf = gu$*.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & I & \xrightarrow{1_I} & I & & \\
 & & & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & X & \xrightarrow{k} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & {\scriptstyle 1_X}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & X & \xrightarrow[l]{} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D & \longrightarrow & 0
\end{array} \tag{3.4}$$

in which (I, i) and (I, j) are the kernels of f and g respectively, and we can assume that they involve the same object I and have $j = pi$ (and $i = sj$) since the square formed by $qf = gp$ is a pullback. Given a morphism $u : A \to B$ with $us = 1_B$, we observe:

- Since f being a normal epimorphism is a cokernel of i, and since

$gui = gusj = gj = 0$, there exists a morphism $v : C \to D$ with $vf = gu$. Moreover, for such a morphism v, we have $vtg = vfs = gus = g$, and since g is an epimorphism, we obtain $vt = 1_D$.
• After this it remains to prove that if u is a groupoid structure on (B, A, p, s, k), and v is a reflexive graph structure on (D, C, q, t, l) with $vf = gu$, it is also a groupoid structure. But this is the case by Theorem 3.1 in [10], since any homological category is Malt'sev and regular, and g, f are both normal (and thus regular) epimorphisms. □

¿From this proposition, using also Proposition 1.4 and the obvious (normal epi, mono)-factorization system in $SplExt(X)$, we obtain:

Corollary 3.5. *Suppose $\mathbb{C}$ is homological. Then $\mathbb{C}$ is action accessible if and only if it is is groupoid accessible.*

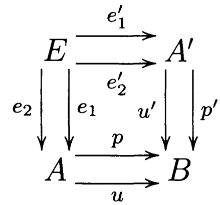
4 The centralizer of an accessible equivalence relation

Here is our main result:

Theorem 4.1. *Let R be an equivalence relation on an object B in a protomodular category $\mathbb{C}$, X an object in $\mathbb{C}$, and $(g, f) : (B, A, p, s, k, u) \to (D, C, q, t, l, v)$ a morphism in $Grpd(X)$, in which (B, A, p, s, k, u) is the equivalence relation R considered as a groupoid in $\mathbb{C}$, and (D, C, q, t, l, v) is faithful. Then the kernel pair $R[g]$ of g is the centralizer of R, i.e. the largest equivalence relation on B with $[R, R[g]] = 0$.*

Proof. The fact the commutator $[R, R[g]]$ is trivial follows from the fact that the morphism $R[f] \to R[g]$ of equivalence relations induced by (p, q) is a discrete fibration of groupoids (see [7] for details).
It remains to prove that if an equivalence relation $R' = (B, A', p', s', u')$ has $[R, R'] = 0$, then R' is less or equal to the kernel pair of g. That is,

we have to prove that $gp' = gu'$ whenever there exists a diagram



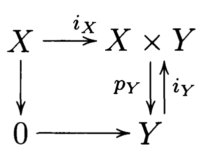
(4.1)

in which each pair of parallel arrows determines an equivalence relation, and the pairs (e_1, p') and (e_2, u') determine discrete fibrations. Since the relevant squares are pullbacks, the two horizontal top arrows in (4.1) determine an object $Grpd(X)$, and both pairs (e_1, p') and (e_2, u') determine a morphism from that object to (B, A, p, s, k, u). Composing these morphisms with (g, f) and using the fact that (D, C, q, t, l, v) is faithful, we obtain the desired equality. □

Corollary 4.2. *All equivalence relations in a groupoid accessible category $\mathbb{C}$ have centralizers. This is the case in particular for any action representative category and any homological action accessible category.*

5 Centralizer of subobjects and centralizer of equivalence relations

As soon as the category $\mathbb{C}$ is pointed protomodular, there is an intrinsic notion of commutation for subobjects; see [5]. Indeed, given any pair (X, Y) of objects, the following downward square is a pullback:



and consequently the pair (i_X, i_Y) is jointly strongly epic. Accordingly given any pair of subobject $x : X \rightarrowtail Z$, $y : Y \rightarrowtail Z$, there is at most one map $\phi : X \times Y \to Z$ such that $\phi i_X = x$ and $\phi i_Y = y$. When this is the case, we say that the subojects commute, call ϕ the cooperator of these two subobjects, and write $[\![X, Y]\!] = 0$ as in [6].

On the other hand, since $\mathbb{C}$ is pointed, any equivalence relation R on Z determines a subobject, namely the "equivalence class" $i_R = d_1 k : I_R \rightarrowtail Z$ of the initial map given by the following pullback:

$$\begin{array}{ccccc} I_R & \xrightarrow{k} & R & \xrightarrow{d_1} & Z \\ \downarrow & & \downarrow{\scriptstyle d_0} & & \\ 0 & \xrightarrow[\alpha_Z]{} & Z & & \end{array}$$

The following lemma is an obvious consequence of protomodularity:

Lemma 5.1. *The normalization function which associates with any equivalence relation on Z its normal subobject*

$$Rel(Z) \to Sub_Z, \ R \mapsto I_R$$

preserves and reflects the order.

