CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 46, nº 4 (2005), p. 301-312

<http://www.numdam.org/item?id=CTGDC_2005__46_4_301_0>

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLVI-4 (2005)

PROJECTIVE FRAMES: A GENERAL VIEW

by B. BANASCHEWSKI

RESUME. Cet article étudie la projectivité dans la catégorie **Frm** des cadres ('frames') par rapport aux homomorphismes surjectifs dont l'adjoint à droite appartient à une sous-catégorie adéquate **K** de la catégorie des semi-treillis avec unité. Les applications à plusieurs sous-catégories **K** familières re-donnent divers résultats connus qui sont ainsi unifiés dans un même schéma naturel.

This note deals with a very general form of projectivity in the category of frames, establishing external and internal characterizations for it which then specialize to several previous results (Banaschewski-Niefield [3], Escardó [5]), exhibiting them as immediate consequences of some fundamental facts of considerably wider scope.

Specifically, the setting here is the category **M** of meet-semilattices (always taken with unit) in which we consider subcategories **K** containing the category **Frm** of frames reflectively, subject to a very simple natural condition. The projectivity in question is then taken relative to the onto frame homomorphisms $h : L \to M$ for which the right adjoint $h_* : M \to L$ ($h(a) \leq b$ iff $a \leq h_*(b)$) belongs to **K**, referred to as **K**-*flat projectivity*.

In addition, we describe a class of subcategories of **M**, consisting of meet-semilattices with suitably prescribed joins together with their homomorphisms preserving these joins, for which we show that **Frm** is a subcategory of the kind in question. This then provides a suggestive class of concrete cases to which our general characterization of **K**-flat projectives apply; the previous results referred to above then fit into this particular context.

For general background concerning frames we refer to Johnstone [8] or Vickers [12].

The condition postulated for the subcategory $\mathbf{K}$ of $\mathbf{M}$ besides the assumption that $\mathbf{Frm}$ is reflective in $\mathbf{K}$ is as follows:

(C) *For any $\varphi : A \to L$ in $\mathbf{K}$ where L is a frame and A arbitrary, the corestriction of φ to any subframe of L containing the image of φ also belongs to $\mathbf{K}$.*

We refer to this by saying that $\mathbf{K}$ is *corestrictive over* $\mathbf{Frm}$.

The following collects some of the basic properties of $\mathbf{K}$ needed in the sequel. Here, $\eta_A : A \to FA$ is the universal map in $\mathbf{K}$ to frames and, correspondingly, $\varepsilon_L : FL \to L$ for any frame L the frame homomorphism such that $\varepsilon_L \eta_L = \mathrm{id}_L$. Further, $\leq$ stands for the usual argumentwise partial order of maps between partially ordered sets, which is evidently preserved by the composition of maps in $\mathbf{K}$.

Lemma 1 *(1) Each FA is generated by the image of η_A.*

(2) $\mathrm{id}_{FL} \leq \eta_L \varepsilon_L$ for any frame L.

(3) For any frame L, if $h : L \to FL$ is right inverse to $\varepsilon_L : FL \to L$ then $h\varepsilon_L \leq \mathrm{id}_{FL}$.

Proof. (1) Let $L \subseteq FA$ be the subframe generated by $\mathrm{Im}(\eta_A)$, $\varphi : A \to L$ the corresponding corestriction of $\eta_A : A \to FA$, and $i : L \to FA$ the identical subframe embedding. Then by (C) we have a frame homomorphism $h : FA \to L$ such that $h\eta_A = \varphi$, hence $ih\eta_A = \eta_A$, and therefore $ih = \mathrm{id}_{FA}$ by the properties of η_A. It follows that i is onto, showing $L = FA$.

(2) To begin with, note that by (1) each $b \in FL$ is the join of all $\eta_L(a) \leq b$ because η_L is a meet-semilattice homomorphism. Now, for any such $a \in L$,

$$\eta_L(a) = (\eta_L \varepsilon_L)\eta_L(a) \leq (\eta_L \varepsilon_L)(b)$$

(since $\varepsilon_L \eta_L = \mathrm{id}_L$) and consequently $b \leq \eta_L \varepsilon_L(b)$, as desired, by the initial observation.

