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FIRM REFLECTIONS GENERATED BY COMPLETE METRIC SPACES

by E. COLEBUNDERS and A. GERLO

RESUME. Nous étudions des catégories concrètes où chaque objet est un sous-espace d'un produit "d'espaces métrisables". Si une telle catégorie est munie d'un opérateur s de fermeture, nous considérons U_s, la classe des immersions denses. Nous traitons les questions suivantes: (1) si les espaces complètement métrisables sont des objets U_s-injectifs, (2) si la classe des sous-objets s-fermés d'un produit d'espaces complètement métrisables est U_s "uniquement" reflective. Nous démontrons que dans notre contexte, ces questions sont équivalentes et nous formulons des conditions pour avoir une réponse affirmative. Le théorème principal permet de traiter un grand nombre d'exemples.

1 Introduction

The category $\mathbf{Unif}_0$ of separated uniform spaces, endowed with the closure operator r determined by the underlying topology, will be our guiding example in the study of completeness in a more general setting. Completely metrizable uniform spaces play an important role in the uniform case, since firstly they are injective objects with respect to the class $\mathcal{U}_r$ of all dense embeddings and secondly the complete uniform spaces are exactly the closed subspaces of products of completely metrizable spaces. Moreover the complete objects form a firmly $\mathcal{U}_r$- reflective subconstruct of $\mathbf{Unif}_0$ in the sense of [3].

We will investigate to what extent these results hold in a more general setting. The general framework we will be working in is the one of metrically generated constructs as introduced in [6]. These are constructs $\mathcal{X}$ for which a natural functor describes the transition from (generalized) metric spaces to objects in the given category $\mathcal{X}$. For example, with a (generalized) metric d one can associate e.g. a (completely regular) topology $\mathcal{T}_d$, a (quasi)uniformity $\mathcal{U}_d$, a proximity $\mathcal{P}_d$ or an approach structure $\mathcal{A}_d$. In each of these examples, a natural functor K from a suitable base category $\mathcal{C}$ consisting of (generalized) metric spaces to the category $\mathcal{X}$ is given. If the functor K fulfills certain conditions (preserves initial morphisms and has an initially dense image) then the category $\mathcal{X}$ is said to be metrically generated. This setting, which covers all the examples above and many others, is convenient for our purpose since in particular every object in $\mathcal{X}$ is a subspace of a product of "metrizable" spaces. We will restrict to T_0-objects and a first attempt will be to endow $\mathcal{X}_0$ with its regular closure operator r and to consider the class $\mathcal{U}_r$ of all r-dense embeddings. The following two questions will be investigated:

1) Are the completely metrizable objects $\mathcal{U}_r$-injective?

2) Is the class of all r-closed subspaces of products of completely metrizable objects firmly $\mathcal{U}_r$-reflective?

In fact we will show that in our setting these questions are equivalent and we will give necessary and sufficient conditions for a positive answer. Our main theorem will apply to a large collection of examples listed in the tables of the next sections. It will become clear that there exist metrically generated constructs $\mathcal{X}$ allowing a $\mathcal{U}_r$-firm reflective subconstruct $\mathcal{R}$ which cannot be generated by complete metric spaces, so for which the questions above nevertheless have a negative answer.

In some cases where the answer to the questions above is negative, we still succeed in defining a smaller non-trivial closure operator for which the answers do become positive.

2 Metrically generated theories

In this section we gather some preliminary material that is needed to introduce the setting of this paper. We use categorical terminology as developed in [1] or [17] and we refer to [9] for material on closure operators.

In [6] it was shown that every metrically generated construct can be isomorphically described as a subconstruct of a certain model category. It will be convenient to deal with these isomorphic copies. So we recall the material on the model categories and fix some notation.

We call a function $d : X \times X \longrightarrow [0,\infty]$ a quasi-pre-metric if it is zero on the diagonal, we will drop "pre" if d satisfies the triangle inequality and we will drop "quasi" if d is symmetric. Note that we do not ask these quasi-pre-metrics to be realvalued or separated. If d is a quasi-metric we denote by d^* its symmetrization $d \vee d^{-1}$.
Denote by **Met** the construct of quasi-pre-metrics and contractions. Recall that a map $f : (X,d) \longrightarrow (X',d')$ is a contraction (also called a nonexpansive map) if for every $x \in X$ and $y \in X$ one has $d'(f(x),f(y)) \leq d(x,y)$ (or shortly if $d' \circ f \times f \leq d$). Further denote by $\mathbf{Met}(X)$ the fiber of **Met** structures on X. The particular full subcategory of **Met** consisting of all quasi-metric spaces [12] will be denoted by $\mathcal{C}^{\Delta}$. Other subconstructs that will be considered are $\mathcal{C}^{\Delta s}$ the construct of metric spaces, $\mathcal{C}^{\Delta s \vartheta}$ the construct of totally bounded metric spaces and $\mathcal{C}^{\mu}$ the construct of ultrametric spaces.

The order on $\mathbf{Met}(X)$ is defined pointwise and as usual a downset in $\mathbf{Met}(X)$ is a non-empty subset $\mathcal{S}$ such that if $d \in \mathcal{S}$ and e is a quasi-pre-metric, $e \leq d$ then $e \in \mathcal{S}$. For any collection $\mathcal{B}$ of quasi-pre-metrics we put $\mathcal{B}{\downarrow} := \{e \in \mathbf{Met}(X) \mid \exists d \in \mathcal{B} : e \leq d\}$. We say that $\mathcal{B}$ is a basis for $\mathcal{M}$ if $\mathcal{B}{\downarrow} = \mathcal{M}$.

M is the construct with objects, pairs $(X,\mathcal{M})$ where X is a set and $\mathcal{M}$ is a downset in $\mathbf{Met}(X)$. $\mathcal{M}$ is called a *meter (on X)* and $(X,\mathcal{M})$ a *metered space*. If $(X,\mathcal{M})$ and $(X',\mathcal{M}')$ are metered spaces and $f : (X,\mathcal{M}) \longrightarrow (X',\mathcal{M}')$ then we say that f is a *contraction* if

$$\forall d' \in \mathcal{M}' : d' \circ f \times f \in \mathcal{M}.$$

It is easily verified that **M** is a well fibred topological construct. We refer to [6] for the detailed constructions of initial and final structures.

A *base category* $\mathcal{C}$ is a full and isomorphism-closed concrete subconstruct of **Met** which satisfies certain stability conditions as formulated in [6].

