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EQUILOGICAL SPACES, HOMOLOGY AND NON-COMMUTATIVE GEOMETRY

*by Marco GRANDIS**

Résumé. On introduit l'homologie singulière pour les *espaces équilogiques* de D. Scott [22]. Son étude montre que ces structures peuvent exprimer des 'quotients formels' d'espaces topologiques, liés à des C*-algèbres non commutatives bien connues, qui ne peuvent être réalisées en tant qu'espaces ordinaires. On utilise aussi une notion de *morphisme local* entre espaces équilogiques, qui généralise les morphismes usuels et pourrait être d'intérêt dans la théorie générale de ces espaces.

Introduction

An unexpected byproduct of a previous work on the homology of cubical sets [12] was finding that such structures also contain models of 'virtual spaces' of interest in noncommutative geometry, namely the *irrational rotation* C*-algebras or 'noncommutative tori' [6], which cannot be realised as topological spaces. Developing a remark in [12, 6.4], we show here how the simpler structure of an *equilogical space* [22] can still express some of those 'formal quotients' of spaces; more effective results will be obtained, in a sequel, with a *directed* version, preordered equilogical spaces.

An equilogical space $X = (X^\#, \sim)$ is a topological space $X^\#$ provided with an equivalence relation $\sim$. This notion becomes important once one defines a *map* of equilogical spaces $X \to Y$ to be a mapping $X^\#/\!\sim\; \to Y^\#/\!\sim$ which admits *some* continuous lifting $X^\# \to Y^\#$. The category **Eql** thus obtained contains **Top** as a full subcategory, identifying the space X with the obvious pair $(X, =_X)$: we are replacing **Top** with a larger category (studied in Section 1), having finer quotients. Notice that - to have this embedding - we are dropping the condition that the support $X^\#$ be a T_0-space, usually required for equilogical spaces (cf. 1.2).

Singular cubes have an obvious extension to equilogical spaces, as maps $\mathbf{I}^n \to X$ defined on the standard euclidean cube: in other words, we take the

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singular cubes of the space $X^\#$ and we identify them when they have the same projection to $X^\#/\sim$. Thus, *singular homology* is extended to equilogical spaces, enjoying similar properties (Sections 3 and 5). But, as shown in Section 4, the homology of an equilogical space X does *not* reduce to the homology of its underlying space $X^\#/\sim$ and can capture properties of the *formal* quotient $(X^\#, \sim)$ which would be missed by the *topological* quotient $X^\#/\sim$: the latter can be trivial while the homology groups $H_n(X)$ are not.

For instance, the subgroup $G_\vartheta = \mathbf{Z}+\vartheta\mathbf{Z} \subset \mathbf{R}$ (ϑ irrational) acts on the line by translations; being dense in the line, it has a *coarse* orbit space $\mathbf{R}/G_\vartheta$. Replacing this trivial space with the quotient cubical set $(\square\,\mathbf{R})/G_\vartheta$, *or equivalently with the equilogical space* $(\mathbf{R}, \equiv_{G_\vartheta})$, we have a non-trivial object, homologically equivalent to the torus $\mathbf{T}^2$ (4.5.2)

$$(1)\quad H_*(\mathbf{R}, \equiv_{G_\vartheta}) \;=\; H_*((\square\,\mathbf{R})/G_\vartheta) \;\cong\; H_*(\mathbf{T}^2).$$

The same result holds for the *equilogical space of leaves* of the Kronecker foliation of the torus, with slope ϑ (4.6). All this agrees with the *irrational rotation* C*-algebra A_ϑ [6, 7, 18, 19], which 'replaces' - in noncommutative geometry - the trivial quotient $\mathbf{R}/G_\vartheta$ and the trivial space of leaves of the foliation, and has the same complex K-groups as $\mathbf{T}^2$ (its definition is recalled in 4.4).

One should note that these models, the cubical sets and even more the equilogical spaces, have a clear geometrical derivation from the original problem - studying the orbit 'space' of the action of G_ϑ on $\mathbf{R}$ or the 'space' of leaves of the Kronecker foliation; much more evident than the corresponding C*-algebra. Without forgetting, however, that C*-algebras seem to cover a wider range of 'virtual spaces'. Note also that, while we have little direct intuition of the formal quotient $(\mathbf{R}, \equiv_{G_\vartheta})$, *homology explores it quite effectively* (cf. 4.5).

Maps of equilogical spaces fail to be *locally defined*: while in **Top** one can verify the continuity of a mapping on any *open cover* of the domain, this is no longer true in **Eql**. As related drawbacks of **Eql**, paths cannot be concatenated and there are different models of the circle, like the topological space $\mathbf{S}^1$ and the orbit equilogical space $(\mathbf{R}, \equiv_\mathbf{Z})$, whose distinction seems to be artificial. We introduce, in Section 2, an extended category **EqL** of equilogical spaces and *local maps* which: (a) makes the previous models of the circle isomorphic (and other models homotopy equivalent; 2.5); (b) allows us to concatenate the new paths (2.6); (c) produces the same homology (Thm. 3.5); (d) makes the topological and equilogical realisation of a cubical set homotopy equivalent (Thm. 5.7). One can keep to the original category **Eql** and use its extension **EqL** as a tool to define local homotopy equivalence and fundamental groups (2.6) in the former.

References for equilogical spaces can be found in 1.2. A *map* between spaces is a continuous mapping. $\mathbf{I} = [0, 1]$ is the standard euclidean interval. The index α takes values 0, 1. Category theory intervenes at an elementary level, essentially reduced to the basic properties of limits, colimits and adjoint functors (see [16]).

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1. Equilogical spaces and homotopy

Equilogical spaces are sort of 'formal quotients' of topological spaces. Homotopy is defined by the standard (topological) interval **I**.

1.1. Equilogical spaces. An *equilogical space* $X = (X^\#, \sim_X)$ will be a topological space $X^\#$ provided with an equivalence relation, written $\sim_X$ or $\sim$. (We are not assuming $X^\#$ to be T_0, cf. 1.2.) The space $X^\#$ will be called the *support* of X, while the quotient $|X| = X^\#/\sim$ is the *underlying space* (or *set*, according to convenience). One can think of the object X *as a set* $|X|$ *covered with a chart* $p: X^\# \to |X|$ containing the topological information.

A *map* of equilogical spaces $f: X \to Y$ (also called an *equivariant mapping* [22]) is a mapping $f: |X| \to |Y|$ which admits *some* continuous lifting $f': X^\# \to Y^\#$. It can also be defined as an equivalence class $[f']$ of continuous mappings $f': X^\# \to Y^\#$ *coherent* with the equivalence relations of X and Y

(1) $\forall\, x, x' \in X:\ \ x \sim_X x' \ .\!\Rightarrow\!.\ f'(x) \sim_Y f'(x')$,

under the associated *pointwise* equivalence relation

(2) $f' \sim f''$ if $(\forall\, x, x' \in X:\ \ x \sim_X x' \ .\!\Rightarrow\!.\ f'(x) \sim_Y f''(x'))$.

The name 'pointwise' looks more appropriate for an equivalent formulation: $f'(x) \sim_Y f''(x)$, for all x. But in the set $\mathbf{Top}(X^\#, Y^\#)$ of *all* continuous mappings between the supports, the two properties are no longer equivalent and (2) can express the whole definition: in this set $\sim$ is a *partial* equivalence relation (symmetric and transitive); the condition $f' \sim f'$ determines the coherent maps; the equivalence classes of the latter are the maps of $\mathbf{Eql}(X, Y)$.

The category **Eql** thus obtained contains **Top** as a full subcategory, identifying the space X with the obvious pair $(X, =_X)$. An equilogical space X is isomorphic to a topological space A if and only if A is a retract of $X^\#$, with a retraction $p: X^\# \to A$ whose equivalence relation is precisely $\sim_X$. We shall see that the new category has relevant new objects (Section 4).

The terminal object of **Eql** is the singleton space $\{*\}$. Therefore, a *point* $x: \{*\} \to X$ is an element of the underlying set $|X| = X^{\#}/\sim$ (*not* an element of the support $X^{\#}$). The (faithful) forgetful functor, with values in **Top** (or in **Set**, when convenient)

(3) $|-|$: **Eql** $\to$ **Top**, $\qquad |X| = X^{\#}/\sim$,

sends $f: X \to Y$ to the underlying mapping $f: |X| \to |Y|$ (also written $|f|$, more precisely). On the other hand, the 'function' $X \mapsto X^{\#}$ is *not* part of a functor, as it does not preserve isomorphic objects; indeed, one can often simplify the support by taking a suitable retract, as already remarked above (see also Proposition 5.9a).

In the category $\mathbf{Eql}_*$ of *pointed equilogical spaces*, an object (X, x_0) is an equilogical space equipped with a base point $x_0 \in |X|$; a map $f: (X, x_0) \to (Y, y_0)$ is a map $X \to Y$ in **Eql** whose underlying mapping $|X| \to |Y|$ takes x_0 to y_0.

1.2. Remarks. Equilogical spaces have been introduced in [22] using T_0-spaces as supports, so that they can be viewed as subspaces of algebraic lattices with the Scott topology (which is always T_0). The category so obtained - a full subcategory of the category **Eql** we are using here - is generally written as **Equ**. As a relevant, non obvious fact, **Equ** is *cartesian closed* (while **Top** is not): one can define an 'internal hom' Z^Y satisfying the exponential law $\mathbf{Equ}(X \times Y, Z) = \mathbf{Equ}(X, Z^Y)$; this has been proved in [1]; see also [2, 4, 20, 21].

Here, we prefer to drop the condition T_0, so that every topological space be an equilogical one. The category **Eql** can be obtained from **Top** by a general construction, as its *regular completion* $\mathbf{Top}_{\mathrm{reg}}$ [5]. This fact can be used to prove that also **Eql** is cartesian closed [21, p. 161].

