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STRUCTURAL PROPERTIES OF ENDOFUNCTORS

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ABSTRACT. Un foncteur $F : \mathbb{K} \longrightarrow \mathbb{L}$ est un DVO-foncteur s'il est naturellement équivalent á tout foncteur $G : \mathbb{K} \longrightarrow \mathbb{L}$ tel que pour tout $\mathbb{K}$-object X, FX soit isomorphe à GX. On démontre que chaque DVO-foncteur $F : \mathbb{SET} \longrightarrow \mathbb{SET}$ est finitaire (c.-à-d., préserve les colimites dirigées).

1. Introduction and Main Theorem

Inspired by [6,7], systems of functorial equations were introduced and investigated in [10]. These are systems of equations of the form

$$\mathbb{F}(\alpha) = \beta$$

where $\mathbb{F}$ is a functorial symbol and α, β are cardinal numbers. A functor $F : \mathbb{SET} \longrightarrow \mathbb{SET}$ is a solution of a system $\mathcal{S}$ if, for every equation $\mathbb{F}(\alpha) = \beta$ of $\mathcal{S}$,

$$\operatorname{card} F(\alpha) = \beta.$$

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Clearly, if F is a solution of $\mathcal{S}$, then every functor naturally equivalent to F is a solution of $\mathcal{S}$ as well.

Following [10], we say that a system $\mathcal{S}$ of functorial equations is solvable (or uniquely solvable) if it has a solution (or a solution unique up to natural equivalence).

In [10], the solvability of the systems of two functorial equations

$$\mathbb{F}(\alpha_1) = \beta_1$$
$$\mathbb{F}(\alpha_2) = \beta_2$$

is discussed in the dependence of the quadruple $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ of cardinal numbers. In 'almost all' cases, the decision whether the system is solvable or not is presented in [10]. For the cases remaining open in [10], it is impossible to give a simple YES/NO answer to the question about the solvability of the system because, as proved in [4], the answer depends on the set-theory used. In contrast to this, the following statement is absolute:

> the solution of an arbitrary uniquely solvable system of functorial equations is a finitary functor (i.e., one which preserves directed colimits).

In fact, every functor $F : \mathbb{SET} \longrightarrow \mathbb{SET}$ determines its canonical system of functorial equations, namely the system

$$\mathbb{F}(\alpha) = \operatorname{card} F(\alpha) \qquad \text{for all cardinal numbers } \alpha.$$

This canonical system extends every system of functorial equations solvable by F. If $\mathcal{S}$ is a uniquely solvable system and F is its solution, then the canonical system of F is also uniquely solvable, i.e., F satisfies the following condition:

> if $G : \mathbb{SET} \longrightarrow \mathbb{SET}$ is a functor with $\operatorname{card} GX = \operatorname{card} FX$ for all sets X, then G is naturally equivalent to F.

The functors satisfying this condition are called DVO-functors (i.e., Determined by their Values on Objects). The DVO-functors are investigated in [2,3,4]. In [4], every DVO-functor is proved to be finitary, which immediately implies that the solution of any uniquely solvable system of functorial equations is finitary. However, in [4] this result is proved only under a specific set-theoretical hypothesis. The aim of the present paper is to give an absolute (unfortunately, more involved) proof. Here we prove the following (absolute!)

Main Theorem. *Every* DVO*-functor* $\mathbb{SET} \longrightarrow \mathbb{SET}$ *is finitary.*

Its converse is false, for there are many finitary functors which are not DVO. On the other hand, there are also many finitary functors which are DVO (see [2,3,4]; the full description of all DVO-functors remains unresolved). Hence there also are many uniquely solvable systems of functorial equations: all the canonical systems of the DVO-functors and, possibly, some of their reducts (but a small system of functorial equations, i.e., one consisting only of a set of equations, is never uniquely solvable, see [10]).

Finally, let us mention that the above field of problems can be easily transformed to a more general setting: for arbitrary categories $\mathbb{K}$, $\mathbb{L}$ a functorial equation

$$\mathbb{F}(X) = Y \qquad \text{with } X \in \operatorname{obj}\mathbb{K}, \quad Y \in \operatorname{obj}\mathbb{L}$$

is solvable by any functor $F : \mathbb{K} \longrightarrow \mathbb{L}$ with FX isomorphic to Y; the concept of solvability and unique solvability of systems of functorial equations is evident. Also, every functor $F : \mathbb{K} \longrightarrow \mathbb{L}$ determines its canonical system of functorial equations; this system is uniquely solvable if and only if F is a DVO-functor (i.e., naturally equivalent to any $G : \mathbb{K} \longrightarrow \mathbb{L}$ with GX isomorphic to FX for every $X \in \operatorname{obj}\mathbb{K}$).

Problem. For which cocomplete categories $\mathbb{K}$ and $\mathbb{L}$ is every DVO-functor $\mathbb{K} \longrightarrow \mathbb{L}$ finitary?

2. The idea of the proof and the preliminaries

2.1 The present paper is completely devoted to the proof of Main Theorem. The general scheme of the proof is quite straightforward: given a functor $H : \mathbb{SET} \longrightarrow \mathbb{SET}$ which is not finitary, one has to find a functor $G : \mathbb{SET} \longrightarrow \mathbb{SET}$, not naturally equivalent to H, such that $\operatorname{card} GX = \operatorname{card} HX$ for all sets X. In fact, we shall construct two functors $G_1, G_2 : \mathbb{SET} \longrightarrow \mathbb{SET}$ which are not naturally equivalent and such that

$$\operatorname{card} HX = \operatorname{card} G_1 X = \operatorname{card} G_2 X \qquad \text{for all sets } X.$$

The reason for doing this is that the internal structure of the given functor H could be very complicated, while only a partial knowledge of it suffices to find many functors $G : \mathbb{SET} \longrightarrow \mathbb{SET}$ with $\operatorname{card} HX = \operatorname{card} GX$ for all sets X. But a direct proof that H is not naturally equivalent to such a functor G is a problem. If we construct two such functors G_1, G_2, both with a relatively simple internal structure, we are able to ensure that they are not naturally equivalent. Then at least one of them is not naturally equivalent to H.

