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HOMOTOPY COLIMITS AND COHOMOLOGY WITH LOCAL COEFFICIENTS

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RESUME. Les auteurs décrivent la structure des ensembles simpliciaux de Eilenberg-MacLane généralisés comme colimite et utilisent cette représentation pour obtenir une démonstration élémentaire du fait qu'ils représentent la cohomologie singulière à coefficients locaux.

1 Introduction

For any given abelian group A and natural number n, the n^{th} Eilenberg-Mac Lane simplicial set $K(A,n)$ represents the n^{th} (singular) cohomology group functor with coefficients in A in the sense that for every simplicial set X the corresponding cohomology group $H^n(X,A)$ can be calculated in terms of (homotopy classes of) maps $X \to K(A,n)$ (see [3]),

$$H^n(X,A) \cong [X, K(A,n)]. \tag{1}$$

For any simplicial set X (indeed, even for any space, although we shall limit ourselves here to the case of simplicial sets) more general cohomology groups can be calculated if one is given, instead of just one abelian group A, a *system of local coefficients* for X, that is, a system of abelian groups associated with the fundamental groupoid, $\Pi_1(X)$, of X in such a way as to determine a $\Pi_1(X)$-module or functor

$$A : \Pi_1(X) \to \mathbf{Ab},$$

to the category $\mathbf{Ab}$ of abelian groups.

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Also in this general case of cohomology with local coefficients it would be possible to establish representation theorems analogous to (1), although a general description of the representing simplicial sets may become quite complicated (see [5], [7]). The role that the Eilenberg-Mac Lane simplicial sets play in the constant coefficient case is played, in the local coefficient case, by fibrations with fibres Eilenberg-Mac Lane simplicial sets. Baues called those fibrations *generalized Eilenberg-Mac Lane spaces* (see [1]).

The main purpose of this paper is to give an explicit description of those fibrations, in the context of simplicial sets. We use a particular representation of homotopy colimits due to Bousfield and Kan, to prove first, that for each n the generalized n^{th} Eilenberg-Mac Lane space of a local coefficient system $A : \Pi_1(X) \to \mathbf{Ab}$ can be defined as the canonical fibration associated to the homotopy colimit of the corresponding "local coefficient Eilenberg-Mac Lane functor", $K(A,n)$ and, second, that the simplicial sets so defined represent the local coefficient cohomology groups $H^n(X,A)$.

Briefly, we see that there is a canonical split fibration

$$L_X(A,n) \;= \text{hocolim}_{\Pi_1(X)}\left(K(A,n)\right) \longrightarrow \text{Ner}\left(\Pi_1(X)\right),$$

whose fibre on any object $x \in \text{Ner}\left(\Pi_1(X)\right)$ is the simplicial set $K(A(x),n)$, and then we give a specific isomorphism

$$H^n(X,A) \;\cong\; [X, L_X(A,n)]. \tag{2}$$

(See section 4 and Corollary 4.6 for the precise statement.)

We shall begin by fixing our notation and briefly reminding the reader of the definition of the homotopy colimit of (a diagram) of simplicial sets, recalling Bousfield and Kan's description of the same and stating the essential properties that will be used later on. Those properties motivate our definition of the generalized Eilenberg-Mac Lane simplicial sets, which is given in section 3. In this section 3 we also establish the basic lemmas which lead to the main theorems. Finally, in section 4, we obtain, as an immediate corollary of a general theorem, the required representation theorem establishing (2).

2 Homotopy colimits of simplicial sets

We shall denote **Sset** the category of simplicial sets, $\mathbf{Sset} = \mathbf{Set}^{\Delta^{\mathrm{op}}}$. The *nerve* functor Ner : **Cat** $\to$ **Sset** associates to each small category $\mathcal{C}$ the simplicial set whose 0-cells are the objects of $\mathcal{C}$, and whose m-cells ($m \geq 1$) are the length-m chains of arrows of $\mathcal{C}$. If $\xi = [x_0 \xrightarrow{u_1} x_1 \xrightarrow{u_2} \cdots \xrightarrow{u_m} x_m]$ is an m-cell in Ner($\mathcal{C}$), its i^{th} face ($0 \leq i \leq m$) is obtained by "dropping" the object x_i (so, for example, if $1 \leq i \leq m-1$ then $d_i\xi = [x_0 \xrightarrow{u_1} \cdots \xrightarrow{u_{i-1}} x_{i-1} \xrightarrow{u_{i+1}u_i} x_{i+1} \xrightarrow{u_{i+2}} \cdots \xrightarrow{u_m} x_m]$), while the degeneracies are defined by $s_i\xi = [x_0 \xrightarrow{u_1} \cdots \xrightarrow{u_i} x_i \xrightarrow{1} x_i \xrightarrow{u_{i+1}} \cdots \xrightarrow{u_m} x_m]$.

