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IVAN KOLÁR

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# A GENERAL POINT OF VIEW TO NONHOLONOMIC JET BUNDLES

*by Ivan KOLÁŘ*

**Résumé.** Un foncteur de jets d'ordre $r$ général sur des variétés fibrées est défini comme étant un sous-foncteur du $r$-ième prolongement non-holonome préservant les produits fibrés et contenant le $r$-ième prolongement holonome. Les foncteurs de jets sont caractérisés en termes d'algèbres de Weil. En utilisant ce modèle algébrique, nous classifions tous les foncteurs de jets du second ordre et en déduisons deux résultats géométriques dans le cas d'ordre supérieur.

Our starting point was the classical description of all product preserving bundle functors on the category $\mathcal{M}f$ of smooth manifolds and smooth maps in terms of Weil algebras, see [7] for a survey. In [8] we characterized in a similar way all fiber product preserving bundle functors on the category $\mathcal{FM}_m$ of fibered manifolds with $m$-dimensional bases and fiber preserving maps with local diffeomorphisms as base maps. As a partial result we obtained a complete description of all bundle functors on the product category $\mathcal{M}f_m \times \mathcal{M}f$ that preserve products in the second factor, where $\mathcal{M}f_m$ denotes the category of $m$-dimensional manifolds and local diffeomorphisms.

In [6] we introduced the general concept of an $r$-th order jet functor on $\mathcal{M}f_m \times \mathcal{M}f$ as a subfunctor of the $r$-th nonholonomic jet functor $\widetilde{J}^r$ that preserves products in the second factor and contains $J^r$. Then we clarified that these functors are in bijection with the invariant Weil subalgebras $A \subset \widetilde{\mathbb{D}}^r_m$ containing $\mathbb{D}^r_m$, where $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$ and $\widetilde{\mathbb{D}}^r_m = \widetilde{J}^r_0(\mathbb{R}^m, \mathbb{R})$. In Section 2 of the present paper we define an $r$-th order jet functor on $\mathcal{FM}_m$ as a fiber product preserving bundle

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subfunctor of the $r$-th nonholonomic prolongation that contains the $r$-th holonomic one. Our algebraic description implies immediately that there is a canonical bijection between the jet functors on $\mathcal{M}f_m \times \mathcal{M}f$ and $\mathcal{FM}_m$. To demonstrate the usefulness of our algebraic model, we prove in Section 3 that the only second order jet functors are $J^2$, $\widetilde{J}^2$ and the semiholonomic one $\overline{J}^2$. Then we deduce two geometric results for the higher order cases. In Section 4 we prove that the only third order jet bundle with the property that its three canonical projections to the first order jet bundle are geometrically independent is the whole third order nonholonomic jet bundle. In Section 5 we describe all jet subfunctors of the functor $\overline{J}^{r,r-1}$ of semiholonomic $r$-th order jets that are holonomic up to the order $r-1$.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [7].

## 1. The foundations

The primary result in this field reads that every product preserving bundle functor on the category $\mathcal{M}f$ is a Weil functor $T^A$, where $A$ is an arbitrary Weil algebra. Moreover, the natural transformations $T^A \to T^{\overline{A}}$ of two such functors are in bijection with the algebra homomorphisms $A \to \overline{A}$, [7].

For a bundle functor $G$ on $\mathcal{M}f_m \times \mathcal{M}f$, an $m$-dimensional manifold $M$ and a manifold $N$, we write, analogously to the jet case,

$$G_x(M,N) \quad \text{or} \quad G(M,N)_y \quad \text{or} \quad G_x(M,N)_y\,, \qquad x \in M, y \in N\,,$$

for the submanifold of $G(M,N)$ with the first projection $x$ or second projection $y$ or both, respectively. We say $G$ is of order $r$ in the first factor, if for every local diffeomorphisms $f_1, f_2 : M \to \overline{M}$ and every map $g : N \to \overline{N}$,

$$j^r_x f_1 = j^r_x f_2 \ \text{ implies } \ G_x(f_1,g) = G_x(f_2,g) : G_x(M,N) \to G_{\overline{x}}(\overline{M},\overline{N})\,,$$