2.2 If H is an endofunctor of a locally finitely presentable category $\mathbb{K}$, then its finitary part H^f is the left Kan extension of the restriction of H to the category of the finitely presentable objects of $\mathbb{K}$. Then H^f is really finitary (i.e., it preserves the directed colimits) and it is a subfunctor of H, i.e., there is a 'canonical' monotransformation of H^f into H (see e.g. [1]).

Clearly, $\mathbb{SET}$ is locally finitely presentable and the finitely presentable objects are just finite sets. Since this paper deals only with endofunctors of $\mathbb{SET}$, we shall use a specific description of the above notions which is more suitable for our computation of the cardinalities.

If $H : \mathbb{SET} \longrightarrow \mathbb{SET}$ is a functor, its subfunctor is any functor $G : \mathbb{SET} \longrightarrow \mathbb{SET}$ such that $GX \subseteq HX$ for all sets X and Gg is the domain-range restriction of Hg for every mapping $g : X \longrightarrow X'$ (thus $Hg(GX) \subseteq GX'$). And the finitary part H^f of H is the subfunctor of H given on a set X by the formula

$$H^f X = \bigcup \{\operatorname{Im} Hg \mid g : Y \longrightarrow X,\, Y \text{ finite}\}$$

(where $\operatorname{Im} k$ denotes the image of a mapping k in question) and $H^f g$ is just the domain-range restriction of Hg for all mappings $g : X \longrightarrow X'$. Since Hg sends the set $H^f X$ into $H^f X'$, this definition is correct.

This set-theoretical description permits us to investigate the sets $HX \setminus H^f X$ and to compute their cardinalities. In fact, the functors G_1 and G_2 mentioned in 2.1, will be constructed (in Section 6 of the present paper) so that H^f is also the finitary part of G_1 and G_2, and

$$\operatorname{card}(HX \setminus H^f X) = \operatorname{card}(G_1 X \setminus H^f X) = \operatorname{card}(G_2 X \setminus H^f X)$$

for all sets X.

2.3 We have to recall some simple properties of endofunctors of $\mathbb{SET}$.

The trivial functor $C_\emptyset$ (=the constant functor to the empty set) is finitary, hence it does not contradict to Main Theorem and we can restrict ourselves only to non-trivial functors. Any non-trivial endofunctor G of $\mathbb{SET}$ sends every non-empty set to a non-empty set and there is a natural transformation

$$\mu : \mathrm{Id} \overset{\cdot}{\longrightarrow} G$$

of the identity functor Id into G. In fact, if $\mathbf{1} = \{*\}$ is a standard one-element set, we choose $a \in G\mathbf{1}$ and for every set X we define $\mu_X : X \longrightarrow GX$ by

$$\mu_X(x) = Gv_x(a)$$

where $v_x : \mathbf{1} \longrightarrow X$ is the mapping sending $*$ to x.

The transformation μ is either a monotransformation or it factorizes as

$$\mathrm{Id} \overset{\cdot}{\longrightarrow} C_{0,1} \longrightarrow G$$

where $C_{0,1}$ is the functor sending $\emptyset$ to $\emptyset$ and all non-empty sets to $\mathbf{1}$.

Every transformation $\tau : C_{0,1} \longrightarrow G$ is called a <u>distinguished point of</u> G in [5,8] and $\tau_X(*)$ is a <u>distinguished point of</u> G <u>in</u> GX for every non-empty set X. Clearly, $Gg(\tau_X(*)) = \tau_{X'}(*)$ for every mapping $g : X \longrightarrow X'$. Hence every distinguished point $p \in GX$ of G in GX lies in $G^f X$ where G^f denotes the finitary part of G.

If A, B are subsets of a set X and $i_A : A \longrightarrow X$, $i_B : B \longrightarrow X$ denote the inclusions, then every $x \in \operatorname{Im} Gi_A \cap \operatorname{Im} Gi_B$ is

> a distinguished point of G in GX whenever $A \cap B = \emptyset$ or an element of $\operatorname{Im} Gi_{A\cap B}$, where $i_{A\cap B} : A \cap B \longrightarrow X$ is the inclusion, whenever $A \cap B \neq \emptyset$ (see [8]).

Hence if $x \in GX$ is not a distinguished point of G in GX (e.g. if $x \in GX \setminus G^f X$), then the system

$$\mathfrak{F}_X^G(x) = \{Z \subseteq X \mid x \in \operatorname{Im} Gi_Z,\ i_Z : Z \to X \text{ is the inclusion}\}$$

is a filter on the set X, see [5,8].

2.4 Given a functor $H : \mathbb{SET} \to \mathbb{SET}$ which is not finitary, the filters just described provide a tool to derive a formula for $\operatorname{card}(HX \setminus H^f X)$ in 3.5. The functors G_1, G_2 mentioned in 2.1-2.2 are constructed in Section 6, and elementary expansions discussed in Section 5 are the building blocks of this construction. Transformation monoids investigated in Section 4 serve to prove that the constructed G_1 and G_2 are not naturally equivalent. Observe that, for any functor $K : \mathbb{SET} \to \mathbb{SET}$, any set X and any $x \in KX$, the system

$$\mathfrak{M}_X^K(x) = \{g : X \to X \mid Kg(x) = x\}$$

is a transformation monoid and, if $\nu : K \to K'$ is a natural equivalence then the transformation monoids $\mathfrak{M}_X^K(x)$ and

$$\mathfrak{M}_X^{K'}(\nu_X(x)) = \{h : X \to X \mid K'h(\nu_X(x)) = \nu_X(x)\}$$

are strongly isomorphic (for details, see Section 4). Transformation monoids which are not strongly isomorphic are inserted at the appropriate places in the construction of G_1 and G_2, and this ensures that G_1 and G_2 are not naturally equivalent (for details see Section 6). This will finish our proof.