In this paper we will only consider base categories $\mathcal{C}$ that are contained in $\mathcal{C}^{\Delta}$ and that satisfy some supplementary conditions from [5] ensuring some results on separation.

In order to deal with completions we will add one more condition which will be assumed on all base categories we encounter.

[B] $\mathcal{C}$ is said to be closed under "r-dense" extensions in $\mathcal{C}^{\Delta}$ whenever $f : (X,d) \to (Y,d')$ is a $\mathcal{T}_{d'^*}$-dense embedding in $\mathcal{C}^{\Delta}$ with (X,d) belonging to $\mathcal{C}$ then also (Y,d') belongs to $\mathcal{C}$.

The subconstructs of **Met** introduced earlier, $\mathcal{C}^{\Delta}$, $\mathcal{C}^{\Delta s}$, $\mathcal{C}^{\Delta s \vartheta}$ and $\mathcal{C}^{\mu}$ are base categories and as we know from [5] the results on separation go through. Note that all of them satisfy [B].

Given a base category $\mathcal{C}$, one considers $\mathcal{C}$-meters, these are meters having a basis consisting of $\mathcal{C}$-metrics. The full reflective subconstruct of **M**, consisting of all metered spaces with meters having a basis consisting of $\mathcal{C}$-metrics is denoted by $\mathbf{M}^{\mathcal{C}}$ and the fiber of $\mathbf{M}^{\mathcal{C}}$ structures on X is denoted by $\mathbf{M}^{\mathcal{C}}(X)$.

An *expander* ξ *on* $\mathbf{M}^{\mathcal{C}}$ provides us for every set X with a function

$$\mathbf{M}^{\mathcal{C}}(X) \to \mathbf{M}^{\mathcal{C}}(X) : \mathcal{M} \mapsto \xi(\mathcal{M})$$

such that the following properties are fulfilled:

[E1] $\mathcal{M} \subset \xi(\mathcal{M})$,

[E2] $\mathcal{M} \subset \mathcal{N} \Rightarrow \xi(\mathcal{M}) \subset \xi(\mathcal{N})$,

[E3] $\xi(\xi(\mathcal{M})) = \xi(\mathcal{M})$,

[E4] if $f : Y \to X$ and $\mathcal{M} \in \mathbf{M}^{\mathcal{C}}(X)$, then: $\xi(\mathcal{M}) \circ f \times f \subset \xi(\mathcal{M} \circ f \times f \downarrow)$

Given an expander ξ on $\mathbf{M}^{\mathcal{C}}$, then $\mathbf{M}^{\mathcal{C}}_{\xi}$ is the full coreflective subconstruct of $\mathbf{M}^{\mathcal{C}}$ with objects, those metered spaces $(X, \mathcal{M})$ for which $\xi(\mathcal{M}) = \mathcal{M}$.

The main result of [6] states that $\mathbf{M}^{\mathcal{C}}$ provides a model for all $\mathcal{C}$-metrically generated theories in the sense that a topological construct $\mathcal{X}$ is $\mathcal{C}$-metrically generated (meaning that there is a functor $K : \mathcal{C} \to \mathcal{X}$ preserving initial morphisms and having an initially dense image) if and only if $\mathcal{X}$ is concretely isomorphic to $\mathbf{M}^{\mathcal{C}}_{\xi}$ for some expander ξ on $\mathbf{M}^{\mathcal{C}}$. Again in order to apply some results on separation we assume two extra technical assumptions [E5],[E6] on the expanders:

[E5] $\xi(\{\mathbf{0}\}) = \{\mathbf{0}\}$, where **0** denotes the zero-metric,

[E6] $\xi(\mathcal{M})$ is saturated for taking finite suprema, for every $\mathcal{M} \in \mathbf{M}^{\mathcal{C}}(X)$.

Without explicit mentioning, we will only consider expanders that satisfy the conditions [E1] up to [E6] from [6] and [5].

For a $\mathcal{C}$-meter $\mathcal{D}$ on a set X, denote $\xi^{\mathcal{C}}(\mathcal{D}) = \{d \in \xi(\mathcal{D}) \mid d\ \mathcal{C}\text{-metric}\} \downarrow$. If we consider the following examples for ξ, we obtain expanders $\xi^{\mathcal{C}}_T, \xi^{\mathcal{C}}_A, \xi^{\mathcal{C}}_U$,

$\xi^{\mathcal{C}}_{UG}, \xi^{\mathcal{C}}_{D}$ and $\iota^{\mathcal{C}}$ on $\mathbf{M}^{\mathcal{C}}$, which will yield important constructs within the framework of metrically generated theories.

- $d \in \xi_T(\mathcal{D})$ iff $\forall x \in X, \forall \varepsilon > 0, \exists d_1, \ldots, d_n \in \mathcal{D}, \exists \delta > 0 : \sup_{i=1}^{n} d_i(x,y) < \delta \Rightarrow d(x,y) < \varepsilon$
- $d \in \xi_A(\mathcal{D})$ iff $\forall x \in X, \forall \varepsilon > 0, \forall \omega < \infty, \exists d_1, \ldots, d_n \in \mathcal{D} : d(x,y) \wedge \omega \leq \sup_{i=1}^{n} d_i(x,y) + \varepsilon$
- $d \in \xi_U(\mathcal{D})$ iff $\forall \varepsilon > 0, \exists d_1, \ldots, d_n \in \mathcal{D}, \exists \delta > 0 : \sup_{i=1}^{n} d_i(x,y) < \delta \Rightarrow d(x,y) < \varepsilon$
- $d \in \xi_{UG}(\mathcal{D})$ iff $\forall \varepsilon > 0, \forall \omega < \infty, \exists d_1, \ldots, d_n \in \mathcal{D} : d(x,y) \wedge \omega \leq \sup_{i=1}^{n} d_i(x,y) + \varepsilon$
- $d \in \xi_D(\mathcal{D})$ iff $d \leq \sup_{e \in \mathcal{D}} e$.
- $d \in \iota(\mathcal{D})$ iff $d \leq \sup_{e \in \mathcal{E}} e$, for a finite $\mathcal{E} \subset \mathcal{D}$.

Whenever it is clear from the context what base category is involved, we will drop the superscript $\mathcal{C}$ in the notations above. We capture many known topological constructs, considering the above expanders on categories $\mathbf{M}^{\mathcal{C}}$, for different base categories $\mathcal{C}$.

