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EQUIVARIANT EVALUATION SUBGROUPS AND RHODES GROUPS

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Résumé

Dans cet article, nous définissons des sous-groupes équivariants d'évaluation du groupe de Rhodes supérieur, et nous étudions leurs relations avec les groupes de Gottlieb-Fox.

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Introduction

While the Gottlieb groups $G_n(X)$ of a CW-space X with a basepoint are important objects of study in homotopy theory, the first Gottlieb group $G_1(X)$ actually originated from Nielsen fixed point theory. In fact, $G_1(X)$ was first introduced by B. Jiang and it is also known as the Jiang subgroup $J(X)$.

In [15], the classical Nielsen fixed point theory was generalized to the equivariant setting under the presence of a group action. Subsequent works related to equivariant fixed point theory include [2] and [7]. To facilitate computation of certain equivariant Nielsen type numbers, an equivariant Jiang condition was introduced in [15]. This condition was

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studied and relaxed by Fagundes and Gonçalves [2] who gave an example in which all equivariant fixed point classes have the same index while the space does not satisfy the equivariant Jiang condition. In [4] equivariant Gottlieb groups, which are analogs of the equivariant Jiang subgroups in higher homotopy groups, were defined and used to compute the Gottlieb groups of orbit spaces.

This paper is organized as follows. In Section 1, we recall the definition of the higher Rhodes groups $\sigma_n(X, x_0, G)$ and prove the following.

Theorem 1.2 *Suppose a group G acts freely on X with a basepoint x_0. Then, $\sigma_n(X, x_0, G) \xrightarrow{p_*} \tau_n(X/G, p(x_0))$ is an isomorphism for all $n \geq 1$.*

This generalizes the result of Rhodes when $n = 1$. Moreover, we establish a split exact sequence for the Rhodes groups, similar to that proved by Fox for the torus homotopy groups. We then introduce new equivariant Gottlieb groups (in contrast to those introduced in [4]) as subgroups of the higher Rhodes groups and discuss some basic properties in Section 2. We prove

Proposition 2.4 *A space X is Gottlieb if and only if it is a Gottlieb-Fox space.*

Relationships among various evaluation subgroups are discussed in Section 3. In particular, n-Gottlieb and equivariant n-Gottlieb spaces are related as follows.

Theorem 3.1 *Suppose a group G acts freely on a spaces X with a basepoint x_0.*

(1) *For $n \geq 2$, if X is equivariant n-Gottlieb then X/G is n-Gottlieb.*
(2) *For $n \geq 1$, if X/G is n-Gottlieb then X is equivariant n-Gottlieb.*
(3) *Suppose X is a finite aspherical G-CW space. If X is equivariant 1-Gottlieb then X/G is 1-Gottlieb.*

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1 Higher Rhodes groups

Throughout, G will denote a finite group acting on a compactly generated Hausdorff path connected space X with a basepoint. The associated pair (X, G) is called in the literature a transformation group.

In [11], Rhodes introduced the notion of the fundamental group $\sigma(X, x_0, G)$ of the pair (X, G), where x_0 is a basepoint in X. A typical element in $\sigma(X, x_0, G)$ is the homotopy class $[\alpha; g]$ consisting of a path α in X and a group element $g \in G$ such that $\alpha(0) = x_0, \alpha(1) = gx_0$. The multiplication in $\sigma(X, x_0, G)$ is given by

$$[\alpha_1; g_1] * [\alpha_2; g_2] := [\alpha_1 + g_1\alpha_2; g_1g_2].$$

It is easy to see that the groups $\pi_1(X, x_0)$, $\sigma(X, x_0, G)$ and G fit into the following short exact sequence

$$1 \to \pi_1(X, x_0) \to \sigma(X, x_0, G) \to G \to 1.$$

Then, Rhodes [12] defined higher homotopy groups $\sigma_n(X, x_0, G)$ of the pair (X, G) for $n \geq 1$ which is an extension of $\tau_n(X, x_0)$ by G so that

$$1 \to \tau_n(X, x_0) \to \sigma_n(X, x_0, G) \to G \to 1 \tag{1}$$

is exact. Here, $\tau_n(X, x_0)$ denotes the n-th torus homotopy group of X introduced by R. Fox [3]. The group $\tau_n = \tau_n(X, x_0)$ is defined to be the fundamental group of the function space $X^{T^{n-1}}$ and is uniquely determined by the groups $\tau_1, \tau_2, \ldots, \tau_{n-1}$ and the Whitehead products, where T^{n-1} is the $(n-1)$-dimensional torus. The group τ_n is non-abelian in general.