3. Abstract filters

3.1 Definition. Let $\mathcal{F}$ be a filter on a set X and $\mathcal{G}$ be a filter on a set Y. We say that they are equivalent if there exist $F \in \mathcal{F}$, $G \in \mathcal{G}$ and a bijection b of F onto G such that, for every $F' \subseteq F$,

$$F' \in \mathcal{F} \quad \text{if and only if} \quad b(F') \in \mathcal{G}.$$

Any class $\mathcal{A}$ of all mutually equivalent filters is called an abstract filter. If a filter $\mathcal{F}$ (on a set X) is an element of an abstract filter $\mathcal{A}$, we say that $\mathcal{F}$ is a location (on the set X) of the abstract filter $\mathcal{A}$. Let us denote $\mathcal{A}(X)$ the set of all locations of $\mathcal{A}$ on X.

Remark. By the above equivalence, the class of all filters (on all sets) is decomposed into classes of mutually equivalent filters. Let $|\mathcal{F}|$ denote $\min\{\operatorname{card} F \mid F \in \mathcal{F}\}$. If $\mathcal{F}$ and $\mathcal{G}$ are locations of an abstract filter $\mathcal{A}$, then, clearly, $|\mathcal{F}| = |\mathcal{G}|$ and $\operatorname{card} \bigcap \mathcal{F} = \operatorname{card} \bigcap \mathcal{G}$. Let us denote $|\mathcal{A}| = |\mathcal{F}|$ and $|\bigcap \mathcal{A}| = \operatorname{card} \bigcap \mathcal{F}$ for a location $\mathcal{F}$ of $\mathcal{A}$ (on a set X).

Observation. If $\mathcal{F}$ is a location of $\mathcal{A}$ on a set X and if $f : X \longrightarrow Y$ is a map injective on some $F \in \mathcal{F}$ then the filter $\mathcal{G}$ with a basis $\{f(F) \mid F \in \mathcal{F}\}$ is a location of $\mathcal{A}$ on Y. We shall write $\mathcal{G} = f(\mathcal{F})$.

3.2 Abstract filters and their locations are useful tool for the examination of functors $\mathbb{SET} \longrightarrow \mathbb{SET}$ and the following lemma will be often used.

Lemma. *For every abstract filter* $\mathcal{A}$ *and every set* Y, $\mathcal{A}(Y) = \emptyset$ *if* $\operatorname{card} Y < |\mathcal{A}|$ *and* $\operatorname{card} \mathcal{A}(Y) \geq \operatorname{card} Y$ *if* $\operatorname{card} Y \geq \max\{|\mathcal{A}|, \aleph_0\}$.

Proof. If $F \in \mathcal{F}$ for a location $\mathcal{F}$ of $\mathcal{A}$ then $\operatorname{card} F \geq |\mathcal{A}|$. Hence if $\mathcal{F}$ is a location of $\mathcal{A}$ on a set Y then $\operatorname{card} Y \geq |\mathcal{A}|$. Thus $\mathcal{A}(Y) = \emptyset$ for all sets Y with $\operatorname{card} Y < |\mathcal{A}|$. If $\operatorname{card} Y \geq \max\{|\mathcal{A}|, \aleph_0\}$ then $\operatorname{card}(Y \times Y) = \operatorname{card} Y$ and since on every fibre $Y \times \{y\}$ there is a location of $\mathcal{A}$, it follows $\operatorname{card} \mathcal{A}(Y) \geq \operatorname{card} Y$. $\square$

3.3 For a functor H and $x \in HX$ let us recall (see 2.3) the family

$$\mathfrak{F}_X^H(x) = \{Y \subseteq X \mid x \in \operatorname{Im} Hi \text{ for the inclusion } i : Y \longrightarrow X\}.$$

If $x \in HX$ is non-distinguished then $\mathfrak{F}_X^H(x)$ is a filter on X.

Clearly, if $x \in HX \setminus H^f X$ then $|\mathfrak{F}_X^H(x)|$ is infinite.

Notation. For an arbitrary functor $H : \mathbb{SET} \longrightarrow \mathbb{SET}$ and for a filter $\mathcal{F}$ on a set X, let us denote

$$p(H, \mathcal{F}) = \{x \in HX \mid x \text{ is non-distinguished, } \mathfrak{F}_X^H(x) = \mathcal{F}\}.$$

3.4 Lemma. *Let $H : \mathbb{SET} \longrightarrow \mathbb{SET}$ be a functor and let $\mathcal{F}$ and $\mathcal{G}$ be locations of an abstract filter $\mathcal{A}$ on X and Y, respectively. Then there exists a mapping $f : X \longrightarrow Y$ such that Hf maps bijectively $p(H, \mathcal{F})$ onto $p(H, \mathcal{G})$. If both $\mathcal{F}$ and $\mathcal{G}$ are locations of $\mathcal{A}$ on a set X and if $\mathcal{F} \neq \mathcal{G}$ then $p(H, \mathcal{F}) \cap p(H, \mathcal{G}) = \emptyset$.*

Proof. If both $\mathcal{F}$ and $\mathcal{G}$ are locations of $\mathcal{A}$, $\mathcal{F}$ on X and $\mathcal{G}$ on Y, then there exists a bijection b of some $F \in \mathcal{F}$ onto some $G \in \mathcal{G}$ such that for $F' \subseteq F$, $F' \in \mathcal{F}$ if and only if $b(F') \in \mathcal{G}$. If $f : X \longrightarrow Y$ is an arbitrary extension of b then $\mathcal{G} = f(\mathcal{F})$ (see 3.1 Observation) and hence $Hf(x) \in p(H, \mathcal{G})$ for all $x \in p(H, \mathcal{F})$, see also [5,9]. Hence Hf maps $p(H, \mathcal{F})$ bijectively onto $p(H, \mathcal{G})$. If $X = Y$ and $x \in p(H, \mathcal{F}) \cap p(H, \mathcal{G})$ then

$$\mathcal{F} = \mathfrak{F}_X^H(x) = \mathcal{G}. \quad \square$$

3.5 Convention. In what follows, the symbol

$$\mathbb{A}$$

denotes the system of all abstract filters $\mathcal{A}$ with $|\mathcal{A}| \geq \aleph_0$.