(3) If $\varepsilon_L h = \mathrm{id}_L$ then $h \leq \eta_L \varepsilon_L h = \eta_L$ by (2) and hence

$$(h\varepsilon_L)\eta_L = h \leq \eta_L$$

which implies the desired result by (1) since $h\varepsilon_L$ is a frame homomorphism. □

Remark 1 *Note that the above (1) is actually equivalent to (C). For any $\varphi : A \to L$ such that $Im(\varphi) \subseteq M$ for some subframe M of L, the frame homomorphism $h : FA \to L$ for which $h\eta_A = \varphi$ maps into M by (1) and the composite of its corestriction to M with η_A belongs to $\mathbf{K}$ – but this is just the corestriction of φ to M.*

Remark 2 *Since $\varepsilon_L \eta_L = id_L$, (2) implies η_L is right adjoint to ε_L. In a similar vein, if $\varepsilon_L h = id_L$ for some $h : L \to FL$ then h is left adjoint to ε_L by (3) and consequently unique.*

The reflectiveness of **Frm** in **K** determines a binary relation on each frame L as follows:

$$x \lhd a \ \text{ iff } \ a \leq \varepsilon_L(b) \ \text{ implies } \ \eta_L(x) \leq b, \ \text{ for all } \ b \in FL.$$

Now we have our first result concerning **K**-flat projectivity.

Proposition 1 *The following are equivalent for any frame L.*
(1) L is $\mathbf{K}$-flat projective.
(2) ε_L has a right inverse.
(3) L is a retract of some FA, $A \in \mathbf{K}$.
(4) For each $a \in L$, $a = \bigvee\{x \in L \mid x \lhd a\}$; further $x \lhd a \wedge b$ whenever $x \lhd a$ and $x \lhd b$, and $e \lhd e$ for the unit $e \in L$.

Proof. (1) $\Rightarrow$ (2). By Remark 2, $(\varepsilon_L)_* = \eta_L$ which belongs to **K** by its definition, and ε_L is onto since $\varepsilon_L \eta_L = \mathrm{id}_L$. Hence, if L is **K**-flat projective we have $h : L \to FL$ such that $\varepsilon_L h = \mathrm{id}_L$.

(2) $\Rightarrow$ (3). Trivial

(3) $\Rightarrow$ (1). Since a retract of a projective in whatever sense is projective in that sense it is enough to show that each FA is **K**-flat projective. Consider, then the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "h_* f \eta_A"'] & FA \arrow[d, "f"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

with frame homomorphisms h and f, h onto and **K**-flat and f arbitrary. Then $h_* f \eta_A \in \mathbf{K}$ and hence we have a frame homomorphism $g : FA \to$

L such that $g\eta_A = h_* f\eta_A$. It follows that $hg\eta_A = f\eta_A$ (h is onto!) and therefore $hg = f$ by the properties of η_A.

(2) $\Rightarrow$ (4). We begin by showing that $x \lhd a$ iff $\eta_L(x) \leq h(a)$ for the given $h : L \to FL$ such that $\varepsilon_L h = \mathrm{id}_L$. Here ($\Rightarrow$) is immediate since $a = \varepsilon_L(h(a))$ and ($\Leftarrow$) follows from Lemma 1(3): if $a \leq \varepsilon_L(b)$ then $h(a) \leq h\varepsilon_L(b) \leq b$, and $\eta_L(x) \leq h(a)$ then implies $\eta_L(x) \leq b$.

Now, by Lemma 1(1), $h(a) = \bigvee\{\eta_L(x) \mid \eta_L(x) \leq h(a)\}$ and hence, acting ε_L and applying what was just shown, we obtain

$$a = \bigvee\{x \in L \mid x \lhd a\}.$$

Further, if $x \lhd a$ and $x \lhd b$ then $\eta_L(x) \leq h(a) \wedge h(b) = h(a \wedge b)$ so that $x \lhd a \wedge b$. Finally, $e \lhd e$ since $\eta_L(e) = e = h(e)$.