Definition 1.1 *Suppose that X is a G-space with a basepoint $x_0 \in X$. Let $C_n = I \times T^{n-1}$. We say that a map $f : C_n \to X$ is of* order *$g \in G$ provided $f(0, t_2, \ldots, t_n) = x_0$ and $f(1, t_2, \ldots, t_n) = g(x_0)$ for $(t_2, \ldots, t_n) \in T^{n-1}$. Denote by $[f; g]$ the homotopy class of a map $f : C_n \to X$ of order g and by $\sigma_n(X, x_0, G)$ the set of all such homotopy classes. We define an operation $*$ similar to the one on $\sigma(X, x_0, G)$ on the set $\sigma_n(X, x_0, G)$, i.e.,*

$$[f'; g'] * [f; g] := [f' + g'f; g'g].$$

This operation makes $\sigma_n(X, x_0, G)$ a group. We write $\sigma_1(X, x_0, G)$ for $\sigma(X, x_0, G)$. Now, the orbit map $p : X \to X/G$ leads to a homomorphism $p_* : \sigma_n(X, x_0, G) \to \tau_n(X/G, p(x_0))$, where $p_*([f; g])$ is represented by the adjoint of the composite $I \to (X/G)^{T^{n-1}}$ of $C_n = I \times T^{n-1} \xrightarrow{f} X \xrightarrow{p} X/G$ for $[f; g] \in \sigma_n(X, x_0, G)$.

It was shown in [11] that $p_* : \sigma_1(X, x_0, G) \to \tau_1(X/G, p(x_0))$ is an isomorphism for a free G-action on X. More generally, we have the following

Theorem 1.2 *Suppose G acts freely on X with a basepoint x_0. Then,*

$$\sigma_n(X, x_0, G) \xrightarrow{p_*} \tau_n(X/G, p(x_0))$$

is an isomorphism for all $n \geq 1$.

Proof: Since the G-action on X/G is trivial, it follows that

$$\sigma_n(X/G, p(x_0), G) \cong \tau_n(X/G, p(x_0)).$$

By induction, we assume that

$$\sigma_k(X, x_0, G) \cong \tau_k(X/G, p(x_0))$$

for $k < n$. We define the projection $\sigma_n \to \sigma_{n-1}$ by $[f; g] \mapsto [\hat{f}; g]$, where $\hat{f} : C_{n-1} \to X$ is given by $f \circ i_n$ with $i_n : C_{n-1} \hookrightarrow C_n$ defined by $i_n(t_1, t_2, \ldots, t_{n-1}) = (t_1, t_2, \ldots, t_{n-1}, 0)$ for $(t_1, \ldots, t_{n-1}) \in C_{n-1}$. The projection $\sigma_n \to \sigma_{n-1}$ has a section $\sigma_{n-1} \to \sigma_n$ via the projection $C_n \to C_{n-1}$ defined by $(t_1, t_2, \ldots, t_n) \mapsto (t_1, t_2, \ldots, t_{n-1})$ for $(t_1, \ldots, t_n) \in C_n$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \tau_n(X, x_0) & \longrightarrow & \sigma_n(X, x_0, G) & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \tau_{n-1}(X, x_0) & \longrightarrow & \sigma_{n-1}(X, x_0, G) & \longrightarrow & G & \longrightarrow & 1, \end{array} \tag{2}$$

where the first two vertical homomorphisms have sections. Since this diagram is commutative for all k, it follows that the canonical projections $\sigma_n(X, x_0, G) \to \sigma_{n-1}(X, x_0, G)$ and $\tau_n(X, x_0) \to \tau_{n-1}(X, x_0)$ have

isomorphic kernels, which we call K. But, the split exact sequence of Fox [3] takes the following form (see [5])

$$0 \to \tau_{n-1}(\Omega X, \overline{x_0}) \to \tau_n(X, x_0) \overset{\leftarrow}{\to} \tau_{n-1}(X, x_0) \to 1. \tag{3}$$

Thus,

$$K \cong \tau_{n-1}(\Omega X, \overline{x_0}),$$

where $\overline{x_0}$ denotes the constant loop at x_0. Now, consider the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & \sigma_n(X, x_0, G) & \to & \sigma_{n-1}(X, x_0, G) & \to & 1 \\ & & \downarrow p_* & & \downarrow p_* & & \downarrow \cong & & \\ 1 & \to & \tau_{n-1}(\Omega(X/G), \overline{p(x_0)}) & \to & \tau_n(X/G, p(x_0)) & \to & \tau_{n-1}(X/G, p(x_0)) & \to & 1. \end{array}$$