$x \in M, \overline{x} = f_1(x) = f_2(x) \in \overline{M}$. We say $G$ preserves products in the second factor, if

$$G(M, N \times Q) = G(M,N) \times_M G(M,Q)\,,$$

where $Q$ is another manifold. In [8] we deduced that the bundle functors on $\mathcal{M}f_m \times \mathcal{M}f$ of order $r$ in the first factor and preserving products in the second factor are in bijection with the pairs $(A, H)$, where $A$ is a Weil algebra and $H : G^r_m \to \operatorname{Aut} A$ is a group homomorphism of the $r$-th jet group in dimension $m$ $G^r_m$ into the group of all algebra automorphisms of $A$. For every two pairs $M$, $N$ and $\overline{M}$, $\overline{N}$, every local diffeomorphism $f : M \to \overline{M}$ and every map $g : N \to \overline{N}$, $G(M, N)$ is an associated bundle to the $r$-th order frame bundle $P^rM$ of $M$

$$G(M,N) = P^rM[T^AN] \quad \text{and} \quad G(f,g) = P^rf[T^Ag]\,.$$

If $\overline{G} = (\overline{A}, \overline{H})$, $\overline{H} : G^r_m \to \operatorname{Aut} \overline{A}$ is another such functor, then the natural transformations $G \to \overline{G}$ are in bijection with the equivariant algebra homomorphisms $A \to \overline{A}$.

Consider a bundle functor $F$ on the category $\mathcal{FM}_m$. The definition of the order of $F$ is based on the concept of $(q, s, r)$-jet, $s \geq q \leq r$, [7]. We say that $F$ is of the order $(q, s, r)$, if for every two fibered manifolds $p : Y \to M$, $\overline{p} : \overline{Y} \to \overline{M}$ and every two $\mathcal{FM}_m$-morphisms $f, g : Y \to \overline{Y}$,

$$j^{q,s,r}_y f = j^{q,s,r}_y g \quad \text{implies} \quad Ff|F_yY = Fg|F_yY\,, \qquad y \in Y\,.$$

The integer $r$ is called the base order of $F$. In [8] we deduced that the fiber product preserving bundle functors on $\mathcal{FM}_m$ of the base order $r$ are in bijection with the triples $(A, H, t)$, where $A$ and $H$ are as above and $t : \mathbb{D}^r_m \to A$ is an equivariant algebra homomorphism. We have

$$FY = \{\{u, Z\} \in P^rM[T^AY], t_M(u) = T^Ap(Z)\}\,,$$

where $t_M : T^r_mM \to T^AM$ is the natural transformation induced by $t$ and the inclusion $P^rM \subset T^r_mM$ is taken into account. Further, $Ff : FY \to F\overline{Y}$ is the restriction and corestriction of $P^r\underline{f}[T^Af]$, $\underline{f}$ being the base map of $f$. Moreover, if $\overline{F} = (\overline{A}, \overline{H}, \overline{t})$ is another such functor, then the natural transformations $F \to \overline{F}$ are in bijection with the equivariant algebra homomorphisms $\mu : A \to \overline{A}$ satisfying $\overline{t} = \mu \circ t$.

Every bundle functor $F$ on $\mathcal{FM}_m$ induces a bundle functor $F_0$ on $\mathcal{M}f_m \times \mathcal{M}f$ by

$$F_0(M,N) = F(M \times N \to M),\ F_0(f,g) = F(f \times g)\,, \\ f : M \to \overline{M}\,,\ g : N \to \overline{N}\,,$$

where $M \times N \to M$ is the product fibered manifold and $f \times g$ is interpreted as an $\mathcal{FM}_m$-morphism $(M \times N \to M) \to (\overline{M} \times \overline{N} \to \overline{M})$. If we have a fiber product preserving bundle functor $F = (A, H, t)$, then

$$F_0 = (A, H)\,.$$

## 2. Jet functors on $\mathcal{FM}_m$

The construction of the bundle $J^r(M,N)$ of $r$-jets of an $m$-dimensional manifold $M$ into a manifold $N$ can be interpreted as a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$, [7]. We have $J^r = (\mathbb{D}^r_m, C)$, where the canonical action $C$ of $G^r_m$ on $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$ is the jet composition, [8]. C. Ehresmann introduced the bundle $\widetilde{J}^r(M,N)$ of nonholonomic $r$-jets of $M$ into $N$, [4]. Even $\widetilde{J}^r$ is a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$. In [3] we deduced $\widetilde{J}^r = (\widetilde{\mathbb{D}}^r_m, \widetilde{C})$, where the action $\widetilde{C}$ of $G^r_m$ on $\widetilde{\mathbb{D}}^r_m = \widetilde{J}^r_0(\mathbb{R}^m, \mathbb{R})$ is the composition of nonholonomic jets, [4]. According to [3],