Cartesian closedness is crucial in the theory of data types, where equilogical spaces originated; here it will play a marginal role: we are essentially interested in the (easy) fact that the path space X^I exists in **Eql**, and coincides with the topological one when X is in **Top** (1.5).

1.3. Limits. The category **Eql** has all limits and colimits. We recall briefly how they are constructed ('precisely' as in [1], for **Equ**); as well-known, we only need to consider products, equalisers and their duals.

Products and sums are obvious: a product ΠX_i is the product of the supports $X_i^{\#}$, equipped with the product of all equivalence relations; a sum (or coproduct) ΣX_i is the sum of the supports $X_i^{\#}$, with the sum of their equivalences.

Now, take two maps f, g: $X \to Y$. For their equaliser $E = (E^\#, \sim)$, take first the (set-theoretical or topological) equaliser E_0 of the underlying mappings f, g: $|X| \to |Y|$; then, the space $E^\#$ is the counterimage of E_0 in $X^\#$, with the restricted topology and equivalence relation; the map $E \to X$ is induced by the inclusion $E^\# \to X^\#$. For the coequaliser C of the same maps, let us form the coequaliser of the underlying mappings f, g: $|X| \to |Y|$ as a quotient $Y^\#/\sim_C$, by an equivalence relation coarser than $\sim_Y$. Then $C = (Y^\#, \sim_C)$, with the map $Y \to C$ induced by the identity of $Y^\#$ (and represented by the canonical projection $|Y| \to |C|$). Notice that we are using coequalisers *in* **Set** *rather than in* **Top** (the latter do not agree with products, which precludes cartesian closedness).

An (equilogical) *subspace*, or regular subobject $A = (A^\#, \sim)$ of X is a topological subspace $A^\# \subset X^\#$ *saturated* with respect to $\sim_X$, with the restricted equivalence relation. The order relation $A \subset B$ (of regular subobjects) amounts to $A^\# \subset B^\#$, or equivalently to $|A| \subset |B|$. We say that the equilogical subspace A is *open* (resp. *closed*) in X if $A^\#$ is open (resp. closed) in $X^\#$; or, equivalently, if the underlying set $|A|$ is open (resp. closed) in the space $|X|$.

An (equilogical) *quotient*, or regular quotient of X is the space $X^\#$ itself, equipped with a *coarser* equivalence relation. A map f: $X \to Y$ has a canonical factorisation through its *coimage* (a quotient of X) and its *image* (a subspace of Y)

(1) $\quad X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y,$

where $\mathrm{Coim}(f) = (X^\#, R)$ is determined by the equivalence relation associated to the composed mapping $X^\# \to |Y|$, while $(\mathrm{Im}(f))^\#$ is the counterimage of $f(|X|)$ in $Y^\#$. Plainly, Im(f) is 'contained' in a subspace B of Y if and only if *every* lifting $f': X^\# \to Y^\#$ has an image contained in $B^\#$.

The (faithful) forgetful functor $|-|$: **Eql** $\to$ **Top** (1.1.3) is left adjoint to the (full) embedding **Top** $\subset$ **Eql**

(2) **Top**($|X|$, T) = **Eql**(X, T) (X in **Eql**; T in **Top**),

since every map $|X| \to T$ can be lifted to $X^\#$. The left adjoint $|-|$ preserves colimits (obviously) and equalisers, but not products, while the embedding **Top** $\subset$ **Eql** preserves limits (obviously) and sums, but not coequalisers (see 1.4).

1.4. Circles and spheres. The category **Eql** has various (non isomorphic) *models of the circle*, i.e., objects whose associated space is homeomorphic to $\mathbf{S}^1$. Similar facts happen with other structures of common use in algebraic topology: simplicial complexes, simplicial sets, cubical sets. We will see that the models we consider here are equivalent up to 'local homotopy' (2.5).

First of all, we have the *topological circle* itself: $\mathbf{S}^1$ is the coequaliser *in* **Top** of the faces of the standard interval $\mathbf{I} = [0, 1]$

(1) $\partial^\alpha\colon \{*\} \rightrightarrows \mathbf{I}, \qquad \partial^\alpha(*) = \alpha \qquad (\alpha = 0, 1),$

and represents loops in **Top** (as maps $\mathbf{S}^1 \to X$); it also lives in **Eql**.

But the coequaliser *in* **Eql** of the faces of the interval is produced by the equivalence relation $R_{\partial\mathbf{I}}$ which identifies the endpoints

(2) $\mathbf{S}^1_e = (\mathbf{I}, R_{\partial\mathbf{I}})$ (the *standard equilogical circle*);

(R_A will often denote the equivalence relation identifying the points of a subset A).

A third model is the orbit quotient of the action of the group **Z** on **R**, in **Eql**

(3) $\bar{\mathbf{S}}^1_e = (\mathbf{R}, \equiv_{\mathbf{Z}})$.

Finally, we consider a sequence of models

(4) $C_k = (k\mathbf{I}, R_k)$ (the k-*gonal equilogical circle*),

where $k\mathbf{I} = \mathbf{I} + ... + \mathbf{I}$ (the sum of k copies) and R_k is the equivalence relation identifying the terminal point of any addendum with the initial point of the following one, circularly. *This can be pictured as circle with* k *corner points* (cf. 2.3), *or as a polygon* for $k \geq 3$; note that $C_1 = \mathbf{S}^1_e$.

There are obvious maps

(5) $... \to C_{k+1} \to C_k \to ... \to C_2 \to C_1 = \mathbf{S}^1_e \to \bar{\mathbf{S}}^1_e \to \mathbf{S}^1$

where $C_{k+1} \to C_k$ collapses the last 'edge'; their underlying map is (at least) a homotopy equivalence. But it is easy to see that any morphism in the opposite direction has an underlying map which is homotopically trivial. This situation will be further analysed below (2.3, 2.5).

Similarly, in dimension $n > 0$, we have the topological n-sphere $\mathbf{S}^n$ and

(6) $\mathbf{S}^n_e = (\mathbf{I}^n, R_{\partial\mathbf{I}^n})$ (the *standard equilogical n-sphere*),

(7) $\bar{\mathbf{S}}^n_e = (\mathbf{R}^n, \sim_n)$,

where the equivalence relation $\sim_n$ is generated by the congruence modulo $\mathbf{Z}^n$ and by identifying all points $(t_1,..., t_n)$ where at least one coordinate belongs to **Z**. Of course, $\mathbf{S}^0 = \mathbf{S}^0_e = (\{0, 1\}, =)$ has the discrete topology. We shall see that all the standard equilogical spheres are pointed suspensions of $\mathbf{S}^0$ (1.6).

1.5. Theorem [Internal homs]. Let A be a Hausdorff, locally compact topological space.

(a) The *equilogical exponential* Y^A, for Y in **Eql**, can be realised as

(1) $Y^A = (Y^{\#A}, \sim_E), \qquad h' \sim_E h'' \text{ if } (\forall\, a \in A,\ h'(a) \sim_Y h''(a)),$

where $Y^{\#A}$ is the topological exponential (i.e., the set of maps **Top**$(A, Y^\#)$ with the compact-open topology) and $\sim_E$ is the pointwise equivalence relation of maps $A \to Y^\#$ (1.1.2).

(b) If also Y is a topological space, the topological and equilogical exponentials Y^A coincide.

(c) For every equilogical space X, $|X{\times}A| = |X|{\times}A$.

(d) More generally, all this holds for every space A *exponentiable in* **Top**.

Proof. (Comments on cartesian closedness can be found in the last remark in 1.2.)

(a) First, let us recall that - in our hypotheses - the endofunctor $-{\times}A$: **Top** $\to$ **Top** has a right adjoint, the endofunctor $(-)^A$, with the obvious natural bijection

(2) φ: **Top**$(S{\times}A, T) \to$ **Top**(S, T^A) (*exponential law*),

$\varphi(f) = g, \qquad f(x, a) = g(x)(a) \qquad (x \in S,\ a \in A).$

Now, if X and Y are equilogical spaces, we can prove that the bijection φ (for their supports) induces a natural bijection ψ (for the equilogical spaces themselves)

(3) φ: **Top**$(X^\#{\times}A, Y^\#) \to$ **Top**$(X^\#, Y^{\#A})$, $\qquad \psi$: **Eql**$(X{\times}A, Y) \to$ **Eql**(X, Y^A).

Indeed, if $\varphi(f') = g'$ and $\varphi(f'') = g''$, the relations $f' \sim f''$ and $g' \sim g''$ (1.1.2) are equivalent

(4) $f' \sim f'' \text{ if } (\forall\, x, x' \in X,\ \forall\, a \in A:\ x \sim_X x' \ .\!\Rightarrow\!.\ f'(x, a) \sim_Y f''(x', a)),$

(5) $g' \sim g'' \text{ if } (\forall\, x, x' \in X:\ x \sim_X x' \ .\!\Rightarrow\!.\ g'(x) \sim_E g''(x')),$

(by definition of $\sim_E$, in (1)). Therefore, φ restricts to a bijection between the *coherent* maps, and then induces a bijection ψ between their equivalence classes.

(b) Follows immediately from (a).

(c) The functor $-{\times}A$ is a left adjoint and preserves coequalisers; therefore, in **Top**

(6) $|X{\times}A| = \mathrm{Coeq}(\sim_X{\times}A \rightrightarrows X{\times}A) = (\mathrm{Coeq}(\sim_X \rightrightarrows X)){\times}A = |X|{\times}A.$

(d) This point will not be used and the proof is omitted, for brevity (it can be found in [13]). For a characterisation of exponentiable topological spaces, see [10]. □

1.6. Elementary homotopy. We consider here an *elementary* notion of paths (and homotopies), as opposed to a more general notion of 'local paths', studied in Section 2. The latter are better behaved, but can be reduced to finite concatenations of the elementary ones, which give therefore a finer information.