The nerve functor has a left adjoint ("categorisation"), a fact which implies that it preserves all limits. Since the category of small groupoids is a reflective subcategory of **Cat**, "nerve of a groupoid" also has a left adjoint ("fundamental groupoid"). So, restricting our attention to groupoids, we have an adjoint pair,

$$\Pi_1 : \mathbf{Sset} \rightleftarrows \mathbf{Gpd} : \mathrm{Ner}, \; \Pi_1 \dashv \mathrm{Ner}.$$

Given a small category $\mathcal{C}$, let us denote $\mathrm{N}_\mathcal{C} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sset}$ the composition of Ner with the functor $\mathcal{C}^{\mathrm{op}} \xrightarrow{(-)/\mathcal{C}} \mathbf{Cat}$ taking an object $x \in \mathcal{C}$ to the slice category $x/\mathcal{C}$, that is, $\mathrm{N}_\mathcal{C}(x) = \mathrm{Ner}(x/\mathcal{C})$. If now we are given a functor $F : \mathcal{C} \to \mathbf{Sset}$ the homotopy colimit of F is the coend of the functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{N}_\mathcal{C} \times F} \mathbf{Sset}$, that is,

$$\mathrm{hocolim}_\mathcal{C}\, F = \int^{x \in \mathcal{C}} \mathrm{Ner}(x/\mathcal{C}) \times F(x). \tag{3}$$

Example 2.1. *Let F be the terminal object in $\mathbf{Sset}^\mathcal{C}$, that is, the constant functor $\mathcal{C} \xrightarrow{1} \mathbf{Sset}$ whose image is the one-point (terminal) simplicial set. Then the homotopy colimit of F is the nerve of $\mathcal{C}$:*

$$\mathrm{hocolim}_\mathcal{C}\, 1 = \mathrm{Ner}(\mathcal{C}).$$

As a consequence, the homotopy colimit of every small diagram of shape $\mathcal{C}$ has a canonical map to the nerve of $\mathcal{C}$:

$$\mathrm{hocolim}_\mathcal{C}\, F \xrightarrow{\ell} \mathrm{Ner}(\mathcal{C}), \qquad \ell = \mathrm{hocolim}_\mathcal{C}(F \to 1). \tag{4}$$

There is a description of the homotopy colimit which we will find useful in this paper. According to [2], the homotopy colimit of a small diagram, $F : \mathcal{C} \longrightarrow$ **Sset**, can be obtained as the diagonal of the *simplicial replacement* $\Psi(F)$ of F,

$$\mathrm{hocolim}_{\mathcal{C}} F \cong \mathrm{Diag}(\Psi(F)), \tag{5}$$

see [2] Chapt. 12, sec. 5.2, and [10].

According to this description, the set of m-cells of $\mathrm{hocolim}_{\mathcal{C}} F$ can be represented as:

$$(\mathrm{hocolim}_{\mathcal{C}} F)_m = \coprod_{\xi \in \mathrm{Ner}(\mathcal{C})_m} F(d_1 \cdots d_m \xi)_m = \coprod_{x_0 \xrightarrow{u_1} x_1 \xrightarrow{u_2} \cdots \xrightarrow{u_m} x_m \in \mathrm{Ner}(\mathcal{C})_m} F(x_0)_m,$$

while its faces and degeneracies are given in terms of those of the $F(x)$ and of $\mathrm{Ner}(\mathcal{C})$ as:

$$d_i(\xi, a) = \begin{cases} (d_0\xi, F(u_1)(d_0 a)) & \text{if } i = 0 \\ (d_i\xi, d_i a) & \text{if } i > 0, \end{cases} \tag{6}$$

(where $\xi = [x_0 \xrightarrow{u_1} x_1 \xrightarrow{u_2} \cdots \xrightarrow{u_m} x_m]$ and $a \in F(x_0)_m$), and

$$s_i(\xi, a) = \big(s_i\xi, s_i a\big). \tag{7}$$

It is immediate to verify that, with this description of the homotopy colimit, the map ℓ is given by $\ell(\xi, a) = \xi$, and to deduce the following known result:

Proposition 2.2. *For any small category $\mathcal{C}$ and any functor $F : \mathcal{C} \longrightarrow$ **Sset**, the fibre, at any object $x \in \mathcal{C}$, of the canonical map*

$$\ell : \mathrm{hocolim}_{\mathcal{C}} F \longrightarrow \mathrm{Ner}(\mathcal{C})$$

is the simplicial set $F(x)$.

Proof. Let us consider the situation at dimension m. We must identify the set $F(x)_m$ with the set of all those $(\xi, a) \in (\mathrm{hocolim}_{\mathcal{C}} F)_m$ such that $\xi = [x \xrightarrow{1_x} x \longrightarrow \cdots x \xrightarrow{1_x} x]$, and show that the faces and degeneracies of

$F(x)$ agree with those of $\text{hocolim}_{\mathcal{C}} F$. Now, in such pairs (ξ, a), a is arbitrary as long as $a \in F(x)$, and ξ is unique. On the other hand, the agreement of the faces d_i for $i > 0$ and of all the degeneracies is evident from (6) and (7). It only remains to consider the zero faces: since $\xi = [x \xrightarrow{1_x} x \longrightarrow \cdots x \xrightarrow{1_x} x]$, we have $d_0(\xi, a) = \big(d_0\xi, F(1_x)(d_0 a)\big) = \big(d_0\xi, d_0 a\big)$. □