(4) $\Rightarrow$ (2). Let $h : L \to FL$ be the set map defined by

$$h(a) = \bigvee\{\eta_L(x) \mid x \lhd a\}.$$

Then $\varepsilon_L h = \mathrm{id}_L$ by the first part of (4) while

$$h\varepsilon_L(b) = \bigvee\{\eta_L(x) \mid x \lhd \varepsilon_L(b)\} \leq b$$

since $x \lhd \varepsilon_L(b)$ implies $\eta_L(x) \leq b$, and hence $h\varepsilon_L \leq \mathrm{id}_{FL}$. It follows that h is left adjoint to ε_L, and as such it preserves arbitrary joins. Further, for any $a, c \in L$,

$$h(a) \wedge h(c) = \bigvee\{\eta_L(x \wedge z) \mid x \lhd a,\ z \lhd c\} \leq h(a \wedge c) :$$

$x \lhd a$ and $z \lhd c$ implies $x \wedge z \lhd a$ and $x \wedge z \lhd c$ by the definition of $\lhd$ and hence $x \wedge z \lhd a \wedge c$ by the second part of (4). Finally, $h(e) = e$ since $e \lhd e$ by hypothesis and $\eta_L(e) = e$. In all, this shows h is a frame homomorphism, right inverse to ε_L. □

Remark 3 *The first part of this proposition, the equivalence of (1), (2), and (3), is obviously rather formal and may be viewed as a variant of the general principle "projective = retract of some free object". In fact, it actually is a special case of a completely general result. For any categories* $\mathbf{F}$ *and* $\mathbf{K}$ *with functors* $S : \mathbf{F} \to \mathbf{K}$ *and* $T : \mathbf{K} \to \mathbf{F}$ *where*

T is left adjoint to S, if $\mathfrak{R}$ is the class of all $h : L \to M$ in $\mathbf{F}$ for which $Sh : SL \to SM$ is right invertible in $\mathbf{K}$ then the relevant parts of the above proof evidently adapt to $\mathfrak{R}$-projectivity in $\mathbf{F}$. On the other hand, for the specific $\mathbf{K}$ considered here, with $\mathbf{F} = \mathbf{Frm}$, $\mathfrak{R}$-projectivity and $\mathbf{K}$-flat projectivity are the same: trivially ($\Rightarrow$), and ($\Leftarrow$) since any FA is actually $\mathfrak{R}$-projective by the above proof of (3) $\Rightarrow$ (1). Needless to say, the rather deeper equivalence of (2) and (4) is quite a different matter.

There is a further characterization of $\mathbf{K}$-flat projectivity involving the comonad in $\mathbf{Frm}$ determined by the reflection functor F. For this, some additional properties of $\mathbf{K}$ are needed, as follows.

Lemma 2 *(1) F preserves the partial order of maps.*
(2) For any $A \in \mathbf{K}$, $F\eta_A \leq \eta_{FA}$.

Proof. (1) If $\varphi, \psi : A \to B$ such that $\varphi \leq \psi$ then

$$(F\varphi)\eta_A = \eta_B\varphi \leq \eta_B\psi = (F\psi)\eta_A,$$

the first and last step by naturality, and hence $F\varphi \leq F\psi$ by Lemma 1(1).

(2) Using that $b = \bigvee\{\eta_A(a) \mid \eta_A(a) \leq b\}$ for any $b \in FA$,

$$\begin{aligned}(F\eta_A)(b) = \bigvee\{(F\eta_A)\eta_A(a) \mid \eta_A(a) \leq b\} &= \bigvee\{\eta_{FA}\eta_A(a) \mid \eta_A(a) \leq b\} \\ &\leq \eta_{FA}(b),\end{aligned}$$

the second step by naturality. □

Now, the *comonad* determined by F (viewed as an endofunctor of $\mathbf{Frm}$) is $(F, \varepsilon, F\eta)$, and its *coalgebras* are the pairs (L, h) where the structure map $h : L \to FL$ satisfies the familiar conditions

$$\text{(U)}\quad \varepsilon_L h = \mathrm{id}_L \quad\text{ and }\quad \text{(A)}\quad (Fh)h = (F\eta_L)h$$

(Mac Lane [9]). Given that it is entirely determined by the extension $\mathbf{K}$ of $\mathbf{Frm}$, we call this comonad the $\mathbf{K}$-comonad. The desired result then is

Proposition 2 *A frame L is* $\mathbf{K}$*-flat projective iff it has a coalgebra structure for the* $\mathbf{K}$*-comonad.*

Proof. Given the properties of $\mathbf{K}$ and the equivalence (1) $\equiv$ (2) in Proposition 1, this follows from general results concerning so-called Kock-Zöberlein monods (Escardó [5]). For the reader's convenience, we include the short alternative proof of this which naturally arises in the present context.