	$\mathcal{C}^{\Delta}$	$\mathcal{C}^{\Delta s}$	$\mathcal{C}^{\Delta s\theta}$	$\mathcal{C}^{\mu}$
$\xi^{\mathcal{C}}_{T}$	**Top**	**Creg**	**Creg**	**ZDim**
$\xi^{\mathcal{C}}_{A}$	**Ap**	**UAp**	**UAp**	**ZDAp**
$\xi^{\mathcal{C}}_{U}$	**qUnif**	**Unif**	**Prox**	**naUnif**
$\xi^{\mathcal{C}}_{UG}$	**qUG**	**UG**	**efGap**	**tUG**
$\xi^{\mathcal{C}}_{D}$	$\mathcal{C}^{\Delta}$	$\mathcal{C}^{\Delta s}$	$\mathcal{C}^{\Delta s\theta}$	$\mathcal{C}^{\mu}$

Top, **Creg** and **ZDim** consist of all topological spaces, of all completely regular and of all zero dimensional topological spaces respectively, with continuous maps as morphisms.
Ap and **UAp** consist of all approach spaces and uniform approach spaces in the sense of [13], with contractions as morphisms. **ZDAp** is the full subconstruct consisting of all zero dimensional approach spaces. These are approach spaces with a gauge basis consisting of ultrametrics or could be equivalently defined as those approach spaces that are subspaces of products in **Ap** of ultrametric spaces.
qUnif consists of all quasi-uniform spaces [12], [8], **Unif** of all uniform spaces, with uniformly continuous maps as morphisms, **Prox** of all proximity spaces and proximally continuous maps [17] and **naUnif** is the full subconstruct of **Unif** consisting of all non-Archimedian uniform spaces in the sense of [16].
qUG consists of all quasi-uniform gauge spaces [7], **UG** of all uniform gauge spaces [14], with uniform contractions, **efGap** of all Effremovic-gap

spaces in the sense of [10] with associated maps and **tUG** is the full subconstruct of **UG** consisting of all transitive uniform gauge spaces.

3 Cogeneration by completely metrizable spaces

Recall that an object (X,d) in $\mathcal{C}^{\Delta}$ is said to be *bicomplete* if (X,d^*) is complete. (Y,q) is a *bicompletion* of a $\mathcal{C}^{\Delta}$-object (X,d) if (Y,q) is a bicomplete space in which (X,d) is q^*-densely embedded. For objects in a base category $\mathcal{C}$, we will use the following analogous definition for completeness and completion.

Definition 3.1.
- *A $\mathcal{C}$-object (X,d) is called* bicomplete *if (X,d^*) is complete.*
- *(Y,q) is a* $\mathcal{C}$-completion *of a $\mathcal{C}$-object (X,d) if (Y,q) is a bicompletion of (X,d) in $\mathcal{C}^{\Delta}$ and (Y,d) belongs to $\mathcal{C}$.*

As usual we denote by $\mathcal{X}_0$ the class of T_0-objects in $\mathcal{X}$ [15]. In particular $\mathcal{C}_0$ is the subconstruct of $\mathcal{C}$ consisting of its T_0-objects.
It is well known that every T_0 quasi-metric space has an (up to isometry) unique $\mathcal{C}_0^{\Delta}$-completion. It easily follows from our assumptions on the base categories that for (X,d) a T_0 $\mathcal{C}$-object, the $\mathcal{C}_0^{\Delta}$-completion of (X,d) is also the unique $\mathcal{C}_0$-completion.

Recall from [4] that a (complete) construct is said to be Emb-*cogenerated* by a subclass $\mathcal{P}$ if every object is embedded in a product of $\mathcal{P}$-objects.

Proposition 3.2. *Assume $\mathcal{C}$ is a base category and let ξ be an expander on $\mathbf{M}^{\mathcal{C}}$. Let*

$$\mathcal{P} = \{(Z,\xi(\{e\}\downarrow)) : (Z,e) \text{ is a bicomplete } \mathcal{C}_0 - \text{space}\}$$

Then $\mathcal{P}$ is an Emb-*cogenerating class for $\left(\mathbf{M}^{\mathcal{C}}_{\xi}\right)_0$.*

Proof. Case 1) of the proof deals with the expander $\iota^{\mathcal{C}}$. Let $(X,\mathcal{D})$ be an arbitrary $\left(\mathbf{M}^{\mathcal{C}}_{\iota^{\mathcal{C}}}\right)_0$-object, with a base $\mathcal{Q}$ of $\mathcal{C}$-metrics.
Note that the source

$$(1_X : (X,\mathcal{D}) \longrightarrow (X,\{q\}\downarrow))_{q\in\mathcal{Q}}$$

is initial in $\mathbf{M}^{\mathcal{C}}_{\iota\mathcal{C}}$. Recall that the T_0-quotient reflection of a quasi-metric space (X,d) is given by the morphism

$$\tau_d : (X,d) \longrightarrow \left(X_d, \overline{d}\right) : x \longmapsto \overline{x}$$

where $\overline{x} = \{y \in X \mid d(x,y) = d(y,x) = 0\}$, $X_d = \{\overline{x} \mid x \in X\}$ and $\overline{d}(\overline{x},\overline{y}) = d(x,y)$ for $x,y \in X$. Using the standing assumptions on $\mathcal{C}$, the T_0-reflection of a $\mathcal{C}$-object is obtained in the same way as in $\mathcal{C}^{\Delta}$. The reflection morphism $\tau_q : (X,q) \longrightarrow \left(X_q, \overline{q}\right) : x \longmapsto \overline{x}$ is initial, which implies that also the source

$$\left(\tau_q : (X, \mathcal{D}) \longrightarrow \left(X_q, \{\overline{q}\}\downarrow\right)\right)_{q \in \mathcal{Q}}$$

is initial in $\mathbf{M}^{\mathcal{C}}_{\iota\mathcal{C}}$. By our standing assumptions on $\mathcal{C}$, for each $q \in \mathcal{Q}$, one can consider the $\mathcal{C}_0$-completion $\left(\widehat{X_q}, \widehat{\overline{q}}\right)$ of the space $\left(X_q, \overline{q}\right)$. So, for every $q \in \mathcal{Q}$, the map $k_q : \left(X_q, \overline{q}\right) \longrightarrow \left(\widehat{X_q}, \widehat{\overline{q}}\right)$ is initial in $\mathcal{C}$. It follows that the contraction $k_q : \left(X_q, \{\overline{q}\}\downarrow\right) \longrightarrow \left(\widehat{X_q}, \{\widehat{\overline{q}}\}\downarrow\right)$ is initial in $\mathbf{M}^{\mathcal{C}}_{\iota\mathcal{C}}$. Finally one obtains the following initial source in $\mathbf{M}^{\mathcal{C}}_{\iota\mathcal{C}}$:

$$\left(k_q \circ \tau_q : (X, \mathcal{D}) \longrightarrow \left(\widehat{X_q}, \{\widehat{\overline{q}}\}\downarrow\right)\right)_{q \in \mathcal{Q}}$$

Due to the T_0 property of $(X, \mathcal{D})$, which means that for any $x,y \in X, x \neq y$, there exists $d \in \mathcal{M} : d(x,y) \neq 0$ or $d(y,x) \neq 0$, this source turns out to be point-separating. Moreover for every $q \in \mathcal{Q}$, the $\mathcal{C}$-space $\left(\widehat{X_q}, \{\widehat{\overline{q}}\}\downarrow\right)$ is a $\mathcal{P}$-object.