$$\widetilde{\mathbb{D}}^r_m = \mathbb{D}^1_m \underbrace{\otimes \cdots \otimes}_{r\text{-times}} \mathbb{D}^1_m\,. \tag{1}$$

In [6] we defined the general concept of an $r$-th order jet functor $G$ on $\mathcal{M}f_m \times \mathcal{M}f$. This is a subfunctor

$$J^r \subset G \subset \widetilde{J}^r$$

that preserves products in the second factor. By Section 1, these functors are in bijection with the $G^r_m$-invariant Weil subalgebras $A$ of $\widetilde{\mathbb{D}}^r_m$ satisfying

$$\mathbb{D}^r_m \subset A \subset \widetilde{\mathbb{D}}^r_m\,.$$

Write $C_A$ for the restriction of the action $\widetilde{C}$ to $A$. Obviously,

$$G = (A, C_A).$$

The injection $i_A : \mathbb{D}^r_m \to A$ is also equivariant, so that $A$ induces a functor $(A, C_A, i_A)$ on $\mathcal{FM}_m$. One verifies easily that the functor

$$G_h = (A, C_A, i_A)$$

on $\mathcal{FM}_m$ coincides with the horizontal extension of $G$ introduced in [6]. In this sense, the classical $r$-jet prolongation of fibered manifolds coincides with $J^r_h$ and the $r$-th nonholonomic prolongation $J^1_h \underbrace{\circ \cdots \circ}_{r\text{-times}} J^1_h$ coincides with $\widetilde{J}^r_h$. To distinguish clearly the bundle functors on $\mathcal{M}f_m \times \mathcal{M}f$ and $\mathcal{FM}_m$, we shall use this somewhat nonstandard notation in what follows.

Now we are in position to introduce the general concept of an $r$-th order jet functor on $\mathcal{FM}_m$.

**Definition.** *A bundle functor $F$ on $\mathcal{FM}_m$ is said to be an $r$-th order jet functor, if $J^r_h \subset F \subset \widetilde{J}^r_h$ and $F$ preserves fiber products.*

By Section 1, we have $F_0 = (A, C_A)$ and

$$F = (A, C_A, i_A).$$

This implies the following general result, which has been known as an observation in several concrete cases.

**Proposition 1.** *Every $r$-th order jet functor $F$ on $\mathcal{FM}_m$ is the horizontal extension $F = (F_0)_h$ of its restriction $F_0$ to $\mathcal{M}f_m \times \mathcal{M}f$.* □

*Remark.* For every functor on $\mathcal{M}f_m \times \mathcal{M}f$ of the form $G = (A, H)$, we can define its vertical extension $G_v$ on $\mathcal{FM}_m$ as

$$G_v = (A, C_A, \nu_A),$$

where $\nu_A : \mathbb{D}^r_m \to A$ is the zero homomorphism. One verifies easily

$$G_v Y = \bigcup_{x \in M} G_x(M, Y_x), \qquad Y_x = p^{-1}(x),$$

see [2] for more details. In the case $G$ is a jet functor, this concept coincides with that one introduced in another way in [6].

## 3. The classification of second order jet functors

The second semiholonomic prolongation $\overline{J}^2_h Y$ of a fibered manifold $Y$ is a subbundle of $\widetilde{J}^2_h Y$ characterized by $J^1_h\beta = \beta_1$, where $\beta : J^1_h Y \to Y$ and $\beta_1 : \widetilde{J}^2_h Y \to J^1_h Y$ are the target jet projections and $J^1_h\beta : \widetilde{J}^2_h Y \to J^1_h Y$ is the induced map. Further one defines $\overline{J}^2(M,N) = \overline{J}^2_h(M \times N \to M)$. Obviously, $\overline{J}^2$ or $\overline{J}^2_h$ is a jet functor on $\mathcal{M}f_m \times \mathcal{M}f$ or $\mathcal{FM}_m$, respectively, and $\overline{J}^2_h$ is the horizontal extension of $\overline{J}^2$.