Note that $\tau_*(\Omega Z)$ is a direct product of higher homotopy groups of Z. Since G acts freely on X, the orbit map $p : X \to X/G$ is a finite cover. Hence, the map $p_* : K \cong \tau_{n-1}(\Omega X, x_0) \to \tau_{n-1}(\Omega(X/G), p(x_0))$ is an isomorphism. Thus, by the Five Lemma and inductive hypothesis, the homomorphism $p_* : \sigma_n(X, x_0, G) \to \tau_n(X/G, p(x_0))$ is an isomorphism. □

Since, in (3), the kernel of $\tau_n \to \tau_{n-1}$ is $\tau_{n-1}(\Omega X, \overline{x_0})$, we have the following result generalizing (3).

Theorem 1.3 *The following sequence*

$$0 \to \tau_{n-1}(\Omega X, \overline{x_0}) \to \sigma_n(X, x_0, G) \overset{\leftarrow}{\to} \sigma_{n-1}(X, x_0, G) \to 1 \tag{4}$$

is split exact.

Remark 1.4 Let $s_{n-1} : \sigma_{n-1} \to \sigma_n$ be the section. Consider the injective homomorphism $s_{n-1} \circ \cdots \circ s_2 \circ s_1 : \sigma_1 \to \sigma_n$ and the action $\Theta_n : \sigma_n \to \mathrm{Aut}(\sigma_n)$ given by conjugation. The action of σ_1 on σ_n is given by

$$\Theta_n \circ s_{n-1} \circ \cdots \circ s_1 : \sigma_1 \to \mathrm{Aut}(\sigma_n).$$

Likewise, for the Fox torus homotopy groups, there is an action of $\pi_1 = \tau_1$ on τ_n. In fact, the action of τ_1 on τ_n is simply the restriction of that of σ_1 on σ_n. Furthermore, since $\tau_{n-1}(\Omega X, \overline{x_0}) \cong \prod^n \pi_i(X, x_0)^{\alpha_i}$, one can show that $\pi_n(X, x_0)$ is a (embedded) normal subgroup of σ_n. By embedding σ_1 in σ_n, one obtains an action of $\sigma_1(X, x_0, G)$ on $\pi_n(X, x_0)$.

2 Evaluation subgroups of Rhodes groups

Given a space X, we defined in [5] the Gottlieb-Fox groups to be the evaluation subgroups $G\tau_n := G\tau_n(X, x_0) := \mathrm{Im}(ev_* : \tau_n(X^X, 1_X) \to \tau_n(X, x_0))$ of the torus homotopy groups τ_n for $n \geq 1$, where $X^X := \mathrm{Map}(X, X)$ is the map space.

In this section, we define and study the analogous evaluation subgroups of the Rhodes groups σ_n for $n \geq 1$.

Definition 2.1 *Given a G-space X, consider the pointwise action of G on the space X^X, i.e., $(gf)(x) := gf(x)$ for $g \in G$, $f \in X^X$ and $x \in X$.*

The evaluation subgroup

$$G\sigma_n := G\sigma_n(X, x_0, G) := \mathrm{Im}(ev_* : \sigma_n(X^X, 1_X, G) \to \sigma_n(X, x_0, G))$$

of σ_n is called the n-th Gottlieb-Rhodes group *of a G-space X.*

To relate the Gottlieb-Rhodes groups with the Gottlieb-Fox groups, we consider the homomorphism $p_n : G\sigma_n \to G$ given by $[f; g] \mapsto g$ for $[f; g] \in G\sigma_n$.

Theorem 2.2 *The following sequence*

$$1 \to G\tau_n \to G\sigma_n \xrightarrow{p_n} G_0 \to 1 \tag{5}$$

is exact. Here, G_0 is the subgroup of G consisting of elements g considered as homeomorphisms of X which are freely homotopic to 1_X.

Proof: Since $[f; g] \in G\sigma_n$, we get $[f; g] = ev_*([F; g])$ for some $[F; g] \in \sigma_n(X^X, 1_X, G)$. This means that $g_* \sim 1_X$ (or equivalently $g \in G_0$) via the homotopy determined by the restriction map $F| : I \times \{1, \ldots, 1\} \to X^X$. Thus, we get $p_n(G\sigma_n) = G_0$. By the naturality of the evaluation map and the fact that $\mathrm{Ker}(p_n \subseteq \tau_n$, it follows that $\mathrm{Ker}\, p_n = G\tau_n$. □

The group $G\sigma_n$ was already defined by M. Woo and Y. Yoon [16] who asked whether $G\sigma_n$ is abelian in general. In view of (5), if G_0 is non-abelain then $G\sigma_n$ is non-abelian. In fact, even when G_0 is abelian, $G\sigma_n$ need not be as we illustrate in the following example.