An (elementary, or global) *path* in an equilogical space X is a map $a: \mathbf{I} \to X$; it has two endpoints *in the underlying space* $|X|$, $\partial^\alpha(a) = a(\alpha)$, and is called an (elementary) *loop* at $x_0 \in |X|$ when these points coincide with x_0. Loops of equilogical spaces are represented as maps $\mathbf{S}^1_e \to X$, defined on the equilogical circle; loops of *pointed* equilogical spaces correspond to pointed maps $(\mathbf{S}^1_e, [0]) \to (X, x_0)$, in $\mathbf{Eql}_*$ (1.1).

Notice that *paths cannot be concatenated*, generally; for instance, this is possible in $\mathbf{S}^1$ (of course) and in $\bar{\mathbf{S}}^1_e = (\mathbf{R}, \equiv_\mathbf{Z})$ (essentially because $\mathbf{R}$ is the universal covering of $\mathbf{S}^1$), but is not possible in $\mathbf{S}^1_e = (\mathbf{I}, R_{\partial\mathbf{I}})$ where paths cannot 'cross' the point $[0] = [1]$. Thus, the *path components* of an equilogical space X are produced by the equivalence relation $\simeq$ in $|X|$ *generated* by being endpoints of a path; $\pi_0(X) = |X|/\simeq$ will denote their set.

The standard interval $\mathbf{I}$ produces in $\mathbf{Eql}$ the *cylinder* functor I and its right adjoint, the *cocylinder* (or path functor) P, which extend the ones of $\mathbf{Top}$ (by 1.3 and 1.5)

(1) $I: \mathbf{Eql} \rightleftarrows \mathbf{Eql}: P, \qquad I(X) = X{\times}\mathbf{I}, \quad P(Y) = Y^\mathbf{I}.$

An (elementary, or global) *homotopy* $f: f_0 \to f_1: X \to Y$ in $\mathbf{Eql}$ is a map $f: X{\times}\mathbf{I} \to Y$ with faces $f{\circ}\partial^\alpha = f_\alpha$, where $\partial^\alpha = X{\times}\partial^\alpha: X \to X{\times}\mathbf{I}$. (A path in Y is a homotopy on the singleton $\{*\}$.) The adjoint functors $\mathbf{Eql} \rightleftarrows \mathbf{Top}$ (1.3.2) preserve homotopies (1.3, 1.5c). Again, homotopies in $\mathbf{Eql}$ cannot be concatenated, in general.

The embedding $\mathbf{Top} \subset \mathbf{Eql}$, preserving P and all limits, preserves also the homotopy limits (including homotopy fibres and loop spaces, in the pointed case). But coequalisers and homotopy colimits are not preserved, and we must distinguish between the *topological* mapping cone, cone and suspension, *in* $\mathbf{Top}$ (written Cf, CX, ΣX) and the corresponding *equilogical* constructs, *in* $\mathbf{Eql}$ (written C_ef, C_eX, Σ_eX), to avoid ambiguity when starting from topological data.

We are not going to study this theory, here. We shall only remark that the *pointed* suspension of a pointed *topological* space, in $\mathbf{Eql}_*$, can be realised as follows (it is a homotopy colimit, cf. [11])

(2) $\Sigma_e(X, x_0) = ((IX, R), [x_0, 0]),$

where R is the equivalence relation which collapses the subspace $(X{\times}\{0, 1\}) \cup$

$(\{x_0\}\times\mathbf{I})$. Applying Σ_e to the discrete space $\mathbf{S}^0 = \{0, 1\}$ (pointed at 0) we obtain the equilogical circle $\mathbf{S}^1_e$ (1.4.2) (while the topological circle $\mathbf{S}^1$ is a suspension in **Top**). Iterating the procedure (which works similarly on equilogical spaces), we obtain the higher equilogical n-spheres

(3) $\mathbf{S}^n_e = (\mathbf{I}^n, R_{\partial \mathbf{I}^n}) = \Sigma^n_e(\mathbf{S}^0, 0)$.

On the other hand, the *unpointed* suspension of $\mathbf{S}^0$ yields the 2-gonal circle C_2 (1.4.4).

All the models of the circle considered in 1.4 are distinct, also *up to homotopy equivalence*: this follows from a previous remark on the sequence 1.4.5, together with the fact that the forgetful functor $|-|$: **Eql** $\to$ **Top** preserves homotopies.

2. Local maps and fundamental groups

We introduce an extension of **Eql** which makes all models of spheres (of a given dimension) equivalent (2.5); the new paths can be concatenated and yield the fundamental groupoid of an equilogical space, or the fundamental group of a pointed one (2.6).

2.1. Local maps and local homotopies. An important feature of topology is the *local character* of continuity: a mapping between two spaces is continuous if and only if it is on a convenient neighbourhood of every point. This local character fails in **Eql**: for instance, the canonical map $(\mathbf{R}, \equiv_\mathbf{Z}) \to \mathbf{S}^1$ has a *topological inverse* $\mathbf{S}^1 \to \mathbf{R}/\equiv_\mathbf{Z}$ which cannot be lifted to a map $\mathbf{S}^1 \to \mathbf{R}$, but can be *locally lifted.*

This suggests us to extend **Eql** to the category **EqL** of equilogical spaces and *locally liftable mappings*, or *local maps*. A *local map* f: X $\twoheadrightarrow$ Y (the arrow is marked with a dot) is a mapping f: $|X| \to |Y|$ between the underlying sets which admits an *open saturated cover* $(U_i)_{i\in I}$ of the space $X^\#$ (by open subsets, saturated for $\sim_X$), so that the mapping f has a partial (continuous) lifting f_i: $U_i \to Y^\#$, for all i

(1) $f[x] = [f_i(x)]$, for $x \in U_i$ and $i \in I$.

Equivalently, for every point $[x] \in |X|$, the mapping f restricts to a map of equilogical spaces on a suitable saturated neighbourhood U of x in $X^\#$. The previous remark on the local character of continuity in **Top** has two consequences: the embedding **Top** $\subset$ **EqL** is (still) *full* and *reflective*, with reflector (left adjoint) $|-|$: **EqL** $\to$ **Top**.

It is easy to see that the *finite* limits and *arbitrary* colimits of **Eql** (as constructed in 1.3) still 'work' in the extension, which is thus cocomplete and finitely complete. A *local isomorphism* will be an isomorphism of **EqL**; a *local path* will be a local map $\mathbf{I} \twoheadrightarrow X$; a *local homotopy* will be a local map $X \times \mathbf{I} \twoheadrightarrow Y$, etc. Items of **Eql** will be called *global* (or *elementary*, for paths) when we want to distinguish them from the corresponding local ones.

2.2. Lemma [Coverings]. Let $p: S \to T$ be a surjective local homeomorphism, in **Top**, and R_p the corresponding equivalence relation on S.

(a) The induced map $(S, R_p) \to T$ is a *local isomorphism* of equilogical spaces.

(b) More generally, if R is an equivalence relation on S, coarser than R_p, and R' the induced equivalence relation on T, then the induced map $(S, R) \to (T, R')$ is a *local isomorphism*.

Proof. It suffices to prove (b). For each $y \in T$, we can choose some $x \in X$ and some open neighbourhood U of the latter such that $p(x) = y$ and the restriction $p': U \to p(U)$ be a homeomorphism with an open neighbourhood of y. This means that we can lift the inverse bijection $T/R' \to S/R$, locally at y, with the inverse homeomorphism $p(U) \to U \subset S$. □

2.3. Some local isomorphisms. Coming back to our models of the circle (1.4), the canonical map $\bar{\mathbf{S}}^1_e = (\mathbf{R}, \equiv_{\mathbf{Z}}) \to \mathbf{S}^1$ is *locally invertible* (2.2), and these models are locally isomorphic. This is not true, in the strict sense, of the canonical map $p: \mathbf{S}^1_e \to \bar{\mathbf{S}}^1_e$: the topological inverse $\mathbf{R}/\mathbf{Z} \to \mathbf{I}/\partial\mathbf{I}$ cannot be locally lifted at $[0]$; but we see below that a *local inverse up to homotopy* exists (2.5).

The fact that $\mathbf{S}^1_e$ and $\mathbf{S}^1$ be not locally isomorphic is consistent with viewing $\mathbf{S}^1_e = (\mathbf{I}, R_{\partial\mathbf{I}})$ as a circle with a corner point (at $[0]$), which elementary paths are not allowed to cross. Thus, elementary homotopy and elementary paths are able to capture properties of equilogical spaces which can be of interest, but are missed by local paths, fundamental groups (2.6) and singular homology (3.7), as well as by any functor invariant up to *local* homotopy.

Also in higher dimension, the canonical map $(\mathbf{R}^n, \sim_n) \to \mathbf{S}^n$ is locally invertible, while this is not true, in the strict sense, for $(\mathbf{I}^n, R_{\partial\mathbf{I}^n}) \to \mathbf{S}^n$ $(n > 0)$.

2.4. Lemma [The concatenation pushout]. Consider the following pushout in **Eql** (or **EqL**)

$$(1)\qquad \begin{array}{ccc} \{*\} & \xrightarrow{\partial^1} & \mathbf{I} \\ {\scriptstyle \partial^0}\downarrow & & \downarrow{\scriptstyle u^0} \\ \mathbf{I} & \xrightarrow[u^1]{} & \mathbf{J} \end{array} \qquad\qquad \begin{array}{l} \mathbf{J} = (\mathbf{I} + \mathbf{I}, R) \\ u^0(1)\, R\, u^1(0) \end{array}$$

which pastes two copies of $\mathbf{I}$ (R identifies the terminal point of the first copy with the initial point of the second). Then the canonical map p, with values in (a realisation of) the topological pushout

$$(2)\quad p: \mathbf{J} \to \mathbf{I}, \qquad p(u^0(t)) = t/2, \quad p(u^1(t)) = (t+1)/2,$$

is a local homotopy equivalence. (Note that $\mathbf{I}$ is a realisation of the underlying space $|\mathbf{J}| = \mathbf{J}/R$.)