The natural transformations $\beta_1$ and $J^1_h\beta$ corresponds to the product projections

$$q_1, q_2 : \widetilde{\mathbb{D}}^2_m = \mathbb{D}^1_m \otimes \mathbb{D}^1_m \to \mathbb{D}^1_m \,.$$

The relation $q_1 = q_2$ characterizes the subalgebra $\overline{\mathbb{D}}^2_m \subset \widetilde{\mathbb{D}}^2_m$ corresponding to $\overline{J}^2$.

**Lemma 1.** *If $A \subset \widetilde{\mathbb{D}}^2_m$ is an invariant subalgebra satisfying $\mathbb{D}^2_m \subset A$ and $q_1|A \neq q_2|A$, then $A = \widetilde{\mathbb{D}}^2_m$.*

*Proof.* Let $1, e^i$ be the standard algebraic generators of $\mathbb{D}^2_m$ and $1, e^i_1$ or $1, e^i_2$ be the standard algebraic generators of the first or second factor of $\mathbb{D}^1_m \otimes \mathbb{D}^1_m$. The injection $\mathbb{D}^2_m \subset \widetilde{\mathbb{D}}^2_m$ is characterized by $e^i \mapsto e^i_1 + e^i_2$. Each element $X \in \widetilde{\mathbb{D}}^2_m$ is of the form

$$X = x_0 + x_{i0}e^i_1 + x_{0i}e^i_2 + x_{ij}e^i_1 e^j_2$$

and we have

$$q_1(X) = x_0 + x_{i0}e^i_1 \,, \quad q_2(X) = x_0 + x_{0i}e^i_2 \,.$$

By equivariancy, each homogenous component of $X \in A$ belongs to $A$ as well. Hence

$$x_{i0}e^i_1 + x_{0i}e^i_2 \in A \,.$$

On the other hand, $x_{i0}(e^i_1 + e^i_2) \in \mathbb{D}^2_m \subset A$. Take an element $X \in A$ satisfying $q_1(X) \neq q_2(X)$. Then

$$0 \neq (x_{i0} - x_{0i})e^i_2 \in A \,.$$

This vector can be transformed by $GL(m,\mathbb{R}) \subset G^2_m$ into an arbitrary non-zero vector of $\mathbb{R}^{m*}$. Hence each $e^i_2 \in A$. Then $e^i_1 + e^i_2 \in A$ implies $e^i_1 \in A$. Since $1, e^i_1$ and $e^i_2$ are the algebraic generators of $\widetilde{\mathbb{D}}^2_m$, we have $A = \widetilde{\mathbb{D}}^2_m$. □

**Proposition 2.** *All second order jet functors are $J^2$, $\overline{J}^2$ and $\widetilde{J}^2$.*

*Proof.* By Lemma 1, we have to discuss the case $q_1 = q_2$. The second order component $X_2$ of $X \in A$ is a bilinear form. Hence we have $X_2 = s(X_2) + a(X_2)$, where $s$ or $a$ denotes the symmetric or antisymmetric part. If $a(X_2) = 0$ for all $X \in A$, we have $A = \mathbb{D}^2_m$. Assume there is an element $X \in A$ with $a(X_2) \neq 0$. By the algebraic Darboux lemma, there exists a basis $\overline{e}^i$ of $\mathbb{R}^{m*}$ such that

$$a(X_2) = \overline{e}^1 \wedge \overline{e}^2 + \cdots + \overline{e}^{2k-1} \wedge \overline{e}^{2k}.$$

Consider the linear transformation $\overline{e}^1_1 \mapsto c\overline{e}^1$, $\overline{e}^i \mapsto \overline{e}^i$, $i = 2, \ldots m$, $0 \neq c \in \mathbb{R}$. By equivariancy,

$$c\overline{e}^1 \wedge \overline{e}^2 + \cdots + \overline{e}^{2k-1} \wedge \overline{e}^{2k} \in A.$$

Hence $\overline{e}^1 \wedge \overline{e}^2 \in A$. This implies $\overline{e}^1_1 \overline{e}^2_2 \in A$ and, by equivariancy, $e^i_1 e^j_2 \in A$ for all $i \neq j$. So we have $A = \overline{\mathbb{D}}^2_m$. □

## 4. The third order nonholonomic case

By omitting $r-1$ terms in the tensor product (1), we obtain $r$ algebra homomorphisms

$$q_s : \widetilde{\mathbb{D}}^r_m \to \mathbb{D}^1_m, \qquad s = 1, \ldots, r.$$

We are going to show that already in the third order we have a more subtle situation than in Lemma 1.