Example 2.3 Let $\mathbb{R}P^3 = \mathbb{S}^3/\{\pm I\}$ be the 3-dimensional real projective space and $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Consider the free action of the quaternionic group $Q_8 \subset \mathbb{S}^3 = Sp(1)$ on the sphere $\mathbb{S}^3$. Then, consider $G = Q_8/\{\pm 1\} \subset Sp(1)/\{\pm 1\} = \mathbb{S}^3/\{\pm I\}$ so that G acts on $\mathbb{R}P^3$ as a quotient of Q_8. Then, $G\sigma_1 = Q_8$, $G\tau_1 = \mathbb{Z}_2$ and $G_0 = G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Using a similar construction with $G = A_5$ acting on $\mathbb{R}P^3$ as a quotient of the icosahedral group of order 120, one obtains $G_0 = A_5$, which is non-abelian.

We have three evaluation subgroups, the classical Gottlieb groups $G_n(X) = G_n(X, x_0)$, the Gottlieb-Fox groups $G\tau_n(X) = G\tau_n(X, x_0)$ and the Gottlieb-Rhodes groups $G\sigma_n(X, G) = G\sigma_n(X, x_0, G)$, where (X, G) is a transformation group. We shall compare these different notions. Recall that a space X is a *Gottlieb space* if $G_n(X) = \pi_n(X) = \pi_n(X, x_0)$ for all $n \geq 1$. Similarly, we say that X is *Gottlieb-Fox* or *Gottlieb-Rhodes* for a G-space X if $G\tau_n(X) = \tau_n(X)$ and $G\sigma_n(X, G) = \sigma_n(X, G)$, respectively for all $n \geq 1$. Certainly, any H-space is Gottlieb-Fox.

Remark 2.4 It is straightforward to see, using Theorem 2.2, that a G-space X is Gottlieb-Rhodes if and only if X is Gottlieb-Fox and $G_0 = G$.

Proposition 2.5 *Suppose that X is a compactly generated Hausdorff path connected space. Then, X is Gottlieb if and only if it is a Gottlieb-Fox space.*

Proof: It was shown in [5] that the Gottlieb-Fox groups are direct products of the classical Gottlieb groups. More precisely, there is an isomorphism

$$G\tau_n(X) \cong \prod_{i=1}^{n} G_i(X)^{\gamma_i},$$

where $\gamma_i = \binom{n-1}{i-1}$. From [3], we also have

$$\tau_n(X) \cong \prod_{i=2}^{n} \pi_i(X)^{\alpha_i} \rtimes \tau_{n-1}(X),$$

where $\alpha_i = \binom{n-2}{i-2}$. Suppose X is Gottlieb. Since $\tau_1 = \pi_1$ and $G\tau_1 = G_1$, we have $\tau_1 = G\tau_1$. By inductive hypothesis, we may assume that $G\tau_n(X) = \tau_n(X)$. Now,

$$\begin{aligned}\tau_{n+1}(X) &\cong \prod_{i=2}^{n+1} \pi_i(X)^{\beta_i} \rtimes G\tau_n(X) \\ &\cong \prod_{i=2}^{n+1} \pi_i(X)^{\beta_i} \rtimes \prod_{i=1}^{n} \pi_i(X)^{\gamma_i},\end{aligned}$$

where $\beta_i = \binom{n-1}{i-2}$. Since the action giving the semi-direct products is given by the Whitehead product and X is Gottlieb, that action is trivial and thus

$$\tau_{n+1}(X) \cong \prod_{i=2}^{n+1} \pi_i(X)^{\beta_i} \times \prod_{i=1}^{n} \pi_i(X)^{\gamma_i}.$$

For $2 \leq i \leq n$,

$$\beta_i + \gamma_i = \binom{n-1}{i-2} + \binom{n-1}{i-1} = \binom{n}{i-1}.$$

It is easy to see that

$$\prod_{i=1}^{n+1} G_i(X)^{\binom{n}{i-1}} \cong G\tau_{n+1}(X) = \tau_{n+1}(X) \cong \prod_{i=1}^{n+1} \pi_i(X)^{\binom{n}{i-1}}.$$

Thus, by induction, X is a Gottlieb-Fox space.