The description (5) of the homotopic colimit is very useful. It is used by Thomason to prove that the homotopy colimit of the nerve of $F : \mathcal{C} \longrightarrow \mathbf{Cat}$ is homotopically equivalent to the nerve of the Grothendieck construction on F (see [9]). Another advantage of this description of the homotopy colimit is that it makes it evident that, when $\mathcal{C}$ is a groupoid, this construction preserves an important property shared by the nerves of groupoids and the Eilenberg-Mac Lane simplicial sets, namely the existence of a natural number n (which is $n = 1$ in the case of the nerve of a groupoid) for which the following property holds,

(P_n) For every $m > n$, if a number $k \in \{0, \ldots, m\}$ is given together with an m-tuple $(a_0, \ldots, a_{k-1}, a_{k+1}, \ldots, a_m)$ of $(m-1)$-cells, which are "k-compatible" (in the sense that for $i < j$ and $i, j \neq k$, $d_i a_j = d_{j-1} a_i$), then there exists a unique m-cell a such that for $i \neq k$, $d_i a = a_i$.

This property is usually expressed by saying that *for every* $m > n$ *the* m*-cells are open horns*, meaning that for any k the canonical map from the m-cells to the k-open horns in dimension m is a bijection. The sense in which homotopy colimits preserve this property is expressed in the following proposition:

Proposition 2.3. *If* $\mathcal{C}$ *is a groupoid and* $F : \mathcal{C} \longrightarrow \mathbf{Sset}$ *is a functor such that for a certain* $n \geq 1$ *every simplicial set* $F(x)$ *(with* x *any object of* $\mathcal{C}$*) satisfies* (P_n) *then also the simplicial set* $\text{hocolim}_{\mathcal{C}} F$ *satisfies* (P_n) *for the same value of* n*.*

Proof. Let $m > n$, $k \in \{0, \ldots, m\}$ and suppose that an m-tuple

$$\big((\xi_0, a_0), \ldots, (\xi_{k-1}, a_{k-1}), (\xi_{k+1}, a_{k+1}), \ldots, (\xi_m, a_m)\big)$$

of k-compatible cells in $(\mathrm{hocolim}_{\mathcal{C}} F)_{m-1}$ is given. Then

$$(\xi_0, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_m)$$

is an m-tuple of k-compatible cells in $\mathrm{Ner}(\mathcal{C})_{m-1}$ and therefore, since the nerve of a groupoid satisfies (P_1), it determines an m-cell ξ in $\mathrm{Ner}(\mathcal{C})$ such that $\xi_i = d_i\xi$. Let $\xi = [x_0 \xrightarrow{u_1} x_1 \xrightarrow{u_2} \cdots \xrightarrow{u_m} x_m]$, then $a_0 \in F(x_1)_{m-1}$ and for $i \neq 0$, $a_i \in F(x_0)_{m-1}$. It is a simple exercise to verify that the m-tuple

$$\big(F(u_1)^{-1}(a_0), a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_m\big)$$

of elements of $F(x_0)_{m-1}$ is k-compatible. Thus, it determines an m-cell $a \in F(x_0)_m$ such that $d_0 a = F(u_1)^{-1}(a_0)$ and for $i > 0$ $(i \neq k)$, $d_i a = a_i$. With this we have obtained an m-cell $(\xi, a) \in (\mathrm{hocolim}_{\mathcal{C}} F)_m$, and it is immediate to verify that $d_i(\xi, a) = (\xi_i, a_i)$, for all $i = 0, \ldots, m$, $i \neq k$. This proves the existence. Uniqueness is evident. □

The fact that (P_n) holds for a given simplicial set K (and in particular for the generalized Eilenberg Mac Lane spaces $L_{\mathcal{C}}(A, n)$ that we deal with below) has the following two main consequences that are essential for our proof establishing the isomorphism (2):

In the first place, a simplicial map $f : X \longrightarrow K$ to such a simplicial set is completely determined by its n-truncation $\mathrm{tr}_n f = (f_0, \ldots, f_n)$. Evidently the maps in this truncation satisfy conditions of compatibility with faces and degeneracies which characterize them as a truncated map, but in this case the component f_n has an additional property called "cocycle condition" which is, in fact, a necessary and sufficient condition for a truncated map $(f_0, \ldots, f_n) : \mathrm{tr}_n X \longrightarrow \mathrm{tr}_n K$ to extend to a simplicial map from X to K. There are, in fact, $n+2$ equivalent ways of expressing the cocycle condition: for any $k \in \{0, \ldots, n+1\}$ the cocycle condition can be stated as *"the unique map $f_{n+1} : X_{n+1} \longrightarrow K_{n+1}$ such that the equation $d_i f_{n+1} = f_n d_i$ holds for all $i \in \{0, \ldots, n+1\}$ but $i \neq k$, is a map that verifies also $d_k f_{n+1} = f_n d_k$.* Briefly: if (P_n) holds for K, the key to define a simplicial map to K is to give a truncated map to $\mathrm{tr}_n K$ with a "good" n-component (see Proposition 3.3).