Proof. (a) The points of $\mathbf{I} + \mathbf{I}$ will be written as $u^\alpha(t) = (t, \alpha)$, for $\alpha = 0, 1$. First, we construct a local map $k: \mathbf{I} \twoheadrightarrow \mathbf{J}$, by an underlying mapping with a stop at the 'pasting point' $[1, 0] = [0, 1]$

$$(3)\quad k: \mathbf{I} \to \mathbf{J}/R,$$

$$k(t) = \begin{cases} [3t, 0] & \text{if } 0 \leq t \leq 1/3, \\ [1, 0] = [0, 1] & \text{if } 1/3 \leq t \leq 2/3, \\ [3t-2, 1] & \text{if } 2/3 \leq t \leq 1, \end{cases} \qquad pk(t) = \begin{cases} 3t/2 \\ 1/2 \\ (3t-1)/2 \end{cases}$$

(so that k can be lifted on the open subsets $[0, 1/2[$, $]1/3, 2/3[$ and $]1/2, 1]$). The composite $pk: \mathbf{I} \to \mathbf{I}$ (computed at the right, above, on the same decomposition of $\mathbf{I}$) is homotopic to $\mathrm{id}\mathbf{I}$, in **Top** and **Eql**.

Finally, also the composite $kp: \mathbf{J} \to \mathbf{J}$ happens to be a *global map*, whose value at $[t, 0]$ or $[t, 1]$ is, respectively:

$$(4)\quad \begin{cases} [3t/2, 0] & \text{if } 0 \leq t \leq 2/3, \\ [1, 0] & \text{if } 2/3 \leq t \leq 1, \end{cases} \qquad \begin{cases} [0, 1] & \text{if } 0 \leq t \leq 1/3, \\ [(3t-1)/2, 1] & \text{if } 1/3 \leq t \leq 1, \end{cases}$$

since it can be lifted to a map $\mathbf{I} + \mathbf{I} \to \mathbf{I} + \mathbf{I}$ (with the same formulas as above, without square brackets). And it is homotopic to $\mathrm{id}\mathbf{J}$ (since its lifting is homotopic to the identity, in **Top**). □

2.5. Proposition [Local homotopy equivalences of spheres]. All the canonical maps linking the models of the circle (1.4.5)

$$(1)\quad \ldots \to C_{k+1} \to C_k \to \ldots \to C_2 \to C_1 = \mathbf{S}^1_e \to \bar{\mathbf{S}}^1_e \to \mathbf{S}^1$$

are *local homotopy equivalences*. The same holds for the higher spheres

(2) $\bar{\mathbf{S}}^n_e \rightarrow \mathbf{S}^n_e \rightarrow \mathbf{S}^n$ $\qquad (\bar{\mathbf{S}}^n_e = (\mathbf{R}^n, \sim_n), \quad \mathbf{S}^n_e = (\mathbf{I}^n, R_{\partial \mathbf{I}^n})).$

Proof. The arguments are similar to the preceding ones, and will only be sketched. For all the models of the circle we use one underlying set, the 1-dimensional torus $\mathbf{T} = \mathbf{R}/\mathbf{Z}$.

We know that the canonical map $p: \mathbf{S}^1_e \rightarrow \bar{\mathbf{S}}^1_e$ is not locally invertible, strictly. But we can build a local map $f: \bar{\mathbf{S}}^1_e \twoheadrightarrow \mathbf{S}^1_e$ so that the composites pf and fp be *locally homotopic* to the identities. For instance, take the following map between the underlying spaces **R/Z**, *locally constant* at [0] (and locally liftable everywhere)

(3) $f: \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{R}/\mathbf{Z}, \qquad f[t] = [0 \vee (3t-1) \wedge 1] \qquad (0 \leq t \leq 1),$

($\wedge$ and $\vee$ are the minimum and maximum in the line; brackets are not needed). Loosely speaking, our map stays at [0] for a third of the time, then runs around the circle for another third, and finally stays again at [0] for the last third. A local map $C_1 \twoheadrightarrow C_2$ is similarly constructed, with a mapping locally constant at [1/2], and so on for $C_k \twoheadrightarrow C_{k+1}$.

In the higher dimensional case, extending (3), a local map $\bar{\mathbf{S}}^1_e \twoheadrightarrow \mathbf{S}^n_e$ is obtained with a mapping locally constant at the point we want to map into $[0] \in \mathbf{I}^n/\partial\mathbf{I}^n$. Since $\bar{\mathbf{S}}^n_e$ and $\mathbf{S}^n$ are locally isomorphic, the argument is done. □

2.6. Concatenation. Let X be an equilogical space, and $a, b: \mathbf{I} \twoheadrightarrow X$ two consecutive local paths: $a(1) = x = b(0) \in |X|$. The concatenation pushout (2.4.1) defines a local map $c': \mathbf{J} \twoheadrightarrow X$ (whose underlying map is the ordinary concatenation of paths, in **Top**). We define the *concatenation* $c = a*b: \mathbf{I} \twoheadrightarrow X$ as the composite $c = c'k$ where $k: \mathbf{I} \twoheadrightarrow \mathbf{J}$ is the local map defined in Lemma 2.4.

In other words, the underlying mapping of c is defined in *three* steps (instead of the usual two)

$$(1) \quad c: \mathbf{I} \rightarrow |X|, \qquad c(t) = \begin{cases} a(3t) & \text{if } 0 \leq t \leq 1/3 \\ a(1) = b(0) & \text{if } 1/3 \leq t \leq 2/3 \\ b(3t-2) & \text{if } 2/3 \leq t \leq 1 \end{cases}$$

allowing for a stop at the concatenation point. One can show directly that this mapping is locally liftable (since, on the open subsets [0, 1/2[,]1/3, 2/3[,]1/2, 1] it essentially reduces to the given local paths, or to a constant mapping on the middle subset).

For n-cubes, one can define a concatenation in each direction i = 1,..., n, using the local map

(2) $k_i = \mathbf{I}^{i-1}\times k\times\mathbf{I}^{n-i}\colon \mathbf{I}^n \twoheadrightarrow \mathbf{I}^{i-1}\times\mathbf{J}\times\mathbf{I}^{n-i}$;

(this is implicitly used below, in dimension 2, for homotopies of paths).

We have thus the *fundamental groupoid* $\Pi_1(X)$ of an equilogical space: a vertex is a point $x \in |X|$ of the underlying set; an arrow $[a]\colon x \to y$ is an equivalence class of local paths from x to y, up to local homotopy with fixed endpoints. Associativity is proved in the classical way (with slight adaptations due to the particular form of (1)); as well as the existence of identities (the classes of constant paths) and inverses (reversing local paths, by precomposing with the reversion $\mathbf{I} \to \mathbf{I}$).

The endoarrows of $\Pi_1(X)$ at a point $x_0 \in |X|$ form the *fundamental group* $\pi_1(X, x_0)$. Globally, we have two functors, defined on equilogical spaces or on the pointed ones

(3) $\Pi_1\colon \mathbf{EqL} \to \mathbf{Gpd}$, $\qquad\qquad \pi_1\colon \mathbf{EqL}_* \to \mathbf{Gp}$,

which are invariant by local homotopies (pointed, in the second case) and extend the analogous functors for topological spaces. Local homotopy equivalence *can* reduce their calculation to the previous ones; for instance, for all our models of the circle, by 2.5; direct computations are also possible, by a version of the Seifert-van Kampen theorem (2.8).

2.7. Proposition [Local and global paths]. A local path $a\colon \mathbf{I} \twoheadrightarrow X$ is always a finite concatenation of elementary paths in X, up to local homotopy with fixed endpoints.

In particular, the path-components of X defined by the equivalence relation produced by elementary paths (1.6) coincide with the ones defined (directly) by local paths, and the functor $\pi_0\colon \mathbf{Eql} \to \mathbf{Set}$ is invariant by local homotopy.

Proof. If the mapping $a\colon \mathbf{I} \to |X|$ can be partially lifted to $X^\#$ on the open cover (U_i) of $\mathbf{I}$, choose a natural number k so that every interval $[(j-1)/k, j/k]$ $(1 \leq j \leq k)$ is contained in some U_i. Letting $a_j\colon \mathbf{I} \to |X|$ be the restriction of a to this interval, reparametrised on the standard one, we have a finite sequence of consecutive elementary paths $a_j\colon \mathbf{I} \to X$, whose concatenation is equivalent to a.

Therefore $\pi_0(X)$ is the set of components of the groupoid $\Pi_1(X)$, and is also invariant by local homotopy. □

2.8. Theorem ['Seifert - van Kampen']. Let the equilogical space X be *covered* *by the interiors of its equilogical subspaces* U, V: $X^\# = \mathrm{int}(U^\#) \cup \mathrm{int}(V^\#)$. If $U^\#$, $V^\#$ and $A^\# = U^\# \cap V^\#$ are path connected and $x_0 \in |A|$, the following square of homomorphisms induced by the inclusions is a pushout of groups

$$(1) \qquad \begin{array}{ccc} \pi_1(A, x_0) & \longrightarrow & \pi_1(U, x_0) \\ \downarrow & & \downarrow \\ \pi_1(V, x_0) & \longrightarrow & \pi_1(X, x_0) \end{array}$$

R. Brown's version for fundamental groupoids [3] can also be extended to equilogical spaces.

Proof. It is the same proof of the classical case, after adapting the subdivision procedure to local paths and local 2-cubes. Indeed, any cube $a: \mathbf{I}^n \to T$ (in **Top**) can be subdivided into a family of 2^n 'subcubes', indexed on the vertices $v \in \{0, 1\}^n$ of $\mathbf{I}^n$

$$(2) \quad sd(a) = (a.u_v), \qquad u_v: \mathbf{I}^n \to \mathbf{I}^n, \quad u_v(t) = (t + v)/2 \qquad (v \in \{0, 1\}^n);$$

and we can iterate this procedure, letting sd operate on each term of the family.