For case 2) of the proof, let $(X, \mathcal{D})$ be an arbitrary $\left(\mathbf{M}^{\mathcal{C}}_{\xi}\right)_0$-object. It suffices to apply the coreflector $\xi : \mathbf{M}^{\mathcal{C}}_{\iota\mathcal{C}} \longrightarrow \mathbf{M}^{\mathcal{C}}_{\xi} : (Y, \mathcal{G}) \longmapsto (Y, \xi(\mathcal{G}))$ to the source $(k_q \circ \tau_q)_{q \in \mathcal{Q}}$. □

We capture some well known results like $\mathbf{Unif}_0$ being Emb-cogenerated by the class

$$\{(Z, \mathcal{U}_d) \mid d \text{ a complete Hausdorff metric on } Z\}$$

and the construct $\mathbf{UAp}_0$ being Emb-cogenerated by the class

$$\{(Z, \delta_d) \mid d \text{ a complete Hausdorff metric on } Z\}.$$

The previous theorem implies analogous results for all the constructs in table of section 2. Note that $\mathbf{Top}_0$ and $\mathbf{Ap}_0$ are cogenerated by a single object. $\mathbf{Top}_0$ is Emb-cogenerated by the Sierpinski space S_2 which is quasi-metrizable by a T_0 bicomplete quasi-metric. $\mathbf{Ap}_0$ is cogenerated by the object $\mathbb{P}$. This object $\mathbb{P}$ however is not (bicompletely) quasi-metrizable. We will come back to these examples in section 5.

4 Construction of complete objects from completely metrizable spaces

In this section we tackle our main problem. We will endow $(\mathbf{M}_\xi^\mathcal{C})_0$ with a closure operator s and we will consider the class $\mathcal{U}_s$ of all s-dense embeddings. The following two questions will be investigated:
1) Are the completely metrizable objects $\mathcal{U}_s$-injective?
2) Is the class of all s-closed subspaces of products of completely metrizable objects firmly $\mathcal{U}_s$-reflective?

For explicit definitions on firmness we refer to [4] and [3]. Here we briefly recall that, given a class $\mathcal{U}$ of $\mathcal{X}$-morphisms, a reflective subconstruct with reflector R is said to be subfirmly $\mathcal{U}$-reflective if it is $\mathcal{U}$-reflective and if for every morphism u in $\mathcal{U}$ the reflection $R(u)$ is an isomorphism. If $\mathcal{U}$ coincides with the class of morphisms for which $R(u)$ is an isomorphism, the subconstruct is said to be firmly $\mathcal{U}$-reflective. Among other things $\mathcal{U}$-firmness implies uniqueness of completion with respect to the class $\mathcal{U}$.

Since the class $\mathcal{U}_s$ we will be dealing with consists of certain embeddings, $\mathcal{U}_s$-firmness will imply that $\mathcal{U}_s$ is contained in the class of all epimorphic embeddings. In all the examples in section 6. we will be dealing with closure operators on $(\mathbf{M}_\xi^\mathcal{C})_0$ that are (pointwise) smaller than the regular closure operator r, describing the epimorphisms. In order to satisfy the standing assumptions on stability of $\mathcal{U}$ with respect to compositions, as put forward in [3], we will assume that the closure operator s is idempotent. The class of $\mathcal{U}_s$-injective objects is denoted by $\mathrm{Inj}\,\mathcal{U}_s$. The proof of the next result uses standard techniques, see for instance [4].

Proposition 4.1. *If s is a weakly hereditary, idempotent closure operator on $\mathcal{X}$, then* $\mathrm{Inj}\,\mathcal{U}_s$ *is closed for taking s-closed subspaces of products in* $(\mathbf{M}_\xi^\mathcal{C})_0$.

In [5] the closure operator r has been explicitely formulated in the following way. For an $\left(\mathbf{M}_{\xi}^{\mathcal{C}}\right)_0$-object $(X,\mathcal{D})$

$$x \in r_X(M) \Longleftrightarrow \forall\, d \in \mathcal{D} : \inf_{m \in M} d(x,m) + d(m,x) = 0.$$

The closure operator r is known to be idempotent and was shown to be hereditary on $\left(\mathbf{M}_{\xi}^{\mathcal{C}}\right)_0$ for all the expanders listed in section 2, i.e. for arbitrary $\mathcal{C}$ in cases where ξ equals any of the expanders $\iota^{\mathcal{C}}, \xi_U^{\mathcal{C}}, \xi_{UG}^{\mathcal{C}}$ or $\xi_D^{\mathcal{C}}$, and for $\mathcal{C} \subset \mathcal{C}^{\Delta s}$ and $\mathcal{C}^{\Delta}$ in the cases $\xi_T^{\mathcal{C}^{\Delta}}, \xi_A^{\mathcal{C}^{\Delta}}$.