Secondly, in a similar way to what happens with the simplicial maps to K, to give a homotopy $h : f \rightarrow g$ between simplicial maps to K is equivalent to giving a truncated homotopy

$$\langle (h_0^0), (h_0^1, h_1^1), \ldots, (h_i^{n-1})_{i=0}^{n-1} \rangle : \mathrm{tr}_{n-1}(f) \to \mathrm{tr}_{n-1}(g) \tag{8}$$

with "good" $n-1$-dimensional components $h_0^{n-1}, \ldots, h_{n-1}^{n-1}$. In this case, a necessary and sufficient condition for a truncated homotopy such as (8) to extend to a homotopy from f to g is called the "homotopy condition" (see Proposition 4.3). Again, this condition simply expreses the fact that the maps $(h_0^n, \ldots, h_n^n)$ uniquely determined by the equations also satisfy the remaining conditions for a homotopy.

3 The generalized Eilenberg-Mac Lane simplicial sets

As we indicated in the introduction we shall adopt the following definition, which, in some sense, enlarges the classical definition of the Eilenberg-Mac Lane simplicial sets. Here, and in the remaining of the paper $\mathcal{C}$ will be a fixed groupoid, A will be a fixed $\mathcal{C}$-module, that is, a functor $A : \mathcal{C} \to \mathbf{Ab}$, and n will be a fixed natural number.

Definition 3.1. *The* n^{th} Eilenberg-Mac Lane simplicial set of $\mathcal{C}$ with coefficients in A, *denoted* $L_{\mathcal{C}}(A, n)$, *is the simplicial set obtained as the homotopy colimit of* F, $\mathrm{hocolim}_{\mathcal{C}} F$, *where* F *is the functor obtained by composing* A *with the classical* n^{th} *Eilenberg-Mac Lane functor:* $F(x) = K(A(x), n)$. *In other words, we define*

$$L_{\mathcal{C}}(A, n) = \mathrm{hocolim}_{\mathcal{C}} K(A, n) = \int^{x \in \mathcal{C}} \mathrm{Ner}(x/\mathcal{C}) \times K(A(x), n).$$

Remarks 3.2. *(a) Since the homotopy colimit comes equipped with the canonical map* (4)*, what this definition provides is really an object of the slice category* $\mathbf{Sset}/\mathrm{Ner}(\mathcal{C})$*. We shall interchangeably speak of the Eilenberg-Mac Lane "space"* $L_{\mathcal{C}}(A, n)$ *as a simplicial set or as an object in* $\mathbf{Sset}/\mathrm{Ner}(\mathcal{C})$ *(in which case we can denote it* $\langle L_{\mathcal{C}}(A, n), \ell \rangle$ *or even just* ℓ*), letting the context clarify which of the two meanings is being used.*

(b) *It is evident that this definition would still make sense if one would allow C to be an arbitrary small category. We would then have the concept of the (generalized) Eilenberg-Mac Lane simplicial sets of a small category relative to a C-module, and Proposition 2.2 would still imply that these "are" fibrations over the nerve of C with fibres the classical Eilenberg-Mac Lane simplicial sets. However, we have prefered to limit our definition here to the case of groupoids since it is only in this case that we are able to prove our main theorem (Theorem 4.5).*

(c) *Since for any object $x \in C$ the simplicial set $K(A(x), n)$ has the property (P_n), Proposition 2.3 implies that the simplicial set $L_C(A, n)$ has the property (P_n). As a consequence, the crucial dimensions of $L_C(A, n)$ when defining simplicial maps to $L_C(A, n)$ or homotopies between such maps, are the lower dimensions up to the $n+1$. For this reason we shall limit ourselves below to the description of those lower dimensions of $L_C(A, n)$.*

Applying the isomorphism (5) to the functor $F(x) = K(A(x), n)$, and using the description of the classical simplicial sets $K(A, n)$ in the dimensions $m \leq n+1$ as:

$$K(A,n)_m = \begin{cases} \{*\} & \text{if } m < n \\ A & \text{if } m = n \\ A^{n+1} & \text{if } m = n+1, \end{cases}$$

one obtains the following explicit description of the simplicial sets $L_C(A, n)$ in those dimensions:

The $(n-1)$-truncation of $L_C(A, n)$ coincides with the $(n-1)$-truncation of the nerve of C, so that for $m < n$, $L_C(A, n)_m = \mathrm{Ner}(C)_m$.

At dimension n,

$$L_C(A,n)_n = \coprod_{x_0 \xrightarrow{u_1} x_1 \to \cdots \xrightarrow{u_n} x_n \in \mathrm{Ner}(C)_n} A(x_0),$$

with faces $d_i : L_C(A, n)_n \to L_C(A, n)_{n-1} = \mathrm{Ner}(C)_{n-1}$, acting only on the nerve, that is, if the pair (ξ, a) represents a generic element in $L_C(A, n)_n$

then $d_i(\xi, a) = d_i\xi$. The degeneracy maps $s_i : \mathrm{Ner}(\mathcal{C})_{n-1} \to L_{\mathcal{C}}(A,n)_n$ act by $s_i\xi = (s_i\xi, 0)$ where 0 denotes the neutral element in the corresponding group $A(x_0)$, $x_0 = d_1 \cdots d_{n-1}\xi$.