If the map a is a local cube $\mathbf{I}^n \twoheadrightarrow X$, there is an open cover (W_i) of $\mathbf{I}^n$ such that a has partial liftings $a_i: W_i \to X^\#$; we can always assume that all these partial liftings take place in $U^\#$ or in $V^\#$ (replacing the previous open subsets with the ones of type $a_i^{-1}(\mathrm{int}(U^\#))$ and $a_i^{-1}(\mathrm{int}(U^\#))$). Choosing a Lebesgue number for this cover, one deduces that there is some $k \in \mathbf{N}$ such that any 'subcube' of $\mathbf{I}^n$ with edge 2^{-k} is contained in some W_i. Therefore, all the 'subdivided' cubes of the family $sd^k(a)$ can be lifted to $U^\#$ or $V^\#$. □

3. Singular homology

Singular homology can be easily extended to equilogical spaces, to study the new objects. Less trivially, we prove that this homology can be equivalently computed by *local cubes* and deduce that it is also invariant under *local* homotopy equivalence.

3.1. Cubes and homology. As in Massey's text [17], we follow the cubical approach instead of the more usual simplicial one (the equivalence with the simplicial definition is proved by acyclic models, see [8, 14]). General motivations for preferring *cubes* essentially go back to the fact that cubes are closed under product, while tetrahedra are not; but a specific motivation will be our use of the natural order on the standard cube $I^n = [0, 1]^n$, in a subsequent paper. References for cubical sets can be found in [12].

Recall that the standard cubes I^n have *faces* δ_i^α and *degeneracies* ε_i (for $\alpha = 0, 1;\ i = 1,..., n$)

$$(1)\quad \delta_i^\alpha: I^{n-1} \to I^n, \qquad \delta_i^\alpha(t_1,..., t_{n-1}) = (t_1,..., t_{i-1}, \alpha, t_i,..., t_{n-1}),$$
$$\varepsilon_i: I^n \to I^{n-1}, \qquad \varepsilon_i(t_1,..., t_n) = (t_1,..., \hat{t}_i,..., t_n).$$

A topological space T has a cubical set $\square T$ of singular cubes, the maps $I^n \to T$; their faces and degeneracies are obtained by pre-composing with the faces and degeneracies of the standard cubes. More generally, an *equilogical* space X has a *cubical set of singular cubes* $\square X$, constructed in the same way in the category **Eql** (instead of **Top**)

$$(2)\quad \square X = ((\square_n X), (\partial_i^\alpha), (e_i)), \qquad \square_n X = \mathbf{Eql}(I^n, X) = (\square_n X^\#)/\sim_n,$$
$$\partial_i^\alpha: \square_n X \to \square_{n-1} X, \qquad \partial_i^\alpha(a) = a \circ \delta_i^\alpha,$$
$$e_i: \square_{n-1} X \to \square_n X, \qquad e_i(a) = a \circ \varepsilon_i \qquad (\alpha = 0, 1;\ i = 1,..., n);$$

therefore, a cube $I^n \to X$ is a mapping $I^n \to |X|$ which can be (continuously) lifted to $X^\#$; or also an equivalence class in the quotient of the set $\square_n(X^\#) = \mathbf{Top}(I^n, X^\#)$ (the n-cubes of the support $X^\#$), modulo the associated equivalence relation $\sim_n$ obtained by projecting such cubes along the canonical projection $X^\# \to |X| = X^\#/\sim$.

Recall also that an (abstract) cubical set K is a sequence of sets K_n, with faces $\partial_i^\alpha: K_n \to K_{n-1}$ and degeneracies $e_i: K_{n-1} \to K_n$ $(\alpha = 0, 1;\ i = 1,..., n)$, satisfying the well-known cubical relations (recalled in [12, 1.2]). Their category will be written as **Cub**.

We have defined in (2) a canonical embedding $\square : \mathbf{Eql} \to \mathbf{Cub}$, acting on a map $f: X \to Y$ of equilogical spaces in the obvious way

$$(3)\quad (\square_n f): \square_n X \to \square_n Y, \qquad (\square_n f)(a) = f \circ a \qquad (\text{for } a: I^n \to X).$$

This embedding produces the (normalised) *singular chain complex* of equilogical spaces and their *singular homology*:

$$(4)\quad C_*: \mathbf{Eql} \to \mathbf{C_*Ab}, \qquad C_*(X) = C_*(\square X),$$
$$H_n: \mathbf{Eql} \to \mathbf{Ab}, \qquad H_n(X) = H_n(\square X) = H_n(C_*(X)),$$

which extends the singular homology of topological spaces, but does *not* reduce to the homology of the underlying space $H_n(|X|)$ (see Section 4).

Using the wider category **EqL** of local maps (2.1), we have the *local cubes* a:

$I^n \twoheadrightarrow X$, the complex of *local chains* $CL_*(X)$ and the *local homology* groups $HL_n(X)$

(5) $\square L_nX = \mathbf{EqL}(I^n, X)$, $\qquad CL_*(X) = C_*(\square LX)$,

$HL_n\colon \mathbf{EqL} \to \mathbf{Ab}$, $\qquad HL_n(X) = H_n(CL_*(X))$.

We will prove that $HL_n(X)$ *coincides with the global homology* $H_n(X)$ (3.5).

3.2. Relative homology. *Relative homology* is defined for a *pair* (X, A), where X is an equilogical space and A an equilogical subspace (determined by a subspace $A^\# \subset X^\#$, *saturated* for $\sim_X$, with the restricted structure; 1.3); it is the homology of the quotient of the associated chain complexes

(1) $H_n\colon \mathbf{Eql}_2 \to \mathbf{Ab}$,

$H_n(X, A) = H_n(C_*(X, A))$, $\qquad C_*(X, A) = C_*(X)/C_*(A)$,

defined on the category $\mathbf{Eql}_2$ of *pairs of equilogical spaces* (in the previous sense). Of course, a map f: (X, A) → (Y, B) consists of a map f: X → Y in **Eql** which takes the regular subobject A into B; in other words, *any* lifting f': $X^\# \to Y^\#$ takes $A^\#$ into $B^\#$ (because these parts are saturated for the equivalence relations of X and Y). Homotopy is extended to $\mathbf{Eql}_2$, by the *relative cylinder*

(2) $I(X, A) = (X\times\mathbf{I}, A\times\mathbf{I})$,

where $A\times\mathbf{I}$ is easily seen to be an equilogical subspace of $X\times\mathbf{I}$ (a product always preserves regular subobjects). For each pair (X, A), the (natural) short exact sequence of chain complexes

(3) $C_*(A) \rightarrowtail C_*(X) \twoheadrightarrow C_*(X)/C_*(A)$

yields the (natural) exact homology sequence of the pair. Again, all this extends to $\mathbf{EqL}_2$.

We shall prove, for both theories (global and local), the Homotopy Invariance Theorem (3.3) and a Subdivision Lemma (3.4). We deduce from the latter the coincidence of global and local homology (3.5); from both, the fact that global homology is also invariant for local homotopy equivalence (3.6). Other 'classical' properties are deferred to Section 5.

3.3. Theorem [Homotopy Invariance]. Homotopic maps of pairs of equilogical spaces induce the same homomorphisms in homology. The same holds for *local homotopy* and *local homology*.

Proof. The classical (cubical) proof [17, II.4.1] extends without problems. Given a homotopy $f: \mathbf{I}{\times}X \to Y$ between $f_0, f_1: X \to Y$ in **Eql**, one constructs in the usual way a homotopy between the associated chain morphisms $C_*(X) \to C_*(Y)$

(1) $\varphi_n: C_n(X) \to C_{n+1}(Y), \qquad \varphi_n(a) = f{\circ}(\mathbf{I}{\times}a) \qquad (a: \mathbf{I}^n \to X).$

This is extended to relative homotopies in $\mathbf{Eql}_2$. The same works in $\mathbf{EqL}_2$, with the relative chain complexes $CL_*(X, A)$. □

3.4. Lemma [The Subdivision operator]. Every equilogical space X has a *subdivision operator*, a natural morphism of chain complexes defined as follows

(1) $Sd: C_*(X) \to C_*(X), \qquad Sd_n(a) = \Sigma_v\, a.u_v \qquad (v \in \{0, 1\}^n),$
$\quad u_v: \mathbf{I}^n \to \mathbf{I}^n, \qquad u_v(t) = (t + v)/2,$

which subdivides any n-dimensional cube into a chain with 2^n cubes, indexed on the vertices $v \in \{0, 1\}^n$ of $\mathbf{I}^n$. This morphism Sd is homotopic to the identity, by a chain homotopy $\varphi = (\varphi_n)$

(2) $\varphi_n: C_n(X) \to C_{n+1}(X), \qquad Sd_n - id = \partial_{n+1}\varphi_n + \varphi_{n-1}\partial_n,$
$\quad \varphi_n(a) = (-1)^{n+1}\, \Sigma_v\, a.\eta_v, \qquad \varphi_0 = 0 \qquad (v \in \{0, 1\}^n,\ n > 0),$

obtained by means of a suitable family of maps $\eta_v: \mathbf{I}^n \to \mathbf{I}^{n+1}$.

The same holds for the local chain complex, $CL_*(X)$: the subdivision operator Sd and the chain homotopy φ are constructed in the same way, and extend the previous ones.

Proof. It is the 'geometric part' of the classical construction used to prove the Subdivision Theorem, for cubical singular homology [17, II.7]. We single it out, to use it independently. □

3.5. Comparison Theorem [Global and local homology]. Let X be an equilogical space. The embedding $C_*(X) \subset CL_*(X)$ induces an isomorphism $H_n(X) \cong HL_n(X)$, natural for global maps.