Conversely, if X is Gottlieb-Fox, then $G\tau_1(X) = \tau_1(X)$ is equivalent to $G_1(X) = \pi_1(X)$. Suppose $G_i(X) = \pi_i(X)$ for $i \leq n$. Then, $\tau_{n+1}(X) \cong \prod_{i=1}^{n+1} \pi_i(X)^{\beta_i} \times \tau_n(X)$ is a direct product since $\tau_n(X) = G\tau_n(X) \cong \prod_{i=1}^{n} G_i(X)^{\gamma_i}$ and Whitehead products with Gottlieb elements vanish. Now,

$$\prod_{i=1}^{n} G_i(X)^{\binom{n}{i-1}} \times G_{n+1}(X) \cong G\tau_{n+1}(X) = \tau_{n+1}(X) \cong \prod_{i=1}^{n} \pi_i(X)^{\binom{n}{i-1}} \times \pi_{n+1}(X)$$

But, by inductive hypothesis, $\pi_i(X) = G_i(X)$ for $i \leq n$. Note that the equality above means that the inclusion $G\tau_{n+1}(X) \hookrightarrow \tau_{n+1}(X)$ is an

isomorphism. This inclusion is given by the product of the respective inclusions $G_{n+1}(X) \subseteq \pi_{n+1}(X)$ and $G_i(X) \subseteq \pi_i(X)$ for $i \leq n$. Since the inclusions $G_i(X) \subseteq \pi_i(X)$, for $i \leq n$, are isomorphisms, it follows that the inclusion $G_{n+1}(X) \hookrightarrow \pi_{n+1}(X)$ must be an isomorphism. Hence, X is a Gottlieb space. □

Example 2.6 Let $X = \mathbb{S}^n$ be the n-sphere and G be a finite group acting freely on X. If n is even, then $G = \mathbb{Z}_2$ in which case $G \neq G_0 = \{e\}$. If n is odd, then for any $g \in G$, the Lefschetz number $L(g_*)$ is zero since g has no fixed points. It follows that $L(g_*) = 1 - \deg g_* = 0$ and this implies that $\deg g_* = 1$. Thus, g_* is homotopic to 1_X. In this case, $G = G_0$.

Example 2.7 Let $X = T^3$ be the 3-dimensional torus and $G = \mathbb{Z}_2 = \langle t \rangle$. The action of G on X is given by $t \cdot (a, b, c) \mapsto (Aa, \bar{b}, \bar{c})$, where A is the antipodal map on $\mathbb{S}^1$ and $\bar{z}$ denotes the complex conjugate of z. The map induced by the generator t is of degree 1 but is not homotopic to the identity so that $G \neq G_0$. It is well known that H-spaces are Gottlieb spaces. This example shows that X being Gottlieb-Fox does not imply X that is Gottlieb-Rhodes without the condition $G = G_0$.

Next, we generalize a result of Gottlieb [6] on the Gottlieb subgroups in a fibration. Let G be a finite group. By a G-fibration, we mean a G-map $p : E \to B$ satisfying the G-Covering Homotopy Property for any G-spaces.

Theorem 2.8 *Let G be a finite group and $E \xrightarrow{p} B$ be a G-fibration such that $B^G \neq \emptyset$. Let $F = p^{-1}(b_0)$ for some $b_0 \in B^G$ and choose $\tilde{b}_0 \in F$. Then, there exists a homomorphism $d_* : \sigma_n(\Omega B, \overline{b_0}, G) \to \sigma_n(F, \tilde{b}_0, G)$ such that*

$$d_*(\sigma_n(\Omega B, \overline{b_0}, G)) \subseteq G\sigma_n(F, \tilde{b}_0, G).$$

Proof: Let ΩB be the space of loops based at b_0. Consider the G-action on $\Omega B \times G$ given by $g \cdot (\lambda, g') = (g\lambda, gg')$ for $(\lambda, g') \in \Omega B \times G$ and $g \in G$. Because the definition of σ_n automatically restricts to the component of the basepoint and the group G is finite, the projection $p_G : \Omega B \times G \to \Omega B$ induces an isomorphism

$$p_{G*} : \sigma_n(\Omega B \times G, (\overline{b_0}, e), G) \to \sigma_n(\Omega B, \overline{b_0}, G),$$

where $e \in G$ denotes the identity element.

Using the G-Covering Homotopy Property, there exists a G-lifting function $\mu_G : \Omega B \times G \to F^F$. Define a map d_* to be the composite

$$\sigma_n(\Omega B, \overline{b_0}, G) \xrightarrow{p_{G*}^{-1}} \sigma_n(\Omega B \times G, (\overline{b_0}, e), G) \xrightarrow{\mu_{G*}} \sigma_n(F^F, 1_F, G) \xrightarrow{ev_*} \sigma_n(F, \tilde{b_0}, G$$

Then, the assertion follows from the definitions of d_* and of $G\sigma_n$. □

Remark 2.9 When $G = \{1\}$, the map d_* becomes the boundary homomorphism induced by the action of ΩB on F so that Theorem 2.8 becomes the first part of [5, Theorem 2.2].