At dimension $n+1$,

$$L_{\mathcal{C}}(A,n)_{n+1} = \coprod_{\xi \in \mathrm{Ner}(\mathcal{C})_{n+1}} A(d_1 \cdots d_{n+1}\xi)^{n+1},$$

and the faces $d_i : L_{\mathcal{C}}(A,n)_{n+1} \to L_{\mathcal{C}}(A,n)_n$ are given by

$$d_i(\xi, \mathbf{a}) = \begin{cases} \big(d_0(\xi), A(u_1)(a_0)\big) & \text{if } i = 0 \\ \big(d_i(\xi), a_{i+1}\big) & \text{if } 0 < i < n+1 \\ \Big(d_n(\xi),\ \sum_{j=0}^{n}(-1)^{n+j}a_j\Big) & \text{if } i = n+1, \end{cases}$$

where $\xi = [x_0 \xrightarrow{u_1} x_1 \to \cdots x_n \xrightarrow{u_{n+1}} x_{n+1}]$, $\mathbf{a} = (a_0, \ldots, a_n) \in A(x_0)^{n+1}$.

From this explicit description it is easy to deduce the "cocycle condition" for a truncated map $(f_1, \ldots, f_n) : \mathrm{tr}_n X \to \mathrm{tr}_n L_{\mathcal{C}}(A,n)$.

Proposition 3.3 (The cocycle condition). *A necessary and sufficient condition for a truncated map*

$$(f_1, \ldots, f_n) : tr_n X \longrightarrow tr_n L_{\mathcal{C}}(A,n)$$

to extend to a simplicial map $f : X \to L_{\mathcal{C}}(A,n)$ is that f_n satisfies the following "cocycle condition": for all $x \in X_{n+1}$,

$$A(u_1)^{-1}\big(q(f_n(d_0x))\big) \\ = q(f_n(d_1x)) - q(f_n(d_2x)) + \cdots + (-1)^n q(f_n(d_{n+1}x)). \quad (9)$$

where q is the function which associates with every n-cell (ξ, a) in $L_{\mathcal{C}}(A,n)$ the component $a \in A(d_1 \cdots d_n\xi)$

Proof. Let f_{n+1} be defined by the properties

$$d_k f_{n+1}(x) = f_n(d_k x), \quad (10)$$

for $k = 0, \ldots, n$. For a given $x \in X_{n+1}$ let $\xi = [x_0 \xrightarrow{u_1} x_1 \xrightarrow{u_2} \cdots x_n \xrightarrow{u_{n+1}} x_{n+1}]$ and $\mathbf{a} = (a_0, \ldots, a_n) \in A(x_0)^{n+1}$ be such that $(\xi, \mathbf{a}) = f_{n+1}(x)$.

Then the cocycle condition $d_{n+1}f_{n+1}(x) = f_n(d_{n+1}x)$ (see page 7) can be expressed as $f_n(d_{n+1}x) = \big(d_{n+1}\xi, a_n - a_{n-1} + \cdots + (-1)^n a_0\big)$, which reduces to

$$a_0 = a_1 - a_2 + \cdots - (-1)^n a_n + (-1)^n q(f_n(d_{n+1}x)). \qquad (11)$$

On the other hand, by the equations (10) one deduces that $f_n(d_0x) = \big(d_0\xi, A(u_1)(a_0)\big)$ and for $k = 1, \ldots, n$, $f_n(d_kx) = \big(d_k\xi, a_k\big)$, so that $a_0 = A(u_1)^{-1}\big(q(f_n(d_0x))\big)$ and for $k = 1, \ldots, n$, $a_k = q(f_n(d_kx))$, which substituted into (11) yields (9). □

4 Cohomology with local coefficients

In what follows X will be a simplicial set and $\varphi : X \to \mathrm{Ner}(\mathcal{C})$ a simplicial map. This simplicial map determines, in the slice category **Sset**/Ner($\mathcal{C}$), an object which will be denoted indistinctly by $\langle X, \varphi\rangle$, X_φ or simply φ. Note that for each n-cell $x \in X_n$ of X, if $\varphi_n(x) = [x_0 \xrightarrow{u_1} x_1 \to \cdots \to x_n] \in \mathrm{Ner}(\mathcal{C})_n$ then the object $x_0 \in \mathcal{C}$ can be calculated as $x_0 = d_1 \cdots d_n\varphi_n(x)$ and the arrow u_1 as $u_1 = d_2 \cdots d_n\varphi_n(x)$.

Let us recall that a *φ-n singular cochain* of X with coefficients in A is a function c which assigns to each element $x \in X_n$ an element $c(x) \in A(d_1 \cdots d_n\varphi_n x)$. A *normalized* singular cochain is one such that $c(x)$ is trivial for each degenerate element $x \in X_n$. The set $C^n_\varphi(X, A)$ of all such normalized φ-n singular cochains is an abelian group under addition of functional values.