It is clear we only have to show that the above identity (U) automatically implies the second condition (A). Now, by Lemma 1(2), $h \leq \eta_L \varepsilon_L h = \eta_L$ so that $(Fh)h \leq (F\eta_L)h$ by Lemma 2(1). For the reverse inequality, we have

$$\begin{aligned}(F\eta_L)h(a) &= \bigvee\{(F\eta_L)h(x) \mid x \lhd a\} \leq \bigvee\{\eta_{FL}h(x) \mid x \lhd a\} \\ &= \bigvee\{(Fh)\eta_L(x) \mid x \lhd a\} \leq (Fh)h(a)\end{aligned}$$

by, respectively, (2) $\Rightarrow$ (4) in Proposition 1, Lemma 2(2), the naturality of η_L, and the fact that $\eta_L(x) \leq h(a)$ whenever $x \lhd a$, as noted in the proof of (2) $\Rightarrow$ (4) of Proposition 1. □

Now consider the following way of specifying subcategories of $\mathbf{M}$. For each $A \in \mathbf{M}$, let $\mathcal{S}A$ by a collection of subsets of A such that $\{a \wedge t \mid t \in S\}$ belongs to $\mathcal{S}A$ for each $a \in A$ and $S \in \mathcal{S}A$, and for each $\varphi : A \to B$ in $\mathbf{M}$, $\varphi[S] \in \mathcal{S}B$ whenever $S \in \mathcal{S}A$. Further, let $\mathbf{S}$ be the subcategory of $\mathbf{M}$ consisting of all A such that $\bigvee S$ exists for each $S \in \mathcal{S}A$ and

$$a \wedge \bigvee S = \bigvee\{a \wedge t \mid t \in S\},$$

the maps being the meet-semilattice homomorphisms which preserve all $\bigvee S$, $S \in \mathcal{S}A$.

Among many obvious examples we have the following familiar, particularly significant cases: for each $A \in \mathbf{M}$, $\mathcal{S}A$ consists of

(1) no S, and $\mathbf{S} = \mathbf{M}$,

(2) $\emptyset$, and $\mathbf{S}$ is the category of *bounded meet-semilattices*,

(3) all finite subsets, and $\mathbf{S} = \mathbf{D}$, the category of *bounded distributive lattices*,

(4) all (at most) countable subsets, and $\mathbf{S} = \sigma\mathbf{Frm}$, the category of *$\sigma$-frames*,

(5) all updirected subsets, and $\mathbf{S} = \mathbf{PrFrm}$, the category of *preframes*, and

(6) all subsets, and $\mathbf{S} = \mathbf{Frm}$.

We note that any category $\mathbf{S}$ of this kind trivially contains $\mathbf{Frm}$ and is evidently corestrictive over the latter: if a meet-semilattice homomorphism $\varphi : A \to L$ preserves any specified joins in A and maps A into a subframe M of L then its corestriction also preserves these joins simply because, for any subset S of A, the join of $\varphi[S]$ in L and in M are the same. But we have more:

Proposition 3 *For any* $\mathbf{S}$*,* $\mathbf{Frm}$ *is reflective in* $\mathbf{S}$*.*

Proof. We give an explicit description of the frame reflection in $\mathbf{S}$.

For any $A \in \mathbf{S}$, let $\mathfrak{D}A$ be the frame of all *downsets* of A, that is, the $U \subseteq A$ such that $a \in U$ whenever $a \leq b$ and $b \in U$ (which includes $U = \emptyset$), and $\mathfrak{S}A$ the closure system in $\mathfrak{D}A$ consisting of all U for which

$$S \subseteq U \quad \text{and} \quad S \in \mathcal{S}A \quad \text{implies} \quad \bigvee S \in U.$$

Note that the principal downsets $\downarrow a = \{x \in A \mid x \leq a\}$ belong to $\mathfrak{S}A$, giving rise to a map $\sigma_A : A \to \mathfrak{S}A$ taking a to $\downarrow a$. We claim that, for any $A \in \mathbf{S}$,

(i) $\mathfrak{S}A$ is a frame, and

(ii) $\sigma_A : A \to \mathfrak{S}A$ is the universal map in $\mathbf{S}$ to frames.