3 Equivariant Gottlieb groups and orbit spaces

In [4], equivariant Gottlieb groups $\{\mathbf{G}_n(X^H, x_0)\}$ or simply $\{\mathbf{G}_n(X^H)\}$ were defined for every subgroup $H \leq G$ of a G-space X, where X^H is the subspace of X given by H-invariant points. These groups are subgroups of the classical homotopy groups $\pi_n(X^H)$. When $n = 1$, $\{\mathbf{G}_1(X^H)\}$ are the same as the G-Jiang subgroups as defined in [15]. In this section, we investigate relationships among the various evaluation groups.

For any positive integer n, a space X is *n-Gottlieb* if $G_n(X) = \pi_n(X)$. Similarly, X is *n-Gottlieb-Fox* if $G\tau_n(X) = \tau_n(X)$. Recall from [4] that for any $n \geq 1$ and for any subgroup $H \leq G$,

$$\mathbf{G}_n(X^H) = \mathrm{Im}(ev_* : \pi_n(\mathrm{Map}_{WH}(X^H, X^H), 1_{X^H}) \to \pi_n(X^H, x_0)),$$

where $\mathrm{Map}_{WH}(X^H, X^H)$ is the space of $WH = NH/H$-maps on X^H, where NH is the normalizer of H in the group G and WH is the Weyl group. For any positive integer $n \geq 1$, a G-space X is said to be *n-Gottlieb-Rhodes* if $G\sigma_n(X, x_0, G) = \sigma_n(X, x_0, G)$. Similarly, we say that X is *equivariant n-Gottlieb* if $\mathbf{G}_n(X^H) = \pi_n(X^H)$ for every subgroup H. While one can define analogously the notion of *equivariant n-Gottlieb-Rhodes*, i.e., $G\sigma_n(X^H, x_0, WH) = \sigma_n(X^H, x_0, WH)$ for every subgroup H, for the rest of the paper we will only study the various evaluation subgroups when the G-action is free.

Theorem 3.1 *Suppose a group G acts freely on a space X with a basepoint x_0.*
(1) *For $n \geq 2$, if X is equivariant n-Gottlieb then X/G is n-Gottlieb.*
(2) *For $n \geq 1$, if X/G is n-Gottlieb then X is equivariant n-Gottlieb.*
(3) *Suppose X is a finite aspherical G-CW space. If X is equivariant 1-Gottlieb then X/G is 1-Gottlieb.*

Proof: Certainly, given a free G-action on X, the subspaces X^H are empty for any non-trivial subgroup H of G and $X^{\{e\}} = X$.

(1) For the obvious map $q : \mathrm{Map}_G(X,X) \to \mathrm{Map}(X/G, X/G)$, consider the following commutative diagram

$$\begin{array}{ccc} \pi_n(\mathrm{Map}_G(X,X), 1_X) & \xrightarrow{ev_*} & \pi_n(X, x_0) \\ \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle p_*} \\ \pi_n(\mathrm{Map}(X/G, X/G), 1_{X/G}) & \xrightarrow{ev_*} & \pi_n(X/G, p(x_0)) \end{array}$$

for any $n \geq 2$. If X is equivariant n-Gottlieb then the evaluation map $ev_* : \pi_n(\mathrm{Map}_G(X,X), 1_X) \to \pi_n(X, x_0)$ must be onto. Furthermore, since $p_* : \pi_n(X, x_0) \to \pi_n(X/G, p(x_0))$ is an isomorphism, the commutativity of the diagram implies that $ev_* : \pi_n(\mathrm{Map}(X/G, X/G), 1_{X/G}) \to \pi_n(X/G, p(x_0))$ is also surjective, i.e., $G_n(X/G; p(x_0)) = \pi_n(X/G, p(x_0))$

(2) The case when $n = 1$ was already proven in [15, Proposition 4.9]. For $n \geq 2$, it was proven in [4, Proposition 3.3] that $p_*(\mathbf{G}_n(X)) \cong G_n(X/G)$. Since $p_* : \pi_n(X) \to \pi_n(X/G)$ is an isomorphism for $n \geq 2$ and $G_n(X/G) = \pi_n(X/G)$ by assumption, it follows that $\mathbf{G}_n(X) = \pi_n(X)$, i.e., X is equivariant n-Gottlieb.