Examples 4.1. *1. The "constant" zero map ζ such that $\zeta(x) = 0$ for all $x \in X_n$ is a normalized φ-n singular cochain.*

2. Let $f : X \to L_{\mathcal{C}}(A, n)$ be a simplicial map such that $f_{n-1} = \varphi_{n-1}$. If c is the function such that $f_n(x) = (\xi, c(x))$ (that is, $c(x) = q(f_n(x))$), then c is a normalized φ-n singular cochain of X with coefficients in A: we have $d_n\xi = d_n(\xi, a) = d_nf_n(x) = f_{n-1}(d_nx) = \varphi_{n-1}(d_nx) = d_n\varphi_nx$ and therefore $a \in A(d_1 \cdots d_n\varphi_nx)$; on the other hand if x is degenerate, say $x = s_ky$, then $(\xi, a) = f_n(s_ky) = s_kf_{n-1}(y) = s_k\overline{\xi} = (s_k\overline{\xi}, 0)$, so that $a = 0$.

One can use the fact that all arrows in $\mathcal{C}$ have an inverse to define a coboundary operator

$$\delta : C^n_\varphi(X, A) \to C^{n+1}_\varphi(X, A)$$

by means of the following alternating sum:

$$\delta c(x) = A(u_1)^{-1}\big(c(d_0 x)\big) - c(d_1 x) + \cdots + (-1)^{n+1} c(d_{n+1} x), \qquad (12)$$

where u_1 denotes the arrow such that $\varphi_n(x) = [x_0 \xrightarrow{u_1} x_1 \to \cdots \to x_n]$, that is, $u_1 = d_2 \cdots d_n \varphi_n(x)$. A singular cochain $c \in C^n_\varphi(X, A)$ such that $\delta c = 0$ is called a cocycle. Note that both cochains in the above examples are cocycles. Definition (12) makes δ to be a group homomorphism and $\delta^2 = 0$, so that $C^*_\varphi(X, A)$ becomes a cochain complex of abelian groups. The homology groups of this complex will be written as $H^n_\varphi(X, A)$.

An important example arising naturally when studying a simplicial set X is $\mathcal{C} = \Pi_1(X)$, the fundamental groupoid of X, and $\varphi = \eta_X : X \to \operatorname{Ner}\Pi_1(X)$, the canonical map (the unit of the adjunction $\Pi_1 \dashv \operatorname{Ner}$). In this case the cohomology groups corresponding to φ are the singular cohomology groups of X with local coefficients on A, which are denoted just by $H^n(X, A)$, see [11].

The cocycles in the cochain complex $C^*_\varphi(X, A)$ can be interpreted as maps in $\mathbf{Sset}/\operatorname{Ner}(\mathcal{C})$ from X_φ to $L_{\mathcal{C}}(A, n)$:

Lemma 4.2. *The n-cocycles in the cochain complex $C^*_\varphi(X, A)$ correspond bijectively to the simplicial maps $f : X \to L_{\mathcal{C}}(A, n)$ such that $\ell f = \varphi$,*

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & L_{\mathcal{C}}(A, n)\,. \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\ell} & \\ & \operatorname{Ner}(\mathcal{C}) & \end{array} \qquad (13)$$

In other words, there is a bijection

$$\text{cocycles in } C^n_\varphi(X, A) \xrightarrow{\ \cong\ } \mathbf{Sset}/\operatorname{Ner}(\mathcal{C})\big(X_\varphi, L_{\mathcal{C}}(A, n)\big).$$

Proof. Since the map ℓ is the identity at dimensions $0, \ldots, n-1$, the lower components $f_0, \ldots, f_{n-1}$ of a simplicial map $f : X \to L_{\mathcal{C}}(A, n)$

such that $\ell f = \varphi$ are uniquely determined by φ (they are $f_k = \varphi_k$). Thus, giving a simplicial map $f : X \longrightarrow L_{\mathcal{C}}(A,n)$ such that $\ell f = \varphi$ is equivalent to giving a map $f_n : X_n \longrightarrow L_{\mathcal{C}}(A,n)_n$ satisfying $\ell_n f_n = \varphi_n$ and the cocycle condition (9) and such that $(\varphi_0, \ldots, \varphi_{n-1}, f_n)$ is a truncated simplicial map. Now, as we point out in the second of examples 4.1, the function c determined by the equation $f_n(x) = (\varphi_n(x), c(x))$ is a φ-n singular cochain of X with coefficients in A, and the cocycle condition for f_n states precisely that this singular cochain is a cocycle. Conversely, let $c \in C^n_\varphi(X,A)$ be a cocycle. Let us define the map $f_n : X_n \longrightarrow L_{\mathcal{C}}(A,n)_n$ as $f_n(x) = (\varphi_n(x), c(x))$ so that it automatically satisfies the cocycle condition and $\ell_n f_n = \varphi_n$. The fact that c is normalized implies that $(\varphi_0, \ldots, \varphi_{n-1}, f_n)$ is a truncated simplicial map. By Proposition 3.3 this can be extended to a simplicial map $f : X \to L_{\mathcal{C}}(A,n)$ such that $\ell f = \varphi$. □