Theorem 4.2. *Assume $\mathcal{C}$ is a base category and let ξ be an expander on $\mathbf{M}^{\mathcal{C}}$. On $\left(\mathbf{M}_{\xi}^{\mathcal{C}}\right)_0$ let s be a weakly hereditary, idempotent closure operator and let $\mathcal{U}_s$ be the class of all s-dense embeddings in $\left(\mathbf{M}_{\xi}^{\mathcal{C}}\right)_0$.*
The following are equivalent:

1. *For every $j : (X,\mathcal{H}) \longrightarrow (Y,\mathcal{D})$ with $j \in \mathcal{U}_s$:*

$$j \in \mathcal{U}_r \text{ and } \mathcal{H} = \mathcal{D} \circ j \times j \downarrow$$

2. *The class $\mathcal{P} = \{(Z,\xi(\{e\}\downarrow)) : (Z,e)$ is a bicomplete $\mathcal{C}_0$-object$\}$ is $\mathcal{U}_s$-injective in $(\mathbf{M}_{\xi}^{\mathcal{C}})_0$ and $\mathcal{U}_s \subset \mathcal{U}_r$;*

3. *The class $\mathcal{R}_s$ of s-closed subobjects of products of $\mathcal{P}$-objects is a subfirm $\mathcal{U}_s$-reflective subcategory of $\left(\mathbf{M}_{\xi}^{\mathcal{C}}\right)_0$.*

Proof. To prove that 1. implies 2. let $(Z,\xi(\{e\}\downarrow))$ be an arbitrary $\mathcal{P}$-object, $j : (X,\mathcal{H}) \longrightarrow (Y,\mathcal{D})$ belong to $\mathcal{U}_s$ and $f : (X,\mathcal{H}) \longrightarrow (Z,\xi(\{e\}\downarrow))$ be a contraction in $\mathbf{M}_{\xi}^{\mathcal{C}}$. Since $e \circ f \times f$ belongs to $\mathcal{H}$ and since by 1. $\mathcal{H} = \mathcal{D} \circ j \times j \downarrow$, we can choose a $\mathcal{C}$-metric $d \in \mathcal{D}$ such that $e \circ f \times f \le d \circ j \times j$. Consider the following situation in $\mathcal{C}^{\Delta}$. The map $j : (X, d \circ j \times j) \longrightarrow (Y,d)$ is a d^*-dense embedding and $f : (X, d \circ j \times j) \longrightarrow (Z,e)$ is a contraction. Since (Z,e) is bicomplete, it is injective in $\mathcal{C}^{\Delta}$ with respect to r-dense embeddings, and hence there is a contraction $\widetilde{f} : (Y,d) \longrightarrow (Z,e)$ such that $\widetilde{f} \circ j = f$. Clearly $\widetilde{f} : (Y,\mathcal{D}) \longrightarrow (Z,\{e\}\downarrow)$ is a contraction in $\mathbf{M}^{\mathcal{C}}$ and since $(Y,\mathcal{D})$ belongs to $\mathbf{M}_{\xi}^{\mathcal{C}}$ the map $\widetilde{f} : (Y,\mathcal{D}) \longrightarrow (Z,\xi(\{e\}\downarrow))$ is a contraction in $\mathbf{M}_{\xi}^{\mathcal{C}}$.

To prove that 2. implies 3., we follow the lines of proof of theorem 1.6 in [4]. First note that by 3. $\mathcal{P} \subseteq \mathrm{Inj}\,\mathcal{U}_s$. Hence, from proposition 4.1 we have that $\mathcal{R}_s \subseteq \mathrm{Inj}\,\mathcal{U}_s$. Next we show that $\mathcal{R}_s$ is a $\mathcal{U}_s$-reflective subconstruct.
Let $\mathbf{X}$ be an arbitrary $\left(\mathbf{M}_\xi^{\mathcal{C}}\right)_0$-object. Proposition 3.2 ensures that there exist objects $\mathbf{P}_i \in \mathcal{P}$ $(i \in I)$ such that we have an embedding $j : \mathbf{X} \hookrightarrow \prod_{i \in I} \mathbf{P}_i$. Consider its $(\mathcal{E}^s, \mathcal{M}^s)$-factorization $j = m \circ e$ where $\mathbf{X} \xrightarrow{e} \mathbf{M} \xrightarrow{m} \prod_{i \in I} \mathbf{P}_i$, with $e \in \mathcal{E}^s$ and $m \in \mathcal{M}^s$. Since j is an embedding, so is e. So we get that $e \in \mathcal{U}_s$ and $\mathbf{M} \in \mathcal{R}_s$.
For $\mathbf{Y} \in \mathcal{R}_s$ and $f : \mathbf{X} \longrightarrow \mathbf{Y}$ an arbitrary contraction, using the $\mathcal{U}_s$-injectivity of $\mathbf{Y}$, we can construct a contraction f^* such that $f^* \circ e = f$ which is unique by the fact that e is an epimorphism.
Moreover, $\mathcal{R}_s$ is subfirmly $\mathcal{U}_s$-reflective. For $\left(\mathbf{M}_\xi^{\mathcal{C}}\right)_0$-objects $\mathbf{X}$ and $\mathbf{Z}$ suppose $g : \mathbf{X} \longrightarrow \mathbf{Z}$ belongs to $\mathcal{U}_s$. Denote by $r_\mathbf{Z} : \mathbf{Z} \longrightarrow R\mathbf{Z}$ and $r_\mathbf{X} : \mathbf{X} \longrightarrow R\mathbf{X}$ the $\mathcal{R}_s$-reflection morphisms. Using the $\mathcal{U}_s$-injectivity of $R\mathbf{X}$ and the fact that g, $r_\mathbf{Z}$ and $r_\mathbf{X}$ belong to $\mathcal{U}_s$, we can conclude that there exists a contraction $h : R\mathbf{Z} \longrightarrow R\mathbf{X}$ such that h and Rg are each others inverses. Finally Rg is an isomorphism.

To prove that 3. implies 1. suppose $\mathcal{R}_s$ is subfirmly $\mathcal{U}_s$-reflective. Then the results in [3] already imply that $\mathcal{R}_s = \mathrm{Inj}\,\mathcal{U}_s$ and that $\mathcal{U}_s \subset \mathcal{U}_r$.
Let $j : (X, \mathcal{H}) \longrightarrow (Y, \mathcal{D})$ belong to $\mathcal{U}_s$ and consider an arbitrary $\mathcal{C}$-metric $e \in \mathcal{H}$. Then, as in the proof of proposition 3.2, the map

$$\alpha_e : (X, \mathcal{H}) \longrightarrow \left(\widehat{X}_e, \widehat{\overline{e}}\right) : x \longmapsto \overline{x}$$

is a contraction in $\mathbf{M}^{\mathcal{C}}$ and therefore $\alpha_e : (X, \mathcal{H}) \longrightarrow \left(\widehat{X}_e, \xi(\{\widehat{\overline{e}}\}\downarrow)\right)$ is a contraction in $\mathbf{M}_\xi^{\mathcal{C}}$. Since $\left(\widehat{X}_e, \xi(\{\widehat{\overline{e}}\}\downarrow)\right)$ is $\mathcal{U}_s$-injective, there exists a contraction $\widetilde{\alpha_e} : (Y, \mathcal{D}) \longrightarrow \left(\widehat{X}_e, \xi(\{\widehat{\overline{e}}\}\downarrow)\right)$, such that $\widetilde{\alpha_e} \circ j = \alpha_e$. Composing $\widetilde{\alpha_e}$ with the $\mathbf{M}^{\mathcal{C}}$-morphism