By 3.4, we get $\operatorname{card} p(H, \mathcal{F}) = \operatorname{card} p(H, \mathcal{G})$ whenever both $\mathcal{F}$ and $\mathcal{G}$ are locations of an abstract filter $\mathcal{A}$; let us denote this cardinal number $p(H, \mathcal{A})$. Then for every $X \neq \emptyset$

$$\operatorname{card}(HX \setminus H^{(f)}X) = \sum_{\mathcal{A} \in \mathbb{A}} p(H, \mathcal{A}) \operatorname{card} \mathcal{A}(X).$$

4. Transformation monoids

4.1 Let us recall that a transformation monoid M on a set X is a set of mappings $f : X \longrightarrow X$ closed with respect to the composition of mappings and containing the identity mapping. We abbreviate the words 'transformation monoid' to 't-monoid'.

If M is a t-monoid on a set X and M' is a t-monoid on a set Y then we say that they are strongly isomorphic if there exists a bijection $b : X \longrightarrow Y$ such that

$$f \mapsto b \circ f \circ b^{-1}$$

is a monoid isomorphism of M onto M'.

4.2 For every functor $G : \mathbb{SET} \longrightarrow \mathbb{SET}$, every $x \in GX$ determines a t-monoid $\mathfrak{M}_X^G(x)$ on X, namely

$$\mathfrak{M}_X^G(x) = \{f : X \longrightarrow X \mid Gf(x) = x\}.$$

If μ is a natural equivalence of G onto a functor G' then, clearly, for every set X and every $x \in GX$,

$$\mathfrak{M}_X^G(x) \quad \text{is strongly isomorphic to} \quad \mathfrak{M}_X^{G'}(\mu_X(x)).$$

The t-monoids form a more subtle tool for examining set functors than filters (e.g. if $x, y \in GX$ and $\mathfrak{F}_X^G(x) = \mathfrak{F}_X^G(y)$, then not necessarily $\mathfrak{M}_X^G(x) = \mathfrak{M}_X^G(y)$), and we shall use them in our construction.

4.3 For a filter $\mathcal{F}$ on a set X, let $\mathfrak{M}(\mathcal{F})$ denote the t-monoid consisting of $f : X \longrightarrow X$ which are injective on a set from $\mathcal{F}$ and $\{f(F) \mid F \in \mathcal{F}\}$ form a basis of $\mathcal{F}$.

One can verify easily that

(1) $\mathfrak{M}(\mathcal{F})$ is really a t-monoid on X;
(2) if $g \in \mathfrak{M}(\mathcal{F})$ is injective on a set $F \in \mathcal{F}$ and $f : X \longrightarrow X$ is a mapping inverse to g on $g(F)$ then $f \in \mathfrak{M}(\mathcal{F})$;
(3) an idempotent mapping $g : X \longrightarrow X$ is in $\mathfrak{M}(\mathcal{F})$ if and only if $\operatorname{Im} g \in \mathcal{F}$;
(4) $\mathcal{F} = \{\operatorname{Im}(f) \mid f \in \mathfrak{M}(\mathcal{F})\}$.

4.4 Now, let us suppose that $\operatorname{card}\bigcap\mathcal{F} \geq 3$. Let us choose distinct $u, v \in \bigcap\mathcal{F}$ and denote

$$\mathfrak{M}(\mathcal{F}, u) = \{f \in \mathfrak{M}(\mathcal{F}) \mid f(u) = u\} \qquad \text{and}$$
$$\mathfrak{M}(\mathcal{F}, u, v) = \{f \in \mathfrak{M}(\mathcal{F}) \mid f(u) = u,\ f(v) = v\}.$$

Proposition. *$\mathfrak{M}(\mathcal{F}, u)$ is not strongly isomorphic to $\mathfrak{M}(\mathcal{F}, u, v)$.*

Proof. We prove that $\{x \in X \mid \forall f \in \mathfrak{M}(\mathcal{F}, u),\ f(x) = x\} = \{u\}$. Since $f(u) = u$ and $f(v) = v$ for all $f \in \mathfrak{M}(\mathcal{F}, u, v)$ the proof will be complete. Consider $x \in X \setminus \bigcap\mathcal{F}$, then $X \setminus \{x\} \in \mathcal{F}$ and therefore every mapping $f : X \longrightarrow X$ such that $f(y) = y$ for all $y \in X \setminus \{x\}$ and $f(x) \neq x$ belongs to $\mathfrak{M}(\mathcal{F}, u)$ (and also to $\mathfrak{M}(\mathcal{F}, u, v)$). A mapping f which is an arbitrary permutation of $\bigcap\mathcal{F}$ and $f(y) = y$ for all $y \in X \setminus \bigcap\mathcal{F}$ belongs to $\mathfrak{M}(\mathcal{F})$. Since $\operatorname{card}\bigcap\mathcal{F} \geq 3$ a suitable choice of a permutation guarantees the required statement. □

Remark. This proposition will be used in the proof of Main Theorem to show that the functors G_1 and G_2, which we shall construct in 6., are not naturally equivalent.

4.5 In the rest of the paragraph we assume that a filter $\mathcal{F}$ on a set X with $\bigcap\mathcal{F} \neq \emptyset$ is given.

Definition. A mapping $f : X \longrightarrow Y$ is called $\mathcal{F}$-simple if there exists a set $F \in \mathcal{F}$ such that f is injective on F.

Fix a set $\emptyset \neq W \subseteq \bigcap\mathcal{F}$. We write that $f_1 \sim_W f_2$ for $\mathcal{F}$-simple mappings $f_1, f_2 : X \longrightarrow Y$ if there exist $F \in \mathcal{F}$ and $g \in \mathfrak{M}(\mathcal{F})$ such that $g(w) = w$ for all $w \in W$ and $f_1 \circ g(x) = f_2(x)$ for all $x \in F$.