The algebraic generators of $\widetilde{\mathbb{D}}^3_m$ are $1, e^i_1, e^i_2, e^i_3$. The part containing the linear combinations of the products of two or three $e$'s will be called the second or third order terms of an element $X \in \widetilde{\mathbb{D}}^3_m$. Hence

$$X = x_0 + x_i e^i_1 + y_i e^i_2 + z_i e^i_3 + \quad \text{second and third order terms.}$$

For every $t \in \mathbb{R}$, we define $(\widetilde{\mathbb{D}}^3_m)_t \subset \widetilde{\mathbb{D}}^3_m$ as the set of all elements satisfying

$$z_i = tx_i + (1-t)y_i\,.$$

One verifies easily that $(\widetilde{\mathbb{D}}^3_m)_t$ is an invariant subalgebra containing $\mathbb{D}^3_m$.

Write $\widetilde{J}^3_t$ for the corresponding jet functor on $\mathcal{M}f_m \times \mathcal{M}f$. It is instructive to describe its horizontal extension $(\widetilde{J}^3_t)_h$ geometrically. The projections

$$q_s : \widetilde{\mathbb{D}}^3_m \to \mathbb{D}^1_m\,, \qquad s = 1,2,3$$

induce, for every fibered manifold $Y$, three projections $q_{sY} : \widetilde{J}^3_h Y \to J^1_h Y$. It is well known that $J^1_h Y \to Y$ is an affine bundle. Hence $Z \in \widetilde{J}^3_h Y$ belongs to $(\widetilde{J}^3_t)_h Y$ if and only if $q_{3Y}(Z)$ is the affine combination

$$q_{3Y}(Z) = tq_{1Y}(Z) + (1-t)q_{2Y}(Z)$$

of $q_{1Y}(Z)$ and $q_{2Y}(Z)$. Thus, the condition $q_1|A$, $q_2|A$ and $q_3|A$ are different for an invariant subalgebra $\mathbb{D}^3_m \subset A \subset \widetilde{\mathbb{D}}^3_m$ does not imply $A = \widetilde{\mathbb{D}}^3_m$.

It is easy to verify that for an arbitrary Weil algebra $B$, the set $\mathrm{Hom}(B, \mathbb{D}^1_m)$ of all algebra homomorphisms is an affine subspace of the vector space $\mathrm{Lin}(B, \mathbb{D}^1_m)$ of all linear maps of $B$ into $\mathbb{D}^1_m$.

**Proposition 3.** *Let $\mathbb{D}^3_m \subset A \subset \widetilde{\mathbb{D}}^3_m$ be an invariant subalgebra. If $q_s|A$, $s = 1,2,3$, do not lie on the same straight line in* $\mathrm{Hom}(A, \mathbb{D}^1_m)$*, then $A = \widetilde{\mathbb{D}}^3_m$.*

*Proof.* Omitting one term in $\widetilde{\mathbb{D}}^3_m = \mathbb{D}^1_m \otimes \mathbb{D}^1_m \otimes \mathbb{D}^1_m$, we obtain 3 algebra homomorphisms

$$q_{12}, q_{13}, q_{23} : \widetilde{\mathbb{D}}^3_m \to \widetilde{\mathbb{D}}^2_m\,.$$

Since all $q_s|A$ are different, Lemma 1 implies $q_{st}(A) = \widetilde{\mathbb{D}}^2_m$ in all three cases. Write $A_1$ for the first order part of $A$. Hence each $X \in A_1$ is of the form

$$X = x_i e^i_1 + y_i e^i_2 + z_i e^i_3\,.$$

If $x_i = y_i = z_i$ for all $X \in A_1$, we have $A_1 = (\widetilde{\mathbb{D}}^3_m)_1$, which corresponds to the one-semiholonomic jets from [6]. If in $A$ there is an

(2)
$$X = x_i e^i_1 + x_i e^i_2 + z_i e^i_3 , \qquad (x_i) \neq (z_i) ,$$

then $\mathbb{D}^3_m \subset A$ implies $(z_i - x_i)e^i_3 \in A$. By equivariancy, each $e^i_3 \in A$ and then each $e^i_1 + e^i_2 \in A$. If all elements of $A_1$ are of the type (2), then $q_1|A = q_2|A$.