Our next objective is to establish a characterization of homotopies between maps in **Sset**/ Ner($\mathcal{C}$) analogous to the one of simplicial maps given in Proposition 3.3. To that end let us recall some notions related to homotopic maps:

Let f, g be two maps in **Sset**/ Ner($\mathcal{C}$) from X_φ to $L_{\mathcal{C}}(A,n)$. In particular these are simplicial maps $f, g : X \longrightarrow L_{\mathcal{C}}(A,n)$. By a homotopy from f to g as maps in **Sset**/ Ner($\mathcal{C}$) it is meant a homotopy $h : f \longrightarrow g$ (as simplicial maps from X to $L_{\mathcal{C}}(A,n)$) satisfying the additional condition

$$\ell_{m+1} h_i^m = s_i \varphi_m \tag{14}$$

for every $m > 0$ and each $i = 0, 1, \ldots, m$.

This additional property implies that for $m < n-1$, $h_i^m = s_i\varphi_m$. Therefore the reasoning surounding equation (8) in page 8 and remark (c) in page 9 imply that giving a homotopy h from f to g as maps in **Sset**/ Ner($\mathcal{C}$) is equivalent to giving just components $h_0^{n-1}, \ldots, h_{n-1}^{n-1}$ satisfying the appropriate homotopy condition. This homotopy condition is given in the following proposition:

Proposition 4.3 (The homotopy condition). *Let $f, g : X_\varphi \longrightarrow L_{\mathcal{C}}(A,n)$ be two maps in* Sset/ Ner($\mathcal{C}$) *as above. The "homotopy condition" that must be satisfied by a truncated homotopy*

$$\langle (h_0^0), (h_0^1, h_1^1), \ldots, (h_i^{n-1})_{i=0}^{n-1} \rangle : tr_{n-1}(f) \longrightarrow tr_{n-1}(g) \tag{15}$$

to extend to a homotopy $h : f \longrightarrow g$ is that the components $(h_i^{n-1})_{i=0}^{n-1}$ satisfy for all $x \in X_n$:

$$q\big(g_n(x)\big) = q\big(f_n(x)\big) + \sum_{k=0}^{n-1} \Big[(-1)^{k+1} A(u_1)^{-1}(c_0^k) + \sum_{j=1}^{n} (-1)^{k+j+1} c_j^k\Big] \quad (16)$$

where q is as in Proposition 3.3, and the $c_j^k \in A\big(d_1 \cdots d_n s_k d_j \varphi_n(x)\big)$ are defined by

$$h_k^{n-1}(d_j x) = (s_k \varphi_{n-1}(d_j x), c_j^k). \quad (17)$$

Proof. Let $x \in X_n$, and $\varphi_n(x) = [x_0 \xrightarrow{u_1} x_1 \to \cdots \to x_n] \in \mathrm{Ner}(\mathcal{C})_n$. We shall first suppose that $h : f \longrightarrow g$ is a homotopy, so that for $k = 0, \ldots, n$ the elements $h_k^n(x) \in L_{\mathcal{C}}(A, n)_{n+1}$ have the form $h_k^n(x) = (\xi^k, \mathbf{a}^k)$, where

$$\begin{aligned} \xi^k &= s_k \varphi_n(x) = [x_0 \xrightarrow{u_1} \cdots \xrightarrow{u_i} x_i \xrightarrow{1} x_i \rightarrow \cdots \xrightarrow{u_n} x_n], \\ \mathbf{a}^k &= (a_0^k, \ldots, a_n^k) \in A(x_0)^{n+1}. \end{aligned}$$

From the homotopy identities satisfied by h_0^n we get $a_0^0 = q\big(f_n(x)\big)$, $a_2^0 = c_1^0$, ..., $a_n^0 = c_{n-1}^0$ so that we can express a_1^0 in terms of $q\big(f_n(x)\big)$, the $(c_i^0)_{i=1}^{n-1}$, and the alternating sum $c_n^0 = q(h_0^{n-1}(d_n x))$. Doing that one obtains:

$$a_1^0 = q\big(f_n(x)\big) + \sum_{j=1}^{n} (-1)^{j+1} c_j^0.$$

In a similar way, the homotopy identities satisfied by h_k^n imply for each $k \in \{1, \ldots, n-1\}$,

$$a_{k+1}^k = a_k^{k-1} + (-1)^k A(u_1)^{-1}(c_0^{k-1}) + \sum_{j=1}^{k-1} (-1)^{k+j} c_j^{k-1} + \sum_{j=k+1}^{n} (-1)^{k+j+1} c_j^k,$$

and a similar analysis of the map h_n^n leads to

$$q\big(g_n(x)\big) = a_n^{n-1} + (-1)^n A(u_1)^{-1}(c_0^{n-1}) + \sum_{j=1}^{n-1} (-1)^{n-j} c_j^{n-1}.$$

Putting these formulas together one obtains (16), proving that the condition is necessary.