For (i), consider the operator k_0 on $\mathfrak{D}A$ such that

$$k_0(U) = U \cup \{\bigvee S \mid S \subseteq U,\ S \in \mathcal{S}A\}.$$

Obviously, $\mathfrak{S}A = \ \mathrm{Fix}(k_0)$, and by general principles $\mathfrak{S}A$ will be a frame if k_0 is a *prenucleus* on $\mathfrak{D}A$, meaning that $k_0(U) \in \mathfrak{D}A$ and

$$U \subseteq k_0(U),\ \ k_0(U) \subseteq k_0(W) \text{ whenever } U \subseteq W,$$

$$\text{and } k_0(U) \cap W \subseteq k_0(U \cap W)$$

for all $U, W \in \mathfrak{D}A$ (Banaschewski [2]). Now, $k_0(U)$ will clearly be a downset provided this holds for its second part, but if $a \leq \bigvee S$ for some $S \subseteq U$ in $\mathcal{S}A$ then $a = \bigvee\{a \wedge t \mid t \in S\}$ by the definition of $\mathbf{S}$, and since the set involved in this join belongs to $\mathcal{S}A$ and is contained in U, a belongs to the set in question. Regarding the last condition, if $a \in k_0(U) \cap W$ belongs to U then trivially $a \in k_0(U \cap W)$ since it belongs to $U \cap W$. On the other hand, if $a = \bigvee S$ where $S \subseteq U$ and $S \in \mathcal{S}A$ then $a \in W$ implies $S \subseteq W$ so that $S \subseteq U \cap W$ and again $a \in k_0(U \cap W)$. Since the other two conditions obviously hold it follows that k_0 is a prenucleus, as desired.

Concerning (ii), it first has to be shown that $\sigma_A : A \to \mathfrak{S}A$ belongs to $\mathbf{S}$. Clearly, it is a meet-semilattice homomorphism: $\wedge$ in $\mathfrak{S}A$ is $\cap$ and $\downarrow a \cap \downarrow b = \downarrow (a \wedge b)$, while $\downarrow e$ is the unit of $\mathfrak{S}A$. Further, for any $S \in \mathcal{S}A$, $S \subseteq \bigvee\{\downarrow t \mid t \in S\}$ (join in $\mathfrak{S}A$) and hence $a \in \bigvee\{\downarrow t \mid t \in S\}$ for $a = \bigvee S$. On the other hand, $t \leq a$ for each $t \in S$ and therefore

$$\downarrow a \subseteq \bigvee\{\downarrow t \mid t \in S\} \subseteq \downarrow a,$$

showing that $\sigma_A(\bigvee S) = \bigvee \sigma_A[S]$.

Finally, regarding the universality property of σ_A, let $\varphi : A \to L$ be any map in $\mathbf{S}$ where L is a frame. Then φ, as a meet-semilattice homomorphism, determines the frame homomorphism $h : \mathfrak{D}A \to L$ such that $h[U] = \bigvee \varphi[U]$; we claim its restriction to $\mathfrak{S}A$ is also a frame homomorphism, which will follow if we show that $hk_0 = h$ for the prenucleus k_0 defining $\mathfrak{S}A$ (Banaschewski [2]). Now, for any $U \in \mathfrak{D}A$,

$$\begin{aligned} hk_0(U) &= \bigvee \varphi[U] \cup \{\bigvee S \mid S \subseteq U,\ S \in \mathcal{S}A\}] \\ &= \bigvee \varphi[U] \vee \bigvee\{\varphi(\bigvee S) \mid S \subseteq U,\ S \in \mathcal{S}A\} \\ &= \bigvee \varphi[U] = h(U), \end{aligned}$$

the next to the last step specifically because $\varphi(\bigvee S) = \bigvee \varphi[S] \leq \bigvee \varphi[U]$ for the S involved.