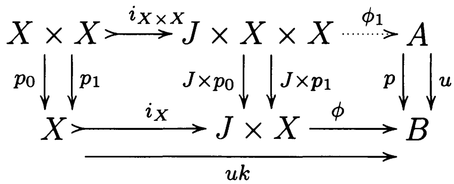
It is also clear that $[R, S] = 0$ implies $[\![I_R, I_S]\!] = 0$. The converse is true for strongly protomodular categories, but not in general, as shown in [6]. We are now going to show that groupoid accessible categories share this converse property with the strongly protomodular ones.

For, let us begin with the following observation. Let R be an equivalence relation on an object B, whose normalization is X and the corresponding X-groupoid is (B, A, p, s, k, u). When $\mathbb{C}$ is groupoid accessible, there is a morphism $(g, f) : (B, A, p, s, k, u) \to (D, C, q, t, l, v)$ in $Grpd(X)$ with the groupoid (D, C, q, t, l, v) being faithful. We have shown that the kernel pair $R[g]$ of g is the centralizer of R, i.e. the largest equivalence relation on B that commutes with R.

Proposition 5.2. *Suppose $\mathbb{C}$ is groupoid accessible. For R and g as above, the kernel morphism $k_g : K_g \rightarrowtail B$ of g (which is the normalization of $R[g]$ as well) is the largest subobject of B commuting with the normalization $uk : X \rightarrowtail B$ of R.*

Proof. Of course $[R, R[g]] = 0$ implies $[\![X, K_g]\!] = 0$; see [1] for instance. Suppose now we have any momomorphism $j : J \rightarrowtail B$ such that $[\![X, J]\!] = 0$; we have to check that J is less or equal to K_g, which is nothing but $gj = 0$. For, we will construct various morphisms of split extensions and then use Observation 1.3 as follows:

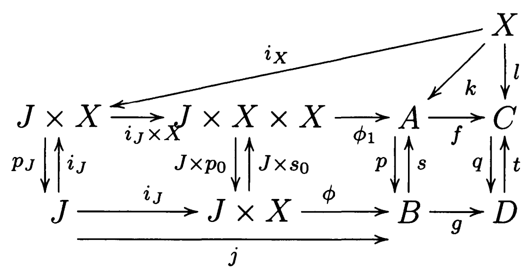
• Let $\phi : J \times X \to B$ be the cooperator of $j : J \rightarrowtail B$ and $uk : X \rightarrowtail B$. First we construct the diagram



where: (a) the left-hand square is a discrete fibration of equivalence relations; (b) since every such discrete fibration (="fibrant morphism") in a protomodular category is cocartesian with respect to the forgetful functor into the ground category (see Lemma 5.1 in [8] or Lemma 6.1.6 in [1]), the new morphism ϕ_1 can be defined as the morphism making the right-hand side of the diagram an internal functor.

• Note that $\phi_1 i_{X\times X} i_1 = k$, since $p\phi_1 i_{X\times X} i_1 = ukp_0 i_1 = 0 = pk$, $u\phi_1 i_{X\times X} i_1 = ukp_1 i_1 = uk$, and p and u are jointly monic.

• Next, using the morphism ϕ_1 above, we construct the diagram



(in obvious notation), which reasonably commutes, i.e.: (a) its top part commutes; (b) its bottom part formed by solid arrows represent morphisms between split epimorphisms (with specified splittings). Accordingly we have a morphism in $SplExt(X)$:

$$(gj, f\phi_1(i_J \times X)) : (J, J \times X, p_J, i_J, i_X) \to (D, C, q, t, l)$$

• Since (D, C, q, t, l) is faithful, Observation 1.3 then tell us that $gj = 0$. □

Observation 5.3. *If (g, f) is the unique morphism from the indiscrete relation $\nabla_X = (X, X \times X, p_0, s_0, p_1)$ into a faithful groupoid, then $R[g]$ is the largest equivalence relation commuting with ∇_X and K_g is nothing but the largest subobject commuting with 1_X, namely the centre ZX of X.*

Remark: When the category $\mathbb{C}$ in question is the category **Rg** of rings, this centre ZX is nothing but the *annihilator* of the ring X.

Theorem 5.4. *Suppose $\mathbb{C}$ is groupoid accessible. Let R and S be two equivalence relations on an object X. Then $[R, S] = 0$ if and only if $[\![I_R, I_S]\!] = 0$.*

Proof. We have already noticed that $[R, S] = 0$ implies $[\![I_R, I_S]\!] = 0$, which is a very general fact. Conversely suppose $[\![I_R, I_S]\!] = 0$. So, by Proposition 5.2, we have $I_S \subset K_g$, and according to Lemma 5.1, we have also $S \subset R[g]$. Whence, according to Theorem 4.1, $[R, S] = 0$. □