4.6 Lemma. *For every set Y, the relation $\sim_W$ on the set of all $\mathcal{F}$-simple mappings $f : X \longrightarrow Y$ is an equivalence.*

Proof. Clearly, $\sim_W$ is reflexive. We prove that $\sim_W$ is symmetric. Let $f_1, f_2 : X \longrightarrow Y$ be $\mathcal{F}$-simple mappings with $f_1 \sim_W f_2$. Then there exist $g \in \mathfrak{M}(\mathcal{F})$ and $F \in \mathcal{F}$ such that $g(w) = w$ for all $w \in W$ and $f_1 \circ g(x) = f_2(x)$ for all $x \in F$. We can assume that g is injective on F because $F \in \mathcal{F}$ and $g \in \mathfrak{M}(\mathcal{F})$. Then $g(F) \in \mathcal{F}$. By 4.3(2), there exists $\bar{g} : X \longrightarrow X \in \mathfrak{M}(\mathcal{F})$ such that $\bar{g} \circ g(x) = x$ for all $x \in F$. Hence $\bar{g}(w) = w$ for all $w \in W$ because $W \subseteq F$. For every $y \in g(F)$, $f_2 \circ \bar{g}(y) = f_1 \circ g \circ \bar{g}(y) = f_1(y)$ and hence $f_2 \sim_W f_1$. Now we show that $\sim_W$ is transitive. Let $f_1 \sim_W f_2 \sim_W f_3$ for $\mathcal{F}$-simple mappings $f_1, f_2, f_3 : X \longrightarrow Y$. Then there exist $g, g' \in \mathfrak{M}(\mathcal{F})$ and $F, F' \in \mathcal{F}$ such that $g(w) = g'(w) = w$ for all $w \in W$, $f_1 \circ g(x) = f_2(x)$ for all $x \in F$ and $f_2 \circ g'(x) = f_3(x)$ for all $x \in F'$. Then $Z = F' \cap (g')^{-1}(F) \in \mathcal{F}$ and $f_1 \circ (g \circ g')(z) = f_2 \circ g'(z) = f_3(z)$ for all $z \in Z$. Clearly, $g \circ g' \in \mathfrak{M}(\mathcal{F})$ and $g \circ g'(w) = w$ for all $w \in W$. Hence $f_1 \sim_W f_3$. □

4.7 Lemma. *Let $f_1, f_2 : X \longrightarrow Y$ be $\mathcal{F}$-simple mappings with $f_1 \sim_W f_2$ and let $h : Y \longrightarrow Z$ be an arbitrary mapping. Then either both $h \circ f_1$ and $h \circ f_2$ are $\mathcal{F}$-simple mappings with $h \circ f_1 \sim_W h \circ f_2$ or neither $h \circ f_1$ nor $h \circ f_2$ is $\mathcal{F}$-simple and $h \circ f_1(w) = h \circ f_2(w)$ for all $w \in W$.*

Proof. We have only to prove that $h \circ f_1$ is $\mathcal{F}$-simple if and only if $h \circ f_2$ is $\mathcal{F}$-simple, the other statements are obvious. Since $f_1 \sim_W f_2$ there exist $g \in \mathfrak{M}(\mathcal{F})$ and $F \in \mathcal{F}$ such that $g(w) = w$ for all $w \in W$ and $f_1 \circ g(x) = f_2(x)$ for all $x \in F$. We can assume that g is injective on F. If $h \circ f_1$ is $\mathcal{F}$-simple then $h \circ f_1$ is injective on a set $F' \in \mathcal{F}$. Consider $F'' = F \cap g^{-1}(F') \in \mathcal{F}$, then $g(F'') \subseteq F'$ and hence $h \circ f_1 \circ g$ is injective on F'' and $h \circ f_1 \circ g(x) = h \circ f_2(x)$ for all $x \in F''$, thus $h \circ f_2$ is $\mathcal{F}$-simple. By symmetry, we obtain that from the fact that $h \circ f_2$ is $\mathcal{F}$-simple it follows that $h \circ f_1$ is $\mathcal{F}$-simple. □

4.8 Lemma. *Let $f_1 \in \mathfrak{M}(\mathcal{F})$. Then $f_1 \sim_W f_2$ for an $\mathcal{F}$-simple mapping $f_2 : X \longrightarrow X$ if and only if $f_2 \in \mathfrak{M}(\mathcal{F})$ and $f_1(w) = f_2(w)$ for all $w \in W$.*

Proof. Observe that if a mapping $f_2 : X \longrightarrow X$ is $\mathcal{F}$-simple and $f_2 \sim_W f_1$ for $f_1 \in \mathfrak{M}(\mathcal{F})$ then $f_2 \in \mathfrak{M}(\mathcal{F})$ (because $\mathfrak{M}(\mathcal{F})$ is closed under composition) and $f_2(w) = f_1(w)$ for all $w \in W$. Conversely, assume that $f_1, f_2 \in \mathfrak{M}(\mathcal{F})$ such that $f_1(w) = f_2(w)$ for all $w \in W$. Then there exist $F_1, F_2 \in \mathcal{F}$ such that f_i is injective on F_i and $f_i(F_i) \in \mathcal{F}$ for $i = 1, 2$. Then $F = f_1(F_1) \cap f_2(F_2) \in \mathcal{F}$ and also $F_i' = F_i \cap f_i^{-1}(F) \in \mathcal{F}$ for $i = 1, 2$. By 4.3(2), there exists $g' \in \mathfrak{M}(\mathcal{F})$

such that $f_1 \circ g'(x) = x$ for all $x \in F$. Clearly $g = g' \circ f_2 \in \mathfrak{M}(\mathcal{F})$ and $f_1 \circ g(x) = f_1 \circ g' \circ f_2(x) = f_2(x)$ for all $x \in F_2'$. Since $W \subseteq \bigcap \mathcal{F} = g'(\bigcap \mathcal{F}) \subseteq F_1' \cap F_2'$ and since f_1 and f_2 are one-to-one on $\bigcap \mathcal{F}$ and $f_1(w) = f_2(w)$ for all $w \in W$ we conclude that $g(w) = w$ for all $w \in W$. Thus $f_1 \sim_W f_2$ and the proof is complete. □

As a consequence we obtain this

Corollary. *The cardinal number of the set $\mathfrak{M}(\mathcal{F})/ \sim_W$ is equal to the cardinal number of the set of all injective mappings from W into $\bigcap \mathcal{F}$.*

5. Expansion of functors

5.1 Let $K : \mathbb{SET} \longrightarrow \mathbb{SET}$ be a functor and X be a set with $\operatorname{card} X > 1$. We are going to construct a functor G which extends K by the addition of one element, say a, to KX. The functor G has to enclose K and a together 'as tightly as possible', i.e., to add new elements to any KY only when it is absolutely necessary, for, in a 'tight enough' extension, we shall be able to control the cardinalities of GY. Moreover, we also need to control the internal structure of G, i.e., the knowledge of the filters and of the t-monoids of the newly added elements. This will be possible whenever the filter and the t-monoid of a in GX are prescribed. However, the filter and the t-monoid will have to have properties which make the whole construction possible.