Proof. We use subdivision in $CL_*(X)$ (3.4), to prove that the induced homomorphism $H_n(X) \to HL_n(X)$ is bijective.

Injectivity. Take a global cycle $z \in C_n(X)$ which annihilates in $HL_n(X)$: $z = \partial c$ for some local chain $c \in CL_{n+1}(X)$. There is an open cover (U_i) of $\mathbf{I}^{n+1}$ such that all the cubes of c have partial liftings $U_i \to X^\#$. Choosing a Lebesgue number for it, one deduces that there is some $k \in \mathbf{N}$ such that any 'subcube' of $\mathbf{I}^n$ with edge 2^{-k} is contained in some U_i, so that $c' = Sd^k(c) \in C_{n+1}(X)$. The composed chain homotopy ψ: id $\simeq$ Sd^k: $CL_*(X) \to CL_*(X)$ (obtained from φ: id $\simeq$ Sd, in 3.4) gives $c = c' - \partial\psi(c) - \psi\partial(c)$ in $CL_{n+1}(X)$. We conclude that $z = \partial c = \partial c' - \partial\psi(z)$ is also a boundary in $C_n(X)$, after taking into account the last remark in 3.4: the chain homotopy φ takes $C_*(X)$ into $C_*(X)$, whence also its composite ψ does.

Surjectivity. Take a local cycle $z \in CL_n(X)$: there is an open cover (U_i) of $\mathbf{I}^n$ such that all the cubes of z have partial liftings $U_i \to X^\#$ and some $k \in \mathbf{N}$ such that $z' = Sd^k(z) \in C_n(X)$. The composed chain homotopy ψ: id $\simeq$ Sd^k: $CL_*(X) \to CL_*(X)$ gives $z' - z = \partial\psi(z)$ in $CL_n(X)$, whence z' is a *global* cycle whose homology class in $CL_n(X)$ coincides with $[z]$. □

3.6. Corollary. Global homology is also invariant for local homotopy. More precisely, for all equilogical spaces X, Y:

(a) if the global maps $f, g: X \to Y$ are locally homotopic, $H_n(f) = H_n(g)$;

(b) if the global map $f: X \to Y$ is locally invertible (possibly up to local homotopy), then $H_n(f)$ is an isomorphism;

(c) if X, Y are locally homotopy equivalent, then $H_n(X) \cong H_n(Y)$.

Proof. Follows from Homotopy Invariance (3.3) and Comparison (3.5), using the following diagrams for (a) and (b)

(1)
$$\begin{array}{ccc} H_n(X) & \xrightarrow{\cong} & HL_n(X) \\ f_* \downarrow\downarrow g_* & & \downarrow f_* = g_* \\ H_n(Y) & \xrightarrow{\cong} & HL_n(Y) \end{array} \qquad\qquad \begin{array}{ccc} H_n(X) & \xrightarrow{\cong} & HL_n(X) \\ f_* \downarrow & & \downarrow \cong \\ H_n(Y) & \xrightarrow{\cong} & HL_n(Y) \end{array}$$
□

3.7. Computations. Recall that topological spaces keep their singular homology. The homology of an equilogical space can often be determined by the previous results, reducing it to a topological space equivalent up to local homotopy. Thus, all the models of the circle or of the n-sphere considered above (1.4) have the classical homology (by Prop. 2.5).

More interesting results, having no simple analogue in **Top**, are dealt with in the next section and deduced from the homology of groups. Finally, our homology can be computed directly, as we shall see in the last section.

4. Formal quotients and noncommutative geometry

Starting from a group action or a foliation on a space X, the *equilogical space of orbits* $(X, \equiv_G)$ or *leaves* $(X, \equiv_F)$ can express 'formal quotients' whose underlying topological space is trivial; this agrees with similar analyses in noncommutative geometry.

4.1. Actions. Let X be a topological space and G a group acting on it; the group is in additive notation, and the action is written as x+g (for $x \in X$ and $g \in G$). The following results show that the orbit space of the action can be *better represented* by the *orbit cubical set* $(\Box X)/G$, or also - under additional hypotheses - by the *orbit equilogical space* $(X, \equiv_G)$.

Recall that G acts *freely* if no operator of G has fixed points, except the neutral one; and acts *properly* if every point has a neighbourhood U such that all subsets U+g are disjoint, a much stronger condition. A classical result says that, if G acts *properly* on the *acyclic* space X (having the homology of the point), then the homology of the orbit space coincides with the homology of the group (cf. [15, IV.11.5])

(1) $H_*(X/G) \cong H_*(G)$.

Dropping acyclicity, there is a spectral sequence converging to $H_*(X/G)$ [15, 12]. But the condition of *proper* action is quite strong, and ordinary topology seems unable to deal with more general situations, where the orbit space X/G has a trivial topology. However, the previous result has been extended in [12, Thm. 3.3] to *free* actions on acyclic *cubical sets*. As a straightforward consequence, if the group G acts *freely* on the *acyclic space* X

(2) $H_*((\Box X)/G) \cong H_*(G)$,

since the cubical set $\Box X$ is also acyclic (its homology is the one of X, by definition of the latter) and - plainly - G acts freely on it as well.

An additional hypothesis will allow us to express (2) with the orbit equilogical space $(X, \equiv_G)$. We say that G acts *pathwise freely* on X if, whenever two paths a, b: $\mathbf{I} \to X$ have the same projection to the orbit space X/G, there is precisely one $g \in G$ such that a = b + g. Then, the same works for all pairs of n-cubes a, b: $\mathbf{I}^n \to X$; the 0-dimensional case shows that the action is free.

4.2. Proposition [Orbit equilogical spaces]. Let the group G act on the topological space X.

(a) If the action of G is proper, then it is pathwise free (4.1).

(b) If the action is pathwise free, the canonical surjection

(1) $(\square X)/G \to \square(X, \equiv_G) = (\square X)/(\equiv_G)_*$,

is a bijection (for a free action, also the converse holds) and

(2) $H_*(X, \equiv_G) = H_*((\square X)/G)$.

(c) If the action is pathwise free *and* the space X is acyclic, then

(3) $H_*(X, \equiv_G) = H_*((\square X)/G) \cong H_*(G)$.

Proof. For (a), let G act properly and let us take two paths a, b: $\mathbf{I} \to X$ such that $a(t) = b(t) + g(t)$, for some function $g: \mathbf{I} \to G$. Working locally at some $t_0 \in \mathbf{I}$, we choose a neighbourhood U of $b(t_0)$ such that all subsets U+g are disjoint. Since a and b are continuous, the point a(t) lies in $U+g(t_0)$ while b(t) lies in U, locally at t_0; therefore, a(t) lies in $U+g(t_0)$ and U+g(t), locally at t_0. But this means that the function $g: \mathbf{I} \to G$ is locally constant, whence constant. Point (b) is a rephrasing of the definition of pathwise free, and (c) follows immediately from 4.1.2. □

4.3. Examples. (a) Various *pathwise free* (non proper) actions will be obtained as follows: the space X is an (additive) topological group and G is a *totally disconnected* subgroup, acting on X by translations x+g. Indeed, if the paths a, b: $\mathbf{I} \to X$ have the same projection to X/G, their difference $a - b: \mathbf{I} \to G$ must be constant. (Then, the action is proper if and only if G is discrete.)

(b) As an example of *a free action which is not pathwise free*, take a (non trivial) group G acting on its underlying set X, equipped with the coarse topology. This space is contractible, hence acyclic. Therefore (by 4.1.2) we have $H_*((\square X)/G) = H_*(G)$, *while the orbit equilogical space gives here a trivial result*, $H_*(X, \equiv_G) = 0$. In particular, with $G = \mathbf{Z}/2$ and X the coarse space on two points, we find the homology of the infinite (real) projective space $\mathbf{P}^\infty$ (cf. [14, 23])

(1) $H_*((\square X)/G) \cong H_*(\mathbf{Z}/2) \cong H_*(\mathbf{P}^\infty)$.

4.4. Irrational rotation algebras. After recalling a well-known 'virtual space' of non-commutative geometry, we will show how it can be interpreted - more geometrically - as a cubical set or an equilogical space.

Consider the subgroup $G_\vartheta = \mathbf{Z}+\vartheta\mathbf{Z}$ (ϑ irrational) of the real line, acting on the latter by translation (not properly, of course). Since G_ϑ is dense in the line, the orbit space $\mathbf{R}/G_\vartheta$ has a trivial topology, the coarse one (the same happens to the space of leaves of the corresponding Kronecker foliation on the torus, see 4.6).

In noncommutative geometry, this quotient is 'interpreted' as the (noncommutative) C*-algebra A_ϑ generated by two unitary elements u, v under the relation $vu = \exp(2\pi i\vartheta).uv$, and called the *irrational rotation algebra* associated with ϑ, or also a *noncommutative torus* [6, 7, 18, 19]. Both its complex K-theory groups are isomorphic to $\mathbf{Z}^2$.

As in [19], one can give a more explicit description of $\mathcal{A}_\vartheta$, within the C*-algebra $B(L_2(\mathbf{T}))$ of bounded operators on the Hilbert space $L_2(\mathbf{T})$, for $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. 'Functions' on $\mathbf{T}$ are viewed as functions on $\mathbf{R}$ with $f(t) = f(t+1)$. Then $\mathcal{A}_\vartheta \subset B(L_2(\mathbf{T}))$ is the norm-closed $*$-subalgebra generated by two operators u, v (satisfying the previous relation): the translation $u(f) = f(t - \vartheta)$, corresponding to a rotation on $\mathbf{T}$, and $v(f) = kf$ with $k(t) = \exp(2\pi it)$. Note that the algebra of continuous functions $C(\mathbf{T})$ is embedded in $B(L_2(\mathbf{T}))$ identifying a function g with the operator $M_g(f) = gf$; then, $v = M_k$ generates $C(\mathbf{T})$ in $B(L_2(\mathbf{T}))$.