Now it is easy to see that it suffices to discuss the case

$$x_i e^i_1 + y_i e^i_2 + z_i e^i_3 \in A_1 , \qquad (x_i) \neq (y_i) \neq (z_i) \neq (x_i) .$$

Then

$$(y_i - x_i)e^i_2 + (z_i - x_i)e^i_3 \in A_1 . \tag{3}$$

Since $q_{23}(A) = \widetilde{\mathbb{D}}^2_m$, by Lemma 1 there exist some $(u_i)$ and $(v_i)$ satisfying

$$u_i e^i_1 + (y_i - x_i)e^i_2 \in A_1 , \quad v_i e^i_1 + (z_i - x_i)e^i_3 \in A_1 . \tag{4}$$

If $(u_i + v_i) \neq 0$, then $(u_i + v_i)e^i_1 \in A_1$ implies each $e^i_1 \in A_1$. Further, by (4) each $e^i_2, e^i_3 \in A_1$, so that $A = \widetilde{\mathbb{D}}^3_m$ .

Assume $(u_i + v_i) = 0$ is the only possibility in (4). Then the induced linear map $A_1 \to \big(q_{23}(A)\big)_1$ has zero kernel. In other words, every $y_i e^i_2 + z_i e^i_3 \in \big(q_{23}(A)\big)_1 = \mathbb{R}^{m*} \times \mathbb{R}^{m*}$ determines a unique $x_i e^i_1 \in \mathbb{R}^{m*}$ with the property $x_i e^i_1 + y_i e^i_2 + z_i e^i_3 \in A_1$. This map $\mathbb{R}^{m*} \times \mathbb{R}^{m*} \to \mathbb{R}^{m*}$ is $GL(m, \mathbb{R})$-equivariant. The standard methods from [7] yield

$$x_i = k_1 y_i + k_2 z_i , \qquad k_1, k_2 \in \mathbb{R} .$$

Since $\mathbb{D}^3_m \subset A$, we have $k_1 + k_2 = 1$. Hence $q_1|A$, $q_2|A$ and $q_3|A$ lie on the same straight line. □

The bundle version of Proposition 3 reads: if $F$ is a third order jet functor and the restrictions $q_s|F$, $s = 1, 2, 3$ do not lie on the same straight line in the above sense, then $F = \widetilde{J}^3$.

## 5. The subbundles of $\overline{J}^{r,r-1}$

Let $\overline{J}^r$ be the $r$-th order semiholonomic jet functor and $\pi^r_{r-1} : \overline{J}^r \to \overline{J}^{r-1}$ be the canonical projection, [4]. In [5] we defined

$$\overline{J}^{r,r-1}(M,N) = \left\{X \in \overline{J}^r(M,N), \pi^r_{r-1}(X) \in J^{r-1}(M,N)\right\} ,$$

see also [9]. This is an $r$-th order jet functor on $\mathcal{M}f_m \times \mathcal{M}f$. In [5] we deduced that for every $X \in \overline{J}^{r,r-1}_x(M,N)_y$ there is a unique $S(X) \in J^r_x(M,N)_y$ satisfying

(5) $\quad S(X) \circ Z = X \circ Z \in J^r_0(\mathbb{R},N)_y \quad$ for all $\quad Z \in J^r_0(\mathbb{R},M)_x .$

Clearly, $S : \overline{J}^{r,r-1} \to J^r$ is a natural transformation.