5.2 So let a filter $\mathcal{F}$ on the set X be given such that $|\mathcal{F}| = \operatorname{card} X$, $\bigcap \mathcal{F} \neq \emptyset$. Moreover, let a non-empty set $W \subseteq \bigcap \mathcal{F}$ be given. Recall the t-monoid $\mathfrak{M}(\mathcal{F})$ defined by $\mathcal{F}$ in 4.3, $\mathcal{F}$-simple mappings $f : X \longrightarrow Y$ and the equivalence $\sim_W$ both defined in 4.5. We need them in our construction. We 'add $Gf(a)$ to KY' for every $\mathcal{F}$-simple mapping $f : X \longrightarrow Y$. On the other hand, we want to map $Gf(a)$ into KY whenever $f : X \longrightarrow Y$ is not $\mathcal{F}$-simple. To do it 'functorially', we need further instruments: a natural transformation $\mu : \mathrm{Id} \longrightarrow K$ of the identity functor Id into K (such μ does exist, see 2.3) and an element $u \in W$. Hence our construction will depend on the quadruple of 'parameters'

$$(\mu, \mathcal{F}, W, u).$$

5.3 Construction. For an $\mathcal{F}$-simple mapping $f : X \longrightarrow Y$, let $[f]$ denote the equivalence class of $\sim_W$ on the set of all $\mathcal{F}$-simple mappings $X \longrightarrow Y$ containing f.

For a set Y, define

$$GY = KY \cup \{[f] \mid f : X \longrightarrow Y \text{ is } \mathcal{F}\text{-simple}\},$$

where we suppose that the union is disjoint. If $h : Y \longrightarrow Z$ is a mapping then for every $y \in GY$ define

$$Gh(y) = \begin{cases} Kh(y) & \text{if } y \in KY, \\ [h \circ f] & \text{if } y = [f] \text{ for } \mathcal{F}\text{-simple } f : X \longrightarrow Y \\ & \quad \text{and } h \circ f \text{ is } \mathcal{F}\text{-simple,} \\ Kh(\mu_Y(f(u))) & \text{if } y = [f] \text{ for } \mathcal{F}\text{-simple } f : X \longrightarrow Y \\ & \quad \text{and } h \circ f \text{ is not } \mathcal{F}\text{-simple.} \end{cases}$$

Observation. By 4.6 and 4.7, G is a correctly defined functor from $\mathbb{SET}$ into itself and K is its subfunctor and the element a mentioned in 5.1 is precisely $[1_X]$, where 1_X is the identity mapping of X. We call it the elementary expansion of K (determined by $(\mu, \mathcal{F}, W, u)$).

5.4 In the lemmas below K, X, $\mathcal{F}$, W, u, μ are as above. Moreover, let $\mathcal{A}$ denote the abstract filter of $\mathcal{F}$ (i.e., $\mathcal{F}$ is a location of $\mathcal{A}$ on the set X, see 3.1).

Lemma. *$\mathfrak{F}^G_Y(y)$ is a location of $\mathcal{A}$ for every $y \in GY \setminus KY$ and for every set Y. Further, $\mathfrak{F}^G_X([f]) = \mathcal{F}$ if and only if $f \in \mathfrak{M}(\mathcal{F})$. Moreover, $\mathfrak{M}^G_X([1_X]) = \{f \in \mathfrak{M}(\mathcal{F}) \mid f(w) = w$ for all $w \in W\}$.*

Proof. Assume that $y = [f]$ for an $\mathcal{F}$-simple mapping $f : X \longrightarrow Y$. Thus there exists $F \in \mathcal{F}$ such that f is injective on F. Consider a set $Z \in f(\mathcal{F})$ then $F' = F \cap f^{-1}(Z) \in \mathcal{F}$. Let $\iota : Z \longrightarrow Y$ be the inclusion mapping, then there exists a mapping $g : X \longrightarrow Z$ such that $f(z) = \iota \circ g(z)$ for all $z \in F'$. Since f is $\mathcal{F}$-simple we conclude that g is $\mathcal{F}$-simple. By 4.3(3), every idempotent mapping $h : X \longrightarrow X$ with $\mathrm{Im}(h) = F'$ belongs to $\mathfrak{M}(\mathcal{F})$, hence $f \sim_W \iota \circ g$ and thus $f(\mathcal{F}) \subseteq \mathfrak{F}^G_Y([f])$. Conversely, if $Z \in \mathfrak{F}^G_Y([f])$ and if $\iota : Z \longrightarrow Y$ is the inclusion then there exists an $\mathcal{F}$-simple mapping $g : X \longrightarrow Z$ such that $\iota \circ g \sim_W f$ and hence there exists $F' \in \mathcal{F}$ with $F' \subseteq F$ and $f(F') \subseteq Z$. Therefore $f(\mathcal{F}) = \mathfrak{F}^G_Y([f])$. The fact that f is $\mathcal{F}$-simple demonstrates that $\mathfrak{F}^G_Y([f])$ is a location of $\mathcal{A}$. From the definition of $\mathfrak{M}(\mathcal{F})$ it follows that $f(\mathcal{F}) = \mathcal{F}$ for a $\mathcal{F}$-simple mapping if and only if $f \in \mathfrak{M}(\mathcal{F})$, and the second statement follows. The third statement is implied by Lemma 4.8. □

5.5 Lemma. *If $W = \{u\}$ and* card $\bigcap \mathcal{F} \geq 3$ *then for every set Y and every $y \in GY \setminus KY$, the set $\{z \in Y \mid f(z) = z$ for all $f \in \mathfrak{M}^G_Y(y)\}$ is a singleton.*