K-theory classifies these algebras, by proving that $K_0(A_\vartheta) \cong \mathbf{Z}+\vartheta\mathbf{Z}$ *as an ordered subgroup of* $\mathbf{R}$, a combined result of Pimsner-Voiculescu and Rieffel [18, 19]. It follows that A_ϑ and $A_{\vartheta'}$ are *strongly Morita equivalent* if and only if ϑ and ϑ' are equivalent modulo the *fractional* action (on the irrationals) of the group $GL(2, \mathbf{Z})$ of invertible integral 2x2 matrices [19, Thm. 4]

(1) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}.t = \dfrac{at+b}{ct+d}$ (a, b, c, d $\in \mathbf{Z}$; $ad - bc = \pm 1$),

The orbit of ϑ is its closure under the transformations $R(t) = t^{-1}$ and $T^{\pm 1}(t) = t \pm 1$, on $\mathbf{R}\setminus\mathbf{Q}$.

4.5. Cubical sets, equilogical spaces and irrational rotations. Similar results have been obtained in [12, 4.2a], replacing the line $\mathbf{R}$ with its singular cubical set $\square\mathbf{R}$ and applying the result recalled above (4.2).

In fact, G_ϑ acts *freely* on $\square\mathbf{R}$ (just translating the cubes) so that the homology of the cubical set $(\square\mathbf{R})/G_\vartheta$ is the same as the homology of the group $G_\vartheta \cong \mathbf{Z}^2$

(1) $H_*((\square\mathbf{R})/G_\vartheta) \cong H_*(G_\vartheta) \cong H_*(\mathbf{T}^2)$.

(For the last isomorphism, it suffices to note that $\mathbf{Z}^2$ acts *properly* on the plane, with orbit space the torus $\mathbf{T}^2$). Now, G_ϑ is totally disconnected, so that its action on the line is also *pathwise free* (4.3a); therefore, the same result holds for the orbit equilogical space $(\mathbf{R}, \equiv_{G_\vartheta})$ (by 4.2.3)

(2) $H_*(\mathbf{R}, \equiv_{G_\vartheta}) = H_*((\square\, \mathbf{R})/G_\vartheta) \cong H_*(\mathbf{T}^2)$.

Moreover, two generators of [a], [b] $\in H_1(\mathbf{R}, \equiv_{G_\vartheta}) \cong \mathbf{Z}^2$ and a generator [A] $\in H_2(\mathbf{R}, \equiv_{G_\vartheta}) \cong \mathbf{Z}$ are given by the following cycles [12, Thm. 4.8]

(3) $a, b\colon \mathbf{I} \to \mathbf{R}$, $\qquad a(t) = t, \quad b(t) = \vartheta t$,

(4) $A\colon [0, 1]^2 \to \mathbf{R}/G_\vartheta$, $\qquad A(t, t') = t\vartheta + t'$,

which yields a sort of 'homological correspondence' between the virtual space $(\mathbf{R}, \equiv_{G_\vartheta})$ and the torus $\mathbf{T}^2$, together with some geometric intuition of the former.

Algebraically, all this is in accord with the 'interpretation' of $\mathbf{R}/G_\vartheta$ as the C*-algebra A_ϑ, which has the same K-theory groups as the torus; but note that here we loose the order information, and we cannot recover ϑ, at any extent. (A deeper accord was obtained in [12, 4.2b], with a different cubical set, $C_\vartheta = (\square \uparrow\mathbf{R})/G_\vartheta$, constructed with the *order-preserving* cubes $\mathbf{I}^n \to \mathbf{R}$: its *directed* homology is able to recover the *ordered* group $\uparrow G_\vartheta$ up to isomorphism and the irrational number ϑ up to the previous equivalence relation, described in 4.4.1. This can also be obtained with an *ordered equilogical space*, $C'_\vartheta = (\mathbf{R}, \leq, \equiv_{G_\vartheta})$, as it will be shown in a sequel.)

4.6. Foliations on the torus. The C*-algebra A_ϑ is also used as an interpretation of a space of leaves with a trivial topology. And again, similar results can be obtained with the corresponding *equilogical space of leaves*.

The *Kronecker foliation* F' of the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, *with irrational slope* ϑ, is induced by the foliation $F = (F_\lambda)$ of the plane whose leaves are the lines of slope ϑ

(1) $F_\lambda = \{(x, y) \in \mathbf{R}^2 \mid y = \vartheta x + \lambda\}$ $\qquad (\lambda \in \mathbf{R})$.

The set $\mathbf{T}^2_\vartheta = \mathbf{T}^2/\equiv_{F'}$ of its leaves comes from the following equivalence relation $\equiv$ on $\mathbf{R}^2$

(2) $(x, y) \equiv (x', y') \Leftrightarrow y + k - \vartheta(x+h) = y' + k' - \vartheta(x'+h')$

(for some $h, k, h', k' \in \mathbf{Z}$)

$\Leftrightarrow (y - \vartheta x) - (y' - \vartheta x') \in \mathbf{Z} + \vartheta\mathbf{Z}$.

Each leaf is dense in $\mathbf{T}^2$, and the set $\mathbf{T}^2_\vartheta$ inherits the coarse topology. In [12, 4.3], we have shown that the quotient cubical set $\square\, \mathbf{T}^2/(\equiv_{F'})_*$ (identifying the cubes of the torus which have the same projection to $\mathbf{T}^2/\equiv_{F'}$) is isomorphic to the previous cubical set $\square\, \mathbf{R}/G_\vartheta$, whence again it has the same homology as $\mathbf{T}^2$.

Now, the argument used there can be adapted, to show that the equilogical spaces $(\mathbf{R}, \equiv_{G_\vartheta})$ and $(\mathbf{R}^2, \equiv)$ are isomorphic. The isomorphism is induced by the following maps (in **Top**):

$$(3)\quad i: \mathbf{R} \to \mathbf{R}^2, \qquad i(t) = (0, t),$$
$$p: \mathbf{R}^2 \to \mathbf{R}, \qquad p(x, y) = y - \vartheta x,$$

which are coherent with the equivalence relations of $(\mathbf{R}, \equiv_{G_\vartheta})$ and $(\mathbf{R}^2, \equiv)$

$$(4)\quad i(t) = (0, t) \equiv (0, t + h + k\vartheta) = i(t + h + k\vartheta),$$
$$p(x, y) - p(x', y') = (y - \vartheta x) - (y' - \vartheta x') \in \mathbf{Z} + \vartheta\mathbf{Z} \quad (\text{for } (x, y) \equiv (x', y'));$$

moreover, $pi = \mathrm{id}\mathbf{R}$ while ip induces the identity, because $ip(x, y) = (0, y - \vartheta x) \equiv (x, y)$.

Finally, using also the fact that the canonical projection $(\mathbf{R}^2, \equiv) \to (\mathbf{T}^2, \equiv_{F'})$ is a *local* isomorphism (by 2.2), the *equilogical set of leaves* $(\mathbf{T}^2, \equiv_{F'})$ gives:

$$(5)\quad H_*(\mathbf{T}^2, \equiv_{F'}) \cong H_*(\mathbf{R}^2, \equiv) \cong H_*(\mathbf{R}, \equiv_{G_\vartheta}) \cong H_*(\mathbf{T}^2).$$

4.7. Other applications. Extending our example, as in [12, 4.4], take an n-tuple of real numbers $\vartheta = (\vartheta_1,..., \vartheta_n)$, linearly independent on the rationals, and consider the additive subgroup $G_\vartheta = \Sigma_j \vartheta_j\mathbf{Z} \cong \mathbf{Z}^n$, acting freely on $\mathbf{R}$. (The previous case corresponds to the pair $(1, \vartheta)$.)

Again, the group G_ϑ is totally disconnected, and its action on the line is pathwise free; applying 4.2.3, we can extend the previous case to the new equilogical space $(\mathbf{R}, \equiv_{G_\vartheta})$

$$(1)\quad H_k(\mathbf{R}, \equiv_{G_\vartheta}) = H_k(\square\, \mathbf{R}/G_\vartheta) = H_k(G_\vartheta) = H_k(\mathbf{T}^n) = \mathbf{Z}^{\binom{n}{k}}.$$

The foliation F of $\mathbf{R}^n$, formed of the hyperplanes $\Sigma_j\, \vartheta_j x_j = \lambda$, induces a foliation F' on the n-dimensional torus; the equilogical space of leaves $(\mathbf{T}^n, \equiv_{F'})$ is locally isomorphic to $(\mathbf{R}, \equiv_{G_\vartheta})$ and has the same homology.

In all these cases, the homology of the equilogical quotients, $(\mathbf{R}, \equiv_{G_\vartheta})$ or $(\mathbf{T}^n, \equiv_{F'})$, is relevant, while the one of the underlying spaces, $\mathbf{R}/G_\vartheta$ or $\mathbf{T}^n/\equiv_{F'}$, is trivial. Other examples can be found in [12, 4.5].

5. Complements on homology

The remaining classical results are easily extended, including the Eilenberg-Steenrod definition of an abstract homology theory. We end considering the topological and equilogical realisations of a cubical set.

5.1. Remark. It will be useful to note a simple fact, which is less obvious than its topological analogue and might be overlooked. Given an equilogical space X and two *equilogical subspaces* U, V, one has the following relation between subcomplexes of $C_*(X)$

(1) $C_*U \cap C_*V = C_*(U \cap V)$.

Indeed, since $U^\#$ *is saturated in* X, a non-degenerate cube $a: \mathbf{I}^n \to X$ belongs to C_*U if and only if *every* lifting $a': \mathbf{I}^n \to X^\#$ has an image contained in $U^\#$.