In all, this provides a frame homomorphism $\mathfrak{S}A \to L$ taking $\downarrow a$ to $\varphi(a)$, obviously unique since the frame $\mathfrak{S}A$ is generated by the $\downarrow a$, $a \in A$, and this proves the proposition. □

In the examples listed above, $\mathfrak{S}A$ consists of the following $U \in \mathfrak{D}A$:
(1) all U,
(2) all U containing 0,

(3) the ideals of A,
(4) the σ-ideals of A,
(5) the Scott-closed U, and
(6) the $\downarrow a$.

Back to the case of general $\mathbf{S}$, we note that the adjunction map $\varepsilon_L : \mathfrak{S}L \to L$, for any frame L, is the join map:

$$\varepsilon_L(U) = \varepsilon_L(\bigvee\{\downarrow s \mid s \in U\}) = \bigvee\{\varepsilon_L(\downarrow s) \mid s \in U\} = \bigvee U.$$

On the other hand, for any $A \in \mathbf{S}$, $x \in A$, and $U \in \mathfrak{S}A$,

$$\eta_A(x) \leq U \text{ iff } \downarrow x \subseteq U \text{ iff } x \in U,$$

and consequently the relation $\lhd$ now has the following concrete form:

$$x \lhd a \text{ iff } a \leq \bigvee U \text{ implies } x \in U, \text{ for all } U \in \mathfrak{S}A.$$

Note that probably the earliest instance of this relation was considered by Raney [11] for $\mathfrak{S}A = \mathfrak{D}A$, amounting to the case of our example (1). A subsequent case of great importance was the familiar way below relation $\ll$ connected with continuous lattices which may be described as $\lhd$ for example (3) (Gierz et al. [6]). Of course, in the case of (6) $\lhd$ is just $\leq$.

Regarding previous results concerning projective frames, Banaschewski [1] dealt with the case of example (1) where $\mathbf{S} = \mathbf{M}$, proving the equivalence (1) $\equiv$ (4) of Proposition 1, and the subject was then revisited by Banaschewski-Niefield [3], offering a more direct proof based on the use of the downset functor $\mathfrak{D}$, and adding the equivalences (4) $\equiv$ (3) and (1) $\equiv$ (2) of Proposition 1. Further, it was noted in [3] that the same kind of arguments provide a natural counterpart of this in the case of example (3) where $\mathbf{S} = \mathbf{D}$ and the functor $\mathfrak{D}$ is replaced by the ideal lattice functor. Here, (1) $\equiv$ (4) says that the $\mathbf{D}$-flat projectives are exactly the stably continuous frames, a result originally due to Johnstone [7].

In a similar vein, Escardó [5] considered the case of example (5), $\mathbf{S} =$ **PrFrm** and $\mathfrak{S}A$ the frame of Scott closed downsets of A, establishing what amounts to Proposition 2 together with the equivalence (1) $\equiv$ (4)

of Proposition 1. Further, it is noted in [5] that analogous arguments prove the same result for example (3). It may be worth pointing out that the Proposition 2 part of this did not occur in the previous work mentioned, but it is related to the result of Day [4] that the stably continuous frames correspond to the algebras of the filter monad in the category of T_0 spaces. For a somewhat different but related version of the equivalence (1) $\equiv$ (4) of Proposition 1, see Section 1 of Paseka [10] which includes, among other things, the cases (1), (3), (4), and (6) of $\mathfrak{S}A$.

We close with a general comment on method. It is obvious that the *concrete nature* of the reflection functor and the associated reflection maps, determined by the category extension **K** of **Frm** considered above, is irrelevant for our purposes: the arguments used here depend only on the formal properties of the entities involved and do not require any kind of explicit description for them. This seems to be at variance with the final remark of Escardó [5] which claims that knowing such a description is crucial for obtaining the desired results. On the other hand, though, there is one aspect of the situation considered here which really does seem crucial for the proofs involved: the fact that **K** is a subcategory of **M** and hence all maps under consideration are meet-semilattice homomorphisms. It may well be that the real problem with the category of complete join semilattices alluded to in [5] is not the lack of explicit constructions but rather the nature of its maps.

Acknowlegements. This note was written during a visit to the Category Theory and Topology Research Group at the University of Cape Town, September-November 2003. Thanks go to this group and to the Department of Mathematics and Applied Mathematics of the University of Cape Town for their kind hospitality, to the Natural Sciences and Engineering Research Council for ongoing support in the form of a research grant, and to the referee for pointing out the work of Paseka [10].

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