Let $1, e^i_1, \ldots, e^i_r$ be the canonical algebraic generators of $\widetilde{\mathbb{D}}^r_m$. If we write $E^i = e^i_1 + \cdots + e^i_r$, then the Weil algebra $\overline{\mathbb{D}}^{r,r-1}_m$ of $\overline{J}^{r,r-1}$ is the subalgebra of $\widetilde{\mathbb{D}}^r_m$ of the form

(6) $\quad X = x_0 + x_i E^i + \cdots + x_{i_1 \ldots i_{r-1}} E^{i_1} \ldots E^{i_{r-1}} + x_{i_1 \ldots i_r} e^{i_1}_1 \ldots e^{i_r}_r .$

This defines an identification

$$\overline{\mathbb{D}}^{r,r-1}_m = \mathbb{D}^{r-1}_m \times \overset{r}{\otimes} \mathbb{R}^{m*} .$$

If we write analogously $\mathbb{D}^r_m = \mathbb{D}^{r-1}_m \times S^r \mathbb{R}^{m*}$, the algebra version $S : \overline{\mathbb{D}}^{r,r-1}_m \to \mathbb{D}^r_m$ of (5) is

$$S(b,B) = \big(b, \mathrm{Sym}(B)\big) , \qquad b \in \mathbb{D}^{r-1}_m , \ B \in \overset{r}{\otimes} \mathbb{R}^{m*} .$$

In particular, every $X = (b,B) \in \overline{\mathbb{D}}^{r,r-1}_m$ is of the form

$$X = S(X) + K , \qquad K = B - \mathrm{Sym}(B) \in \overset{r}{\otimes} \mathbb{R}^{m*} .$$

By (6), for another $Z \in \overline{\mathbb{D}}^{r,r-1}_m$, $Z = S(Z) + L$, we have

(7) $\quad XZ = \big(S(X) + K\big)\big(S(Z) + L\big) = S(XZ) + x_0 L + z_0 K .$

This formula implies directly

**Lemma 2.** *If $P \subset \overset{r}{\otimes}\mathbb{R}^{m*}$ is a linear subspace containing $S^r\mathbb{R}^{m*}$, then $\mathbb{D}_m^{r-1} \times P$ with the multiplication (7) is a subalgebra of $\overline{\mathbb{D}}_m^{r,r-1}$.*

Write $\overline{C}$ for the action of $G_m^r$ on $\overline{\mathbb{D}}_m^{r,r-1}$ determined by the functor $\overline{J}^{r,r-1}$. Using the coordinate formula for the composition of semi-holonomic jets, [1], we find that for each $X + K \in \overline{\mathbb{D}}_m^{r,r-1}$, $X \in \mathbb{D}_m^r$, $K \in \overset{r}{\otimes}\mathbb{R}^{m*}$ and $g \in G_m^r$,

$$\overline{C}(g)(X + K) = C(g)(X) + l(g_1)(K)\,, \tag{8}$$

where $g_1 = \pi_1^r(g) \in G_m^1 = GL(m, \mathbb{R})$ and $l$ is the tensor action of $GL(m, \mathbb{R})$ on $\overset{r}{\otimes}\mathbb{R}^{m*}$. If the subspace $P$ is $GL(m, \mathbb{R})$-invariant, then (7) implies that $\mathbb{D}_m^{r-1} \times P$ is an invariant subalgebra of $\overline{\mathbb{D}}_m^{r,r-1}$ containing $\mathbb{D}_m^r$. Conversely, if $\mathbb{D}_m^{r-1} \times P \subset \overline{\mathbb{D}}_m^{r,r-1}$ is an $G_m^r$-invariant subalgebra containing $\mathbb{D}_m^r$, then $P \subset \overset{r}{\otimes}\mathbb{R}^{m*}$ is an $GL(m, \mathbb{R})$-invariant linear subspace containing $S^r\mathbb{R}^{m*}$. Thus, we have proved

**Proposition 4.** *The jet subbundles of $\overline{J}^{r,r-1}$ are in bijection with the $GL(m, \mathbb{R})$-invariant linear subspaces of $\overset{r}{\otimes}\mathbb{R}^{m*}$ containing $S^r\mathbb{R}^{m*}$.*

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IVAN KOLÁŘ, DEPARTMENT OF ALGEBRA AND GEOMETRY, FACULTY OF SCIENCE, MASARYK UNIVERSITY, JANÁČKOVO NÁM. 2A, 662 95 BRNO, CZECH REPUBLIC

*E-mail address*: kolar@math.muni.cz