5.2. Subdivision Theorem. Let X be an equilogical space and $\mathcal{A} = (A_i)_{i \in I}$ a family of parts of its support $X^\#$ whose interiors cover $X^\#$. Let $C_*(X; \mathcal{A}) \subset C_*(X)$ denote the subcomplex generated by those cubes $a: \mathbf{I}^n \to X$ which have some lifting $\mathbf{I}^n \to A_i$. Then, this embedding induces isomorphism in homology.

Proof. It is the obvious extension of the proof of the classical Subdivision Theorem, for cubical singular homology [17, II.7]. We have already extended the geometric construction, in the lemma on the subdivision $Sd: C_*(X) \to C_*(X)$ (3.4), showing that there is a chain homotopy $\varphi: id \simeq Sd$ (3.4.2). Note that, if $\hat{a}: \mathbf{I}^n \to X^\#$ is any lifting of a, $Im(\hat{a}.\eta_v) \subset Im(\hat{a})$ (with the notation of 3.4.2); it follows that φ is *consistent with* $\mathcal{A}$: it sends $C_n(X; \mathcal{A})$ into $C_{n+1}(X; \mathcal{A})$.

Now, a (global) cube $a: \mathbf{I}^n \to X$ has a continuous lifting $\hat{a}: \mathbf{I}^n \to X^\#$, producing an open cover $(\hat{a}^{-1}(A_i))_{i \in I}$ of $\mathbf{I}^n$; by the Lebesgue Lemma, there is some $k \in \mathbf{N}$ such that any 'subcube' of $\mathbf{I}^n$ with edge 2^{-k} is contained in some subset $\hat{a}^{-1}(A_i)$, so that the iterated subdivision $Sd^k(a)$ belongs to $C_*(X; \mathcal{A})$). Since Sd^k is also homotopic to the identity, by a homotopy consistent with $\mathcal{A}$, one shows that the homomorphism $H_n(C_*(X; \mathcal{A})) \to H_n(X)$ is injective and surjective, as in the classical proof (and much in the same way as in the previous proof of the Comparison Theorem, 3.5). □

5.3. Theorem [The Mayer-Vietoris sequence]. Let the equilogical space X be *covered by the interiors of its equilogical subspaces* U, V: $X^\# = int(U^\#) \cup int(V^\#)$. Then we have an exact sequence

$$(1) \quad \ldots\ H_n(U \cap V) \xrightarrow{(i_*,\, j_*)} (H_nU) \oplus (H_nV) \xrightarrow{[u_*,\, -v_*]} H_n(X) \xrightarrow{\Delta} H_{n-1}(U \cap V)\ \ldots$$

with the obvious meaning of brackets.

The maps $u: U \to X$, $v: V \to X$, $i: U \cap V \to U$, $j: U \cap V \to X$ are inclusions and the *connective* Δ is:

(2) $\Delta[c] = [\partial_n a]$, $\qquad c = a + b \qquad (a \in C_n(U),\ b \in C_n(V))$.

The sequence is natural, for a map $f: X \to X'$ which restricts to $U \to U'$, $V \to V'$.

Proof. As in the topological case, one applies Subdivision (5.2) to the cover $\mathcal{A} = (U^{\#}, V^{\#})$, together with the algebraic theorem of the 'long' exact homology sequence, for the usual short exact sequence of chain complexes

$$(3) \qquad 0 \longrightarrow C_*(U \cap V) \xrightarrow{(i_*,\, j_*)} (C_*U) \oplus (C_*V) \xrightarrow{[u_*,\, -v_*]} C_*(X; \mathcal{A}) \longrightarrow 0$$

The exactness of the latter needs one non-trivial verification, at its centre, and depends on a previous remark on $C_*(U \cap V)$ (5.1.1). Take $a \in C_nU$, $b \in C_nV$ and assume that $u_*(a) = v_*(b)$; therefore, each cube really appearing in a (and b) belongs to $C_*U \cap C_*V = C_*(U \cap V)$; globally, there is (one) normalised chain $c \in C_n(U \cap V)$ such that $i_*(c) = a$, $j_*(c) = b$. □

5.4. Theorem [Excision]. Let X be an equilogical space, with equilogical subspaces U, A such that the closure of the subspace $U^{\#}$ is contained in the interior of $A^{\#}$ (with respect to the space $X^{\#}$). Then the inclusion of $(X \setminus U, A \setminus U)$ into (X, A) induces isomorphisms in homology.

Proof. The subsets $(X \setminus U)^{\#} = X^{\#} \setminus U^{\#}$ and $A^{\#} \setminus U^{\#}$ are also saturated for $\sim_X$. Applying Subdivision to the family $\mathcal{A} = (A^{\#}, X^{\#} \setminus U^{\#})$ (whose interiors cover $X^{\#}$) and the Five Lemma, one reduces to considering the usual Noether isomorphism of chain complexes:

$$(1) \quad C_*(X; \mathcal{A})/C_*(A) = (C_*(A) + C_*(X \setminus U))/C_*(A)$$
$$= C_*(X \setminus U)/(C_*(A) \cap C_*(X \setminus U)) = C_*(X \setminus U)/(C_*(A \setminus U)).$$

where the last equality comes from 5.1.1: $C_*(A) \cap C_*(X \setminus U) = C_*(A \setminus U)$. □

5.5. Homology theories. A *homology theory for equilogical spaces* (and local maps) can be defined as a sequence of functors and natural transformations

(1) $H_n: \mathbf{EqL}_2 \to \mathbf{Ab}$, $\qquad \partial_n: H_n(X, A) \to H_{n-1}(A, \emptyset)$,

satisfying the Eilenberg-Steenrod axioms [9], in the form already verified for singular homology: Homotopy Invariance (for local homotopies, 3.3), Exactness

(3.2), Excision (5.4) and Dimension. It restricts on $\mathbf{Top_2}$ to a homology theory for topological spaces, and is therefore uniquely determined on all equilogical spaces having the local homotopy type of a finite CW complex.

In this sense, singular homology of equilogical spaces and local maps is a homology theory with integral coefficients, which extends the classical singular homology of topological spaces. One obtains a (co)homology theory with coefficients in an abelian group G, in the usual way

(2) $C_*(X, A; G) = C_*(X, A)\otimes G, \qquad C^*(X, A; G) = \mathrm{Hom}(C_*(X, A), G),$

and the Universal Coefficient Theorem holds (its proof being purely algebraic).

5.6. Realisations. A cubical set K has a well-known *geometric realisation*

(1) $\mathcal{R}: \mathbf{Cub} \to \mathbf{Top}, \qquad \mathcal{R}(K) = (\Sigma_a \mathbf{I}^{n(a)})/\!\sim,$

which is a colimit in **Top**, the pasting - along faces and degeneracies - of a copy of a standard cube $\mathbf{I}^{n(a)}$ for each cube a of K of dimension n(a). The equivalence relation $\sim$ is generated by identifying the points which corresponds themselves, along the mappings induced by faces (δ_i^α) and degeneracies (ε_i)

(2) $\delta_i^\alpha: \mathbf{I}^{n(b)} \to \mathbf{I}^{n(a)}$ (for $b = \partial_i^\alpha a$), $\qquad \varepsilon_i: \mathbf{I}^{n(a)} \to \mathbf{I}^{n(b)}$ (for $a = e_i b$).

(This colimit can be obtained as a *coend*, cf. [16]). The functor $\mathcal{R}$ is left adjoint to the functor $\square : \mathbf{Top} \to \mathbf{Cub}$ of singular cubical sets.

Similarly, we can construct the *equilogical realisation*, left adjoint to the functor $\square : \mathbf{Eql} \to \mathbf{Cub}$, by the 'same' colimit *in* **Eql**

(3) $\mathcal{E}: \mathbf{Cub} \to \mathbf{Eql}, \qquad \mathcal{E}(K) = (\Sigma_a \mathbf{I}^{n(a)}, \sim).$

Thus, the topological realisation is the space underlying the equilogical realisation

(4) $\mathcal{R}(K) = (\Sigma_a \mathbf{I}^{n(a)})/\!\sim \; = |\mathcal{E}(K)|,$

and the following theorem proves that these objects are homotopically equivalent in **EqL**.

5.7. Theorem [Comparing realisations]. The projection $p: \mathcal{E}(K) \to \mathcal{R}(K)$, induced by the identity of supports, is a local homotopy equivalence.

Proof. It is a generalisation of constructions we have already used in 2.5, and will be omitted, for brevity. It can be found in the Preprint [13]. □

5.8. Comments. One is clearly tempted to view the local homotopy equivalence of all equilogical circles $C_k = (k\mathbf{I}, R_k)$ (1.4.4) as a particular case of the preceding theorem. This is essentially true, but not straightforward.

Indeed, the *k-gonal cubical set* c_k (with k vertices and k edges) has an equilogical realisation which is isomorphic to C_k but has a much bigger support: $k\mathbf{I}^0 + 2k\mathbf{I}^1 + \ldots$ Of course, most of it is due to degenerate cubes. But, even restricting to the non-degenerate ones, we would get a support $k\mathbf{I}^0 + k\mathbf{I}^1$ with k singletons which are not in C_k (and are not necessary). The drawing below represents $4\mathbf{I}^0 + 4\mathbf{I}^1$, at the left, and the support of $C_4 = (4\mathbf{I}, R_4)$ at the right (with dashed lines for their equivalence relations)

(1) 

One can prove that, by similar reductions (omitting degenerate cubes and appropriate faces), we get an *isomorphic* equilogical realisation, so that c_k gives precisely C_k, and the cubical set generated by one cube of dimension n gives precisely $\mathbf{I}^n$ (see Proposition 5.9 in the Preprint [13]). Then, Thm. 5.7 shows that all C_k are locally homotopy equivalent to the topological realisation, $\mathbf{S}^1$.

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