(3) Since X is a finite aspherical CW space, so is X/G. The fundamental groups $\pi_1(X)$ and $\pi_1(X/G)$ are finitely generated and torsion-free. Since X is equivariant 1-Gottlieb, $\mathbf{G}_1(X) = \pi_1(X)$ and consequently $\pi_1(X) \cong \mathbb{Z}^d$ for some integer $d > 0$. By [4, Proposition 3.3], $p_*(\mathbf{G}_1(X)) \subseteq G_1(X/G)$. Moreover, a result of Gottlieb asserts that $G_1(X/G)$ is the center of $\pi_1(X/G)$. It then follows that $p_*(\pi_1(X)) = p_*(\mathbf{G}_1(X)) \subseteq G_1(X/G)$ is central in $\pi_1(X/G)$. In other words, the exact sequence

$$0 \to \mathbb{Z}^d \cong \pi_1(X) \to \pi_1(X/G) \to G \to 1$$

determined by the orbit map $p : X \to X/G$ is a central extension. By [14], the group $\pi_1(X/G) \cong \mathbb{Z}^d$ is free abelian of the same rank as $\pi_1(X)$. Then, since X/G is aspherical, $G_1(X/G) = \text{Center}(\pi_1(X/G) = \pi_1(X/G))$ and the space X/G is 1-Gottlieb. □

For $n = 1$, (1) of Theorem 3.1 does not hold as we show in the following examples.

Example 3.2 Let $X = \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$, $G = \mathbb{Z}_2 = \langle t \rangle$, where the free action of G on X is given by $t \cdot (x, y, z) = (-x, \overline{y}, \overline{z})$. Here, $\overline{w}$ denotes the conjugate of w in $\mathbb{S}^3$ which is regarded as the unit quaternions. Note that $\mathbf{G}_1(X) = \{1\}$ since $\pi_1(X) = \{1\}$, and $G_1(X/G) = \{1\}$ while $\pi_1(X/G) = \mathbb{Z}_2$. To see this, observe that the elements of $G_1(X/G)$ act trivially on $\pi_3(X/G) \cong \pi_3(X)$ but the automorphism $t_\sharp \in \text{Aut}(\pi_3(X))$ for $t_\sharp(1 \oplus 1 \oplus 1) = (1 \oplus (-1) \oplus (-1))$ is non-trivial.

Example 3.3 Let $X = \mathbb{S}^3$ and G be a finite subgroup of $\mathbb{S}^3$. The free action of G on X is multiplication in $\mathbb{S}^3$ so that X/G is the coset space $\mathbb{S}^3/G$. By [9], $G_n(\mathbb{S}^3/G) \cong G_n(\mathbb{S}^3)$ for all $n \geq 1$. Since $\mathbb{S}^3$ is a topological group, it is n-Gottlieb for all $n \geq 1$, i.e., $\mathbb{S}^3$ is a Gottlieb space. Using [4, Proposition 3.3], we have $\mathbf{G}_n(\mathbb{S}^3) \xrightarrow[\cong]{p_*} G_n(\mathbb{S}^3/G)$ for $n \geq 2$ so that $\mathbf{G}_n(\mathbb{S}^3) \cong G_n(\mathbb{S}^3) \cong \pi_n(\mathbb{S}^3)$. For $n = 1$, $\mathbf{G}_1(\mathbb{S}^3) = \{1\} = \pi_1(\mathbb{S}^3)$. Therefore, $\mathbb{S}^3$ is equivariant n-Gottlieb for all $n \geq 1$. However, if G is not abelian, then $\mathbb{S}^3/G$ is not 1-Gottlieb.

Remark 3.4 J. Oprea [10] proved that if G is a finite group acting freely on $\mathbb{S}^{2n+1}$ then $G_1(\mathbb{S}^{2n+1}/G)$ is the center of G. In particular, when $n = 1$, Oprea's result asserts that $\mathbb{S}^3/G$ is Gottlieb if and only if G is abelian. This result cannot be generalized to arbitrary simply connected Gottlieb spaces. In fact, Example 3.2 serves as a counter-example in which X is a simply-connected Gottlieb space (since it is a topological group) admitting a free $G = \mathbb{Z}_2$ action but X/G is not a Gottlieb space.

J. Siegel [13] gave the first example of a finite dimensional Gottlieb space that is not an H-space. Following our discussion in this section, one can construct similar but even simpler examples as follows.

Example 3.5 Let $X = \mathbb{S}^3/\mathbb{Z}_4$ be the coset space of $\mathbb{S}^3$ by the finite subgroup $\mathbb{Z}_4$. Then, X is a Gottlieb space but not an H-space. The fact that X is a Gottlieb space follows from the same argument used in Example 3.3 except now $G = \mathbb{Z}_4$ is abelian so that X is indeed 1-Gottlieb. The fact that X does not admit an H-structure follows from [1, Theorem 4, p. 87].

We end this section by studying the Gottlieb-Rhodes groups of a free G-space and the Gottlieb groups of the orbit space when the space is aspherical.

Proposition 3.6 *Suppose a finite group G acts freely on a finite aspherical G-CW space X with a basepoint. Then X/G is Gottlieb if and only if X is a Gottlieb-Rhodes space.*

Proof: By Theorem 3.1, X/G is Gottlieb if and only if X is equivariant Gottlieb. Since the G-action is free, an equivariant Gottlieb space is the same as a Gottlieb G-space. Using Remark 2.4 and Proposition 2.5, X is Gottlieb-Rhodes if and only if X is Gottlieb and $G_0 = G$. Thus, if X is Gottlieb-Rhodes then X/G is Gottlieb.

To prove the converse, it suffices to show that if X/G is Gottlieb then $G_0 = G$. Since X/G is Gottlieb the groups $\pi_1(X/G)$ and $\pi_1(X)$ are abelian. Hence, because the space X is path connected, there is a canonical induced G-action on $\pi_1(X)$. But $p : X \to X/G$ is a finite cover, so it follows that the short exact sequence $0 \to \pi_1(X) \to \pi_1(X/G) \to G \to 1$ is a central extension so that G acts trivially on $\pi_1(X)$. That is, every $g \in G$ induces the identity homomorphism on $\pi_1(X)$. Since X is aspherical, it follows that g is homotopic to the identity map 1_X. This means that $G_0 = G$. □

Remark 3.7 In general, X being equivariant Gottlieb does not imply that X is Gottlieb-Rhodes even when G acts freely on X. In fact, Example 3.2 furnishes such an example since $G_0 = \{1\} \neq G = \mathbb{Z}_2$ and X is equivariant Gottlieb.

References

[1] Borel, F., Sur les groupes fondamentaux des H-spaces, *Comment. Math. Helv.* **53** (1978), no. 1, 73–91.

[2] Fagundes, P. and Gonçalves D., Indices of equivariant maps of certain Jiang spaces, *Topol. Methods Nonlinear Anal.* **14** (1999), 151–158.

[3] Fox, R., Homotopy groups and torus homotopy groups, *Ann. of Math.* **49** (1948), 471-510.

[4] Golasiński, M. and Gonçalves, D., Equivariant Gottlieb groups, *Cahiers Topologie Géom. Différentielle Catég.* **42** (2001), 83–100.

[5] Golasiński, M., Gonçalves, D. and Wong, P., Generalizations of Fox homotopy groups, Whitehead products, and Gottlieb groups, *Ukrain. Math. Zh.* **57** (2005), no. 3, 320–328 (translated in Ukainian Mat. J. **57** (2005) no. 3, 382-393).

[6] Gottlieb, D., Evaluation subgroups of homotopy groups, *Amer. J. Math.* **91** (1969), 729-756.

[7] Guo, J., Relative and equivariant coincidence theory, Ph.D. Thesis, Memorial University, Newfoundland (December 1996).

[8] Jiang, B., Lectures on Nielsen Fixed Point Theory, Contemporary Mathematics 14, Amer. Math. Soc., Providence R.I. (1983).

[9] Lang, G., Evaluation subgroups of factor spaces, *Pacific J. Math.* **42** (1972), 701–709.

[10] Oprea, J., Finite group actions on spheres and the Gottlieb group, *J. Korean Math. Soc.* **28** (1991), 65–78.

[11] Rhodes, F., On the fundamental group of a transformation group, *Proc. London Math. Soc.* **16** (1966), 635–650.

[12] Rhodes, F., Homotopy groups of transformation groups, *Canad. J. Math.* **21** (1969), 1123–1136.

[13] Siegel, J., G-spaces, H-spaces and W-spaces, *Pacific J. Math.* **31** (1969), 209–214.

[14] Vasquez, A.T., Flat Riemannian manifolds, *J. Differential Geometry* **4** (1970), 367–382.

[15] Wong, P., Equivariant Nielsen numbers, *Pacific J. Math.* **159** (1993), 153–175.

[16] Woo, M. and Yoon, Y., Certain subgroups of homotopy groups of a transformation group, *J. Korean Math. Soc.* **20** (1983), 223–233.

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