Let's now suppose that (16) holds for the truncated homotopy (15). Evidently condition (14) forces the $h_k^n(x)$ to be of the form

$$h_k^n(x) = (s_k\varphi_n(x), \mathbf{a}^k)$$

for some $\mathbf{a}^k = (a_0^k, \dots, a_n^k) \in A(x_0)^{n+1}$. Furthermore, the calculations in the first part of the proof show that the elements $a_j^k \in A(x_0)$ should be defined (inductively) as:

$$a_0^0 = q\big(f_n(x)\big), \quad a_0^k = A(u_1)^{-1}(c_0^{k-1}), \quad a_k^k = a_k^{k-1},$$

and for $j = k+1$,

$$a_{k+1}^k = a_k^{k-1} + (-1)^k a_0^k + \sum_{\substack{j=1 \\ j\neq k,k+1}}^{n+1} (-1)^{j-k} a_j^k,$$

where

$$a_j^k = \begin{cases} c_j^{k-1} & \text{if } j < k \\ c_{j-1}^k & \text{if } j > k+1. \end{cases}$$

Those definitions make the $h_k^n(x)$ satisfy all the homotopy identities except perhaps the equation $d_{n+1}h_n^n(x) = g_n(x)$. But this is equivalent to the equation $a_n^n - a_{n-1}^n + \cdots + (-1)^n a_0^n = q(g_n(x))$, which, using the above definitions, works out to be nothing but equation (16). Thus (16) is a sufficient condition for the extension of the truncated homotopy to the next dimension. Since $L_C(A,n)$ satisfies property (P_n) this extension guarantees the extension to all dimensions and this proves the proposition. □

We are now interested in the particular case of Proposition 4.3 in which g is the zero simplicial map $\zeta : \langle X, \varphi\rangle \to \langle L_C(A,n), \ell\rangle$. In this case we obtain,

Lemma 4.4. *A map $f : \langle X, \varphi\rangle \to \langle L_C(A,n), \ell\rangle$ is homotopic to the zero map $\zeta : \langle X, \varphi\rangle \to \langle L_C(A,n), \ell\rangle$ if and only if for each $k \in \{0, 1, \dots, n-$*

$1\}$ *there is a function* c^k *asigning to each* $x \in X_{n-1}$ *an element* $c^k(x) \in \mathcal{A}(d_1 \cdots d_n s_k \varphi(x))$ *and such that*

$$q\big(f_n(x)\big) + \sum_{k=0}^{n-1} \Big[(-1)^{k+1} A(u_1)^{-1}\big(c^k(d_0x)\big) + \sum_{j=1}^{n}(-1)^{k+j+1} c^k(d_jx)\Big] = 0. \tag{18}$$

Proof. Evidently, if a homotopy exists, we can define the functions c^k by $h_k^{n-1}(x) = (s_k\varphi_{n-1}(x), c^k(x))$, as in Proposition 4.3. As it was shown there, these functions satisfy (18).

Conversely, let c^k be functions such that (18) holds. We can then define a truncated homotopy $h : \mathrm{tr}_{n-1} f \to \mathrm{tr}_{n-1}\zeta$ for which $d_i h_k^{n-1}(x) = (s_k d_i \varphi_{n-1}(x), c^k(d_ix))$. By (18) the $(h_k^{n-1})_{k=0}^{n-1}$ satisfy the homotopy condition, therefore by Proposition 4.3, f and ζ are homotopic. □

From this lemma we immediately obtain

Theorem 4.5. *There is a natural bijection*

$$H^n_\varphi(X, A) \cong [X_\varphi, L_\mathcal{C}(A, n)],$$

between the elements of n^{th} *cohomology group of* $\langle X, \varphi\rangle$ *with coefficients in* A *and the homotopy classes of maps from* $\langle X, \varphi\rangle$ *to* $\langle L_\mathcal{C}(A,n), \ell\rangle$ *in* **Sset**/ Ner$(\mathcal{C})$.

Proof. After the Lemma 4.2 we only have to see that maps corresponding to cohomologous cocycles are homotopic and vice versa. This is the same as proving that a cocycle is a coboundary if and only if its corresponding map $f : X_\varphi \to L_\mathcal{C}(A,n)$ is homotopic to the zero map $\zeta : X_\varphi \to L_\mathcal{C}(A,n)$.

Let's first suppose that the n-cocycle $c \in C^n_\varphi(X, A)$ is a coboundary. Then there is $c' \in C^{n-1}_\varphi(X, A)$ such that $\delta c' = c$, that is, for all $x \in X_n$, by definition (12), we have (putting $u_1 = d_2 \cdots d_n \varphi_n(x)$):

$$c(x) = A(u_1)^{-1}\big(c'(d_0x)\big) + \sum_{i=1}^{n}(-1)^i c'(d_ix). \tag{19}$$

Define functions $c^0, \ldots, c^{n-1}$ on X_{n-1} by $c^0 = c'$ and $c^k(x) = 0 \in A(d_1 \cdots d_n s_k \varphi(x))$ for $k > 0$, and let f be the simplicial map corresponding to the cocycle c. Then (18