Proof. Consider $y = [g] \in GY \setminus KY$ and let $U = \{z \in Y \mid f(z) = z$ for all $f \in \mathfrak{M}^G_Y([g])\}$. If $h \in \mathfrak{M}^G_Y([g])$ then $Gh([g]) = [g]$ implies that $h \circ g \sim_W g$ and from the definition of $\sim_W$ it follows that $h(g(u)) = g(u)$. Therefore $g(u) \in U$. By Lemma in 5.4, $\mathfrak{F}^G_Y([g])$ is a location of $\mathcal{A}$ and hence card $\bigcap \mathfrak{F}^G_Y(y) =$ card $\bigcap \mathcal{F} \geq 3$. One can easily see that if $t \in Y \setminus \bigcap \mathfrak{F}^G_Y([g])$ then the mapping $h : Y \longrightarrow Y$ such that $h(t) \neq t$ and $h(s) = s$ for all $s \in Y$ with $s \neq t$ satisfies $Gh([g]) = [g]$ and hence $t \notin U$ (see also [5,9]). If $t \in \bigcap \mathfrak{F}^G_Y([g])$ with $t \neq g(u)$, then there exists $t' \in \bigcap \mathcal{F}$ with $g(t') = t$ and $t' \neq u$. Let $h : X \longrightarrow X$ be a mapping such that $h(x) = x$ for all $x \in X \setminus \bigcap \mathcal{F}$, the restriction of h on $\bigcap \mathcal{F}$ is a permutation of $\bigcap \mathcal{F}$ with $h(u) = u$ and $h(t') \neq t'$. Since g is $\mathcal{F}$-simple there exists $F \in \mathcal{F}$ such that g is injective on F and therefore there exists a mapping $h' : Y \longrightarrow Y$ such that $g \circ h(x) = h' \circ g(x)$ for all $x \in F$. Hence $h'(t) \neq t$ and $g \sim_W h' \circ g$. Thus $Gh'([g]) = [g]$ and $t \notin U$. □

5.6 Summary. *Let K : $\mathbb{SET} \longrightarrow \mathbb{SET}$ be a functor, let G be an elementary expansion of K determined by the quadruple $(\mu, \mathcal{F}, W, u)$, and let $\mathcal{A}$ be the abstract filter containing $\mathcal{F}$. Then*

(1) *if* $|\mathcal{F}|$ *is infinite and* W *is finite then*

$$\operatorname{card}(GY \setminus KY) = \operatorname{card} \mathcal{A}(Y)$$

for every set Y *whenever* $\mathfrak{F}_Y^K(y)$ *is a location of* $\mathcal{A}$ *for no* $y \in KY$;

(2) *there exists* $a \in GX \setminus KX$ *such that* $\mathfrak{M}_X^G(a) = \{f \in \mathfrak{M}(\mathcal{F}) \mid f(w) = w \text{ for all } w \in W\}$;

(3) *if* $\operatorname{card} \bigcap \mathcal{F} \geq 3$ *and* $W = \{u\}$ *then for every set* Y *and every* $y \in GY \setminus KY$, $\mathfrak{M}_Y^G(y)$ *has exactly one fix-point (i.e., there exists exactly one* $v \in Y$ *with* $f(v) = v$ *for all* $f \in \mathfrak{M}_Y^G(y)$*).*

Proof. If $\mathfrak{F}_Y^K(y)$ is a location of $\mathcal{A}$ for no $y \in KY$ then, by 3.5 and 5.4, $\operatorname{card}(GY \setminus KY) = p(G, \mathcal{A}) \operatorname{card} \mathcal{A}(Y)$. By Lemma and Corollary in 4.8,

$$p(G, \mathcal{A}) = \operatorname{card}(\bigcap \mathcal{F})^W \leq |\mathcal{A}|$$

because $\mathcal{A} \in \mathbb{A}$. From 3.2 and $|\mathcal{A}| \geq \aleph_0$ it follows that

$$p(G, \mathcal{A}) \operatorname{card} \mathcal{A}(Y) = \operatorname{card} \mathcal{A}(Y)$$

and (1) is proved. Lemma 5.4 implies (2) and Lemma 5.5 implies (3). □

6. The construction of G_1 and G_2

6.1 An amalgam $\mathfrak{A} = \{G^{(j)} \mid j \in J\}$ of functors with a base K is a system of functors such that K is a subfunctor of G_j for all $j \in J$ and

$G^{(j_1)}X \cap G^{(j_2)}X = KX$ for all sets X and all $j_1, j_2 \in J$ with $j_1 \neq j_2$.

If, for every set X, $\bigcup_{j \in J} G^{(j)}X$ is a set, we can define the sum of the amalgam $\mathfrak{A}$ by the simple rule

$GX = \bigcup_{j \in J} G^{(j)}X$ and each $G^{(j)}$ is a subfunctor of G.

Clearly, G is a correctly defined functor and, for every set X,

$$\operatorname{card}(GX \setminus KX) = \sum_{j \in J} \operatorname{card}(G^{(j)}X \setminus KX).$$

6.2 Now we are going to complete the proof of Main Theorem. Let a functor $H : \mathbb{SET} \longrightarrow \mathbb{SET}$ which is not finitary be given. Then, by 3.5,

$$\operatorname{card}(HX \setminus H^{(f)}X) = \sum_{\mathcal{A} \in \mathbb{A}} p(H, \mathcal{A}) \operatorname{card} \mathcal{A}(X),$$

where $\mathcal{A}$ and $\mathcal{A}(X)$ are as in 3.1, $p(H, \mathcal{A})$ and $\mathbb{A}$ are as in 3.5. Since H is not finitary, $p(H, \mathcal{A}) \neq 0$ for at least one $\mathcal{A} \in \mathbb{A}$.