6 A characterization of antiadditivity

Recall that a morphism $k : X \to A$ is said to be *central*, if there is a (necessarily unique) cooperator $\phi : X \times A \to A$ such that $\phi i_X = k$ and $\phi i_A = 1_A$; we use here the terminology of [5] again, although this concept of centrality (and of commutator) was originally studied by S. A. Huq [11] (in a slightly different context). In accordance with the terminology of [5], let us call an object A *antiadditive* if there are no nonzero central morphisms into it; that is, a pointed protomodular category is antiadditive in the sense of [5], see also [1], if and only if every object in it is antiadditive in our sense. If $\mathbb{C}$ is antiadditive, any abelian object is trivial. When the ground category $\mathbb{C}$ is homological, an object A is antiadditive if and only if A *has a trivial centre*, and $\mathbb{C}$ is antiadditive if and only if $\mathbb{C}$ *has no non trivial abelian objects.*

Theorem 6.1. *An object A in a pointed protomodular category $\mathbb{C}$ is antiadditive if and only if the split extension*

$$0 \longrightarrow A \overset{\langle 0,1 \rangle}{\rightarrowtail} A \times A \underset{s_0}{\overset{p_0}{\rightleftarrows}} A \longrightarrow 0 \qquad (6.1)$$

is faithful.

Proof. Suppose the split extension (6.1) is faithful and consider any central morphism $k : X \to A$ with cooperator ϕ. Since both

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{i_A}{\rightarrowtail} & X \times A & \underset{i_X}{\overset{p_X}{\rightleftarrows}} & X & \longrightarrow & 0 \\ & & \downarrow 1_A & & \downarrow \langle kp_X, \phi\rangle & & \downarrow k & & \\ 0 & \longrightarrow & A & \underset{\langle 0,1\rangle}{\rightarrowtail} & A \times A & \underset{s_0}{\overset{p_0}{\rightleftarrows}} & A & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{i_A}{\rightarrowtail} & X \times A & \underset{i_X}{\overset{p_X}{\rightleftarrows}} & X & \longrightarrow & 0 \\ & & \downarrow 1_A & & \downarrow 0\times 1_A & & \downarrow 0 & & \\ 0 & \longrightarrow & A & \underset{\langle 0,1\rangle}{\rightarrowtail} & A \times A & \underset{s_0}{\overset{p_0}{\rightleftarrows}} & A & \longrightarrow & 0 \end{array}$$

are morphisms in $SplExt(A)$, we obtain $k = 0$.

Conversely, suppose A antiadditive and suppose we have a morphism (k, f) in $SplExt(A)$ whose codomain is the split extension (6.1). Then, since the square formed by f, k, and the appropriate arrows between them is a pullback, the domain of (k, f) must be isomorphic to the split extension

$$0 \longrightarrow A \overset{i_A}{\rightarrowtail} X \times A \underset{\langle 1_X, k\rangle}{\overset{p_X}{\rightleftarrows}} X \longrightarrow 0 \tag{6.2}$$

Therefore two parallel morphisms (k, f) and (l, g) into the split extension (6.1) will create an isomorphism $h : X \times A \to X \times A$ with $p_X h = p_X$, $h\langle 1_X, k\rangle = \langle 1_X, l\rangle$, and $hi_A = i_A$. Composing h with the projection $p_A : X \times A \to A$ we then obtain a morphism $\phi : X \times A \to A$ with $\phi\langle 1_X, k\rangle = l$ and $\phi i_A = 1_A$. The second identity makes ϕi_X central, and so $\phi i_X = 0$. Together with $\phi i_A = 1_A$ this implies $\phi = p_A$, and then $l = \phi\langle 1_X, k\rangle = p_A\langle 1_X, k\rangle = k$. Therefore the split extension (6.1) as faithful, as desired. □

Corollary 6.2. *A pointed protomodular category $\mathbb{C}$ is antiadditive if and only if the split extension (6.1) is faithful for each object A in $\mathbb{C}$.*

Let us now assume that the category $\mathbb{C}$ is action representable and call the morphism $A \to [A]$ corresponding to the split extension (6.1) *canonical.* From the previous results and Corollary 1.5 we obtain:

Corollary 6.3. *Let $\mathbb{C}$ be a (pointed protomodular) action representative category. Then:*
(a) an object A in $\mathbb{C}$ is antiadditive if and only if the canonical morphism $A \to [A]$ is a monomorphism;
(b) the category $\mathbb{C}$ is antiadditive if and only if the canonical morphism $A \to [A]$ is a monomorphism for each object A in $\mathbb{C}$.

Note that:
• Corollary 6.3(a) applied to the category of groups becomes the following obvious and yet nice observation: a group A has trivial centre if and only if the canonical homomorphism $A \to Aut(A)$ is injective.
• In several action representative categories, such as the dual $\mathbf{Set}_*^{\mathrm{op}}$ of the category of pointed sets, or the categories **BooRg** and **vNRg** of Boolean rings and von Neumann regular rings, the canonical morphisms $A \to [A]$ have been independently shown to be monomorphisms for all object A. As we see now, this can be used as a proof of their antiadditivity – even though direct proofs are also easy.

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1) Université du Littoral, Calais, France
Dominique.Bourn@lmpa.univ-littoral.fr
2) University of Cape Town, Cape Town, South Africa
George.Janelidze@uct.ac.za