$$j' : (X, \mathcal{D} \circ j \times j\downarrow) \longrightarrow (Y, \mathcal{D}) : x \longmapsto j(x)$$

we get that

$$\widetilde{\alpha_e} \circ j' : (X, \mathcal{D} \circ j \times j\downarrow) \longrightarrow \left(\widehat{X}_e, \xi(\{\widehat{\overline{e}}\}\downarrow)\right)$$

is a morphism in $\mathbf{M}^{\mathcal{C}}$. Consequently: $e = \widehat{\overline{e}} \circ (\widetilde{\alpha_e} \circ j') \times (\widetilde{\alpha_e} \circ j')$ belongs to $\mathcal{D} \circ j \times j\downarrow$. □

If moreover we assume the closure operator s to be hereditary, we can strenghten 3. in the equivalences of theorem 4.2.

Corollary 4.3. *Assume $\mathcal{C}$ is a base category and let ξ be any expander on $\mathbf{M}^{\mathcal{C}}$. On $(\mathbf{M}^{\mathcal{C}}_{\xi})_0$ let s be a hereditary, idempotent closure operator and let $\mathcal{U}_s$ be the class of all s-dense embeddings in $(\mathbf{M}^{\mathcal{C}}_{\xi})_0$.*
The following are equivalent:

1. *For every $j : (X, \mathcal{H}) \longrightarrow (Y, \mathcal{D})$ with $j \in \mathcal{U}_s$:*

$$j \in \mathcal{U}_r \text{ and } \mathcal{H} = \mathcal{D} \circ j \times j \downarrow$$

2. *$\mathcal{P} = \{(Z, \xi(\{e\}\downarrow)) : (Z, e)$ is a bicomplete $\mathcal{C}_0$-object$\}$ is $\mathcal{U}_s$-injective in $(\mathbf{M}^{\mathcal{C}}_{\xi})_0$ and $\mathcal{U}_s \subset \mathcal{U}_r$;*

3. *The class $\mathcal{R}_s$ of s-closed subobjects of products of $\mathcal{P}$-objects is a firm $\mathcal{U}_s$-reflective subcategory of $(\mathbf{M}^{\mathcal{C}}_{\xi})_0$.*

Proof. The only non-trivial implication is 2. implies 3. In view of the fact that by theorem 4.2 the class $\mathcal{R}_s$ is already subfirmly $\mathcal{U}_s$-reflective, it is sufficient to show that $\mathcal{U}_s$ is coessential [3]. Suppose both u and $u \circ f$ belong to $\mathcal{U}_s$ then clearly f is an embedding. The hereditariness of s and the fact that $u \circ f$ is s-dense imply that f is s-dense.

□

5 Examples

Remark that if one of the equivalent claims of propositions 4.2 or 4.3 holds for the regular closure operator r of $(\mathbf{M}^{\mathcal{C}}_{\xi})_0$, then it also holds for every idempotent, (weakly) hereditary closure s on $(\mathbf{M}^{\mathcal{C}}_{\xi})_0$ with $s \leq r$. For this reason we start investigating concrete situations of categories endowed with the regular closure r.

5.1 $\mathcal{U}_r$-firmly reflective subconstructs: the case of the expanders ξ equal to $\iota^{\mathcal{C}}$, $\xi_U^{\mathcal{C}}$, $\xi_{UG}^{\mathcal{C}}$ or $\xi_D^{\mathcal{C}}$.

Let $\mathcal{C}$ be any base category. As was shown in [5] the regular closure r on $(\mathbf{M}_\xi^{\mathcal{C}})_0$, built with the expanders listed above, is idempotent and hereditary. We will show that the first claim in 4.3 (and thus also property 2. and 3.) holds.

Proposition 5.1. *For any expander listed in the subtitle 6.1.,let* $j : (X, \mathcal{H}) \longrightarrow (Y, \mathcal{D})$ *a morphism in* $(\mathbf{M}_\xi^{\mathcal{C}})_0$ *such that* $j \in \mathcal{U}_r$, *then we have*

$$\mathcal{H} = \mathcal{D} \circ j \times j \downarrow .$$

Proof. Remark that the proof of the statement for the expanders $\xi_D^{\mathcal{C}}$ and $\iota^{\mathcal{C}}$ is based on the fact that in both cases subobjects in $\mathbf{M}_\xi^{\mathcal{C}}$ coincide with subobjects in $\mathbf{M}^{\mathcal{C}}$.
We give an explicit proof for the case ξ equal to $\xi_{UG}^{\mathcal{C}}$. The remaining case where ξ equals $\xi_U^{\mathcal{C}}$ will follow from it, since $\mathbf{M}_{\xi_U^{\mathcal{C}}}^{\mathcal{C}}$ is a bireflective subconstruct of $\mathbf{M}_{\xi_{UG}^{\mathcal{C}}}^{\mathcal{C}}$. Let $j : (X, \mathcal{H}) \longrightarrow (Y, \mathcal{D})$ a morphism in $(\mathbf{M}_{\xi_{UG}^{\mathcal{C}}}^{\mathcal{C}})_0$, and suppose $j \in \mathcal{U}_r$. First apply the symmetrizer in the sense of [5] to $(X, \mathcal{H}), (Y, \mathcal{D})$ and to j. It is a coreflector in this case. Then compose it with the restriction of the uniform coreflector. Using isomorphic descriptions of the objects we denote $\mathcal{U}(\mathcal{H}^*)$ and $\mathcal{U}(\mathcal{D}^*)$ for the objects obtained and again $j : (X, \mathcal{U}(\mathcal{H}^*)) \longrightarrow (Y, \mathcal{U}(\mathcal{D}^*))$ for the image through the composed functor. j now is a dense embedding in $\mathbf{Unif}_0$.
Let $e \in \mathcal{H}$ be an arbitrary $\mathcal{C}$-metric. Then e is uniformly continuous on $X \times X$ endowed with the product of the uniformities $\mathcal{U}(\mathcal{H}^*)$. In view of the density assumption, there is a unique uniformly continuous quasimetric g on $Y \times Y$ endowed with the product structure of $\mathcal{U}(\mathcal{D}^*)$ and satisfying $g \circ j \times j = e$. An explicit formulation of g is given by

$$g : Y \times Y \longrightarrow [0, \infty] : (y, y') \longmapsto \sup_{d \in \mathcal{D}, \varepsilon > 0} e(j^{-1}(B_{d^*}(y, \varepsilon)), j^{-1}(B_{d^*}(y', \varepsilon))).$$

Since we have that $j : (X, e) \hookrightarrow (Y, g)$ is an r-dense embedding in $\mathcal{C}^{\Delta}$ the quasi-metric g is a $\mathcal{C}$-metric.

The only thing left to prove is that g belongs to $\mathcal{D}$.
Let $\varepsilon > 0$ and $\omega < \infty$ be arbitrary. Since $\mathcal{H} = \xi^{\mathcal{C}}_{UG}(\mathcal{D} \circ j \times j \downarrow)$ there exists a $\mathcal{C}$-metric $d \in \mathcal{D}$ such that $e(z,w) \wedge \omega \leq d \circ j \times j(z,w) + \frac{\varepsilon}{3}$ for every $z, w \in X$. Take $y, y' \in Y$ arbitrarily. We will show that $g(y,y') \wedge \omega \leq d(y,y') + \varepsilon$.
Let $p \in \mathcal{D}$, $\zeta > 0$ be arbitrary. Choose $x, x' \in X$ such that $(p \vee d)^*(y, j(x)) < \zeta \wedge \frac{\varepsilon}{3}$ and $(p \vee d)^*(y', j(x')) < \zeta \wedge \frac{\varepsilon}{3}$. Then we have

$$e(j^{-1}(B_{p^*}(y,\zeta)), j^{-1}(B_{p^*}(y',\zeta))) \wedge \omega \leq e(x,x') \wedge \omega \leq d(y,y') + \varepsilon.$$

□

The previous results imply that for a metrically generated construct $\mathcal{X}_0$, which is one of the examples **qUnif**$_0$,**Unif**$_0$, **Prox**$_0$, **naUnif**$_0$, **qUG**$_0$, **UG**$_0$, **efGap**$_0$, **tUG**$_0$, $\mathcal{C}_0$, or $\left(\mathbf{M}^{\mathcal{C}}_{\iota}\right)_0$, there exists a $\mathcal{U}_r$-firmly reflective subcategory $\mathcal{R}_r$ of complete objects. Moreover the complete objects are "generated" by the completely metrizable objects in the construct, meaning that an object in $\mathcal{X}_0$ is complete if and only if it is an r-closed subset of a product of objects in the image of the class of bicomplete $\mathcal{C}_0$-objects under the functor $K : \mathcal{C} \longrightarrow \mathcal{X}$.

In the table below we associate to each subconstruct $\mathcal{R}_r$ in the list of examples some known subconstruct of complete objects described in the literature.

	$\mathcal{R}_r$ is generated by bicompletely metrizable objects
qUnif$_0$	bicomplete T_0 quasi-uniform spaces
Unif$_0$	complete Hausdorff uniform spaces
Prox$_0$	Effremovic proximity spaces with compact Hausdorff underlying topology
naUnif$_0$	complete non-Archimedian uniform spaces
UG$_0$	complete T_0-Uniform Gauge spaces
efGap$_0$	Gap-spaces with compact Hausdorff underlying topology
tUG$_0$	complete transitive T_0-Uniform Gauge spaces
$\mathcal{C}_0^{\Delta}$	bicomplete T_0 quasi-metric spaces
$\mathcal{C}_0^{\Delta s}$	complete Hausdorff metric spaces
$\mathcal{C}_0^{\Delta s\theta}$	compact metric spaces
$\mathcal{C}_0^{\mu}$	complete T_0 ultrametric spaces

5.2 $\mathcal{U}_r$-firmly reflective subconstructs: the case of the expanders $\xi_T^{\mathcal{C}}$ and $\xi_A^{\mathcal{C}}$.

In case ξ equals $\xi_T^{\mathcal{C}}$ or $\xi_A^{\mathcal{C}}$, things do not work in the same way as in the previous examples.

We first deal with base categories $\mathcal{C}$ contained in $\mathcal{C}^{\Delta s}$ and we refer to table in section 2 for the isomorphic descriptions of the constructs.
It is well known that in $\mathbf{Creg}_0$ there doesn't exist a $\mathcal{U}_r$-subfirm subconstruct $\mathcal{R}_r$. It is shown in [4] that $\mathbf{Creg}_0$ does not have $\mathcal{U}_r$-injective objects, except for the singleton spaces. The argument uses the r-dense embedding $j : (\mathbb{N}, \mathcal{T}) \longrightarrow (\mathbb{N}^*, \mathcal{T}^*)$ of the discrete space of natural numbers into its Alexandroff compactification. On $(\mathbb{N}, \mathcal{T})$ a two valued continuous function, which is 0 on even numbers and 1 on odd numbers, has no continuous extension to $(\mathbb{N}^*, \mathcal{T}^*)$. Since both $(\mathbb{N}, \mathcal{T})$ and $(\mathbb{N}^*, \mathcal{T}^*)$ are zero dimensional, the same argument shows that in $\mathbf{ZDim}_0$ there cannot exist a $\mathcal{U}_r$-subfirm subconstruct either. Considering $(\mathbb{N}, \mathcal{T})$ and $(\mathbb{N}^*, \mathcal{T}^*)$ as topological approach spaces gives the same negative result for $\mathbf{UAp}_0$. Showing that these spaces are moreover zero dimensional approach spaces, yields that there is no $\mathcal{U}_r$-subfirm subconstruct in $\mathbf{ZDAp}_0$ either.

Next we deal with the base category $\mathcal{C}^{\Delta}$. The expanders ξ_T and ξ_A provide isomorphic descriptions of the constructs **Top** and **Ap** respectively. It is well known that the construct **TSob** of sober topological spaces is a $\mathcal{U}_r$-firmly reflective subconstruct of $\mathbf{Top}_0$. However **TSob** is not generated by bicompletely quasi - metrizable objects. In fact for the class

$$\mathcal{P} = \{(Z, \mathcal{T}_e) \mid e \ T_0 \text{ bicomplete quasi-metric}\}$$

we have that $\mathcal{P} \not\subseteq \mathbf{TSob}$.
In order to illustrate this, consider the quasi-metric e on $\mathbb{N}$ given by $e(n,m) = 0$ and $e(m,n) = \infty$ if $n < m$. Note that e is a T_0 quasi-metric such that e^* is discrete and therefore complete. For $\varepsilon > 0$ and $n \in \mathbb{N}$ we have $B_e(n, \varepsilon) = \{n, n+1, \ldots\}$. It now easily follows that $\mathbb{N}$ is irreducible and that it can't be written as the closure of a singleton.

An analogous situation appears in $\mathbf{Ap}_0$. In [11] it was shown that the construct **ASob** of sober approach spaces is $\mathcal{U}_r$-firm in $\mathbf{Ap}_0$. Again

$$\mathcal{P} = \{(Z, \delta_e) \mid e \ T_0 \text{ bicomplete quasi-metric}\} \not\subseteq \mathbf{ASob}$$

and by corollary 4.3 this implies that **ASob** is not generated by bicompletely quasi-metrizable objects. Indeed, consider the same bicomplete T_0 quasi-metric space $(\mathbb{N}, e)$ as in the previous argument. The fact that $(\mathbb{N}, \mathcal{T}_e)$ is not sober as a topological space, implies that $(\mathbb{N}, \delta_e)$ is not sober as an approach space.

	$\mathcal{R}_r$
$\mathbf{Creg}_0$	non existing
$\mathbf{ZDim}_0$	non existing
$\mathbf{Top}_0$	Sober topological spaces; not generated by completely metrizable obj.
$\mathbf{UAp}_0$	non existing
$\mathbf{ZDAp}_0$	non existing
$\mathbf{Ap}_0$	Sober approach spaces; not generated by completely metrizable obj.

5.3 $\mathcal{U}_s$-firmly reflective subconstructs for the closure operator determined by the metric coreflection

In this section, instead of considering the closure operator r we look for a natural closure operator that is smaller. For $(X, \mathcal{D})$ an $\left(\mathbf{M}_\xi^{\mathcal{C}}\right)_0$-object, and $x, y \in X$, put

$$\varphi(x,y) = \sup_{d \in \mathcal{D}} d(x,y).$$

Then, consider the topological closure cl^{φ^*} associated with the symmetrization φ^*. Clearly cl^{φ^*} is an idempotent closure operator which is smaller than the regular closure r.

In case $\xi = \xi_D^{\mathcal{C}}$, the closure cl^{φ^*} clearly coincides with the regular closure r, so the completion theory coincides with the one we investigated in 6.1.

If ξ equals $\xi_{UG}^{\mathcal{C}}$ or $\iota^{\mathcal{C}}$, then cl^{φ^*} is the closure of the symmetrization of the coreflection into $\mathcal{C}_0$ and cl^{φ^*} can be seen to be hereditary. Since proposition 5.1 holds for ξ_{UG} (ι) and the regular closure r, the same is true for cl^{φ^*}. It follows that the subcategory $\mathcal{R}_{cl^{\varphi^*}}$ consisting of all cl^{φ^*}-closed subobjects of products of bicompletely metrizable objects forms a $\mathcal{U}_{cl^{\varphi^*}}$-firm subconstruct of $\left(\mathbf{M}_{\xi_{UG}^{\mathcal{C}}}^{\mathcal{C}}\right)_0$ $\left(\left(\mathbf{M}_{\iota^{\mathcal{C}}}^{\mathcal{C}}\right)_0\right)$. Via the expander ξ_{UG} we get isomorphic descriptions of $\mathbf{qUG}_0$, $\mathbf{UG}_0$, $\mathbf{efGap}_0$, and $\mathbf{tUG}_0$ for which the $\mathcal{U}_{cl^{\varphi^*}}$-completion theory was not yet considered in the literature.

Note that if ξ equals $\xi_T^{\mathcal{C}}$ or $\xi_U^{\mathcal{C}}$, then cl^{φ^*} is the discrete closure and so the cl^{φ^*}-dense embeddings coincide with the isomorphisms in $(\mathbf{M}_\xi^{\mathcal{C}})_0$. So the completion theory with respect to $\mathcal{U}_{cl^{\varphi^*}}$ becomes trivial in these constructs. For example, in $\mathbf{Top}_0$, $\mathbf{Creg}_0$, $\mathbf{ZDim}_0$, $\mathbf{qUnif}_0$, $\mathbf{Unif}_0$, $\mathbf{Prox}_0$ and $\mathbf{naUnif}_0$, all objects are $\mathcal{U}_{cl^{\varphi^*}}$-complete.

If ξ equals $\xi_A^{\mathcal{C}}$ then cl^{φ^*} is the closure of the symmetrization of the coreflection into $\mathcal{C}_0$ and cl^{φ^*} is hereditary. We consider the constructs $\mathbf{UAp}_0$, $\mathbf{ZDAp}_0$ for which the completion theory with respect to the regular closure failed and $\mathbf{Ap}_0$ for which the firm $\mathcal{U}_r$-reflective subconstruct **ASob** is not generated by bicompletely metrizable objects. The subconstruct $\mathbf{cUAp}_0$ consisting of complete objects in $\mathbf{UAp}_0$, as introduced in [13], is firm with respect to $\mathcal{U}_{cl^{\varphi^*}}$, as can be deduced from the result on uniqueness of completion there. Moreover it also follows from [13] that the completely metrizable objects are $\mathcal{U}_{cl^{\varphi^*}}$-injective. So by corollary 4.3 we can conclude that the objects in $\mathbf{cUAp}_0$ are cl^{φ^*}-closed subobjects of products of complete metric approach spaces. Similar results can easily be obtained for the objects in $\mathbf{cZDAp}_0$, the construct of all complete zero dimensional approach spaces.

In [2] a bicompletion theory for $\mathbf{Ap}_0$ was developed. A subconstruct $\mathbf{bicAp}_0$ of so called bicomplete approach spaces was constructed which was shown to be $\mathcal{U}_{cl^{\varphi^*}}$-firm and the bicomplete quasi-metric spaces were shown to be $\mathcal{U}_{cl^{\varphi^*}}$-injective. Again this yields the conclusion that the objects in $\mathbf{bicAp}_0$ are generated by bicomplete quasi-metric spaces.

	$\mathcal{R}_{cl^{\varphi^*}}$ is generated by bicompletely metrizable objects
$\mathbf{UAp}_0$	$\mathbf{cUAp}_0$
$\mathbf{ZDAp}_0$	$\mathbf{cZDAp}_0$
$\mathbf{Ap}_0$	$\mathbf{bicAp}_0$

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