We aim to construct functors $G_1, G_2 : \mathbb{SET} \longrightarrow \mathbb{SET}$ which are not naturally equivalent and satisfy

$$G_1^f = H^f = G_2^f \qquad \text{and}$$
$$\operatorname{card}(G_1X \setminus G_1^fX) = \sum_{\mathcal{A} \in \mathbb{A}} p(H, \mathcal{A}) \operatorname{card} \mathcal{A}(X) = \operatorname{card}(G_2X \setminus G_2^fX)$$

for all sets X. Both G_1 and G_2 will be obtained as sums of suitable amalgams with a base H^f. These amalgams consist of suitable elementary expansions $G_1^{(j)}$ and $G_2^{(j)}$ of H^f. However, to get the quadruples $(\mu, \mathcal{F}, W, u)$ from which the elementary expansions will be constructed (see Section 5), we need one more simple trick. For any filter $\mathcal{F}$ on a set X with $|\mathcal{F}| \geq \aleph_0$, put

$$\Phi\mathcal{F} = \begin{cases} \mathcal{F} & \text{if } \bigcap \mathcal{F} \text{ is infinite,} \\ \{F \cup Q \mid F \in \mathcal{F}\} & \text{if } \bigcap \mathcal{F} \text{ is finite (including } \bigcap \mathcal{F} = \emptyset), \end{cases}$$

where Q is a set with $\operatorname{card} Q = 3$ and $X \cap Q = \emptyset$. Clearly, if $\mathcal{F}$ is equivalent (in the sense of 3.1) to $\mathcal{G}$ then $\Phi\mathcal{F}$ is equivalent to $\Phi\mathcal{G}$; hence we have determined $\Phi\mathcal{A}$ for every abstract filter $\mathcal{A}$ and $\operatorname{card} \bigcap \mathcal{F} \geq 3$ for every location $\mathcal{F}$ of $\Phi\mathcal{A}$.

Lemma. *If* $\mathcal{A} \in \mathbb{A}$ *then*

$$\operatorname{card} \mathcal{A}(Y) = \operatorname{card} \Phi\mathcal{A}(Y) \qquad \text{for all sets } Y.$$

Proof. Since $\mathcal{A} \in \mathbb{A}$, $|\mathcal{A}|$ is infinite. If $\operatorname{card} Y < |\mathcal{A}|$ then $\operatorname{card} \mathcal{A}(Y) = 0 = \operatorname{card} \Phi\mathcal{A}(Y)$. If $\operatorname{card} Y \geq |\mathcal{A}| = |\Phi\mathcal{A}|$ then, clearly,

$$\operatorname{card} \Phi\mathcal{A}(Y) \leq \operatorname{card} \mathcal{A}(Y) \operatorname{card} Y^3.$$

Since $\operatorname{card} Y^3 = \operatorname{card} Y \leq \operatorname{card} \mathcal{A}(Y)$, see 3.2, we conclude that

$$\operatorname{card} \Phi\mathcal{A}(Y) \leq \operatorname{card} \mathcal{A}(Y).$$

The reverse inequality is evident. □

6.3 Now we are ready to describe the quadruples used in Section 5. First we choose a natural transformation μ from the identity functor to $H^{(f)}$. For every $\mathcal{A} \in \mathbb{A}$, choose one location $\mathcal{F}$ of $\Phi\mathcal{A}$ on a set X with $\operatorname{card} X = |\mathcal{A}|$ and two distinct elements $u, v \in \bigcap \mathcal{F}$. Let $G_1^{\mathcal{A}}$ be the elementary expansion of $H^{(f)}$ determined by the quadruple $(\mu, \mathcal{F}, \{u\}, u)$ and $G_2^{\mathcal{A}}$ be the elementary expansion of $H^{(f)}$ determined by the quadruple $(\mu, \mathcal{F}, \{u, v\}, u)$. Let us denote $p(H, \mathcal{A}) \cdot G_i^{\mathcal{A}}$ the sum of the amalgam of $\mathfrak{A}_i = \{G_i^{(j)} \mid j \in J\}$ for $i = 1, 2$ where $\operatorname{card} J = p(H, \mathcal{A})$, $G_i^{(j)}$ is naturally equivalent to the elementary expansion $G_i^{\mathcal{A}}$ of $H^{(f)}$ for all $j \in J$ and $i = 1, 2$ and $G_i^{(j)} X \cap G_i^{(j')} X = H^{(f)} X$ for all distinct $j, j' \in J$, for all sets X and for $i = 1, 2$. Then, by 5.6, for every set Y and $i = 1, 2$,

$$\begin{aligned}\operatorname{card}((p(H, \mathcal{A}) \cdot G_i^{\mathcal{A}})Y \setminus H^{(f)}Y) = & p(H, \mathcal{A}) \operatorname{card} \Phi\mathcal{A}(Y) = \\ & p(H, \mathcal{A}) \operatorname{card} \mathcal{A}(Y).\end{aligned}$$

Finally, let G_i be the sum of the amalgam $\{p(H, \mathcal{A}) \cdot G_i^{\mathcal{A}} \mid \mathcal{A} \in \mathbb{A}\}$, for $i = 1, 2$. Then, for every set Y and for $i = 1, 2$,

$$\operatorname{card} G_i Y = \operatorname{card} H^{(f)} Y + \sum_{\mathcal{A} \in \mathbb{A}} p(H, \mathcal{A}) \operatorname{card} \mathcal{A}(Y) = \operatorname{card} HY,$$

by the equation in 3.5.

6.4 It remains to show that G_1 is not naturally equivalent to G_2. Since $H \neq H^{(f)}$, there exists $\mathcal{A}_0 \in \mathbb{A}$ such that $p(H, \mathcal{A}_0) \neq 0$. Let $\mathcal{F}$ be a location of $\Phi\mathcal{A}_0$ on a set X with $\operatorname{card} X = |\mathcal{A}_0|$. Assume that ν is a natural equivalence of G_1 onto G_2. Then ν maps the finitary part $H^{(f)}$ of G_1 onto the finitary part $H^{(f)}$ of G_2, hence ν_X maps $G_1X \setminus H^{(f)}X$ bijectively onto $G_2X \setminus H^{(f)}X$. Then for every $x \in G_1X \setminus H^{(f)}X$, the t-monoid $\mathfrak{M}_X^{G_1}(x)$ must be strongly isomorphic to $\mathfrak{M}_X^{G_2}(\nu_X(x))$, see 4.2. But for every $x \in G_1X$, the t-monoid $\mathfrak{M}_X^{G_1}(x)$ has at most one fix-point, see 5.5, and $\mathfrak{M}_X^{G_2}[1_X]$ has at least two fix-points, u and v. This is a contradiction, and therefore G_1 and G_2 are not naturally equivalent.

The proof of Main Theorem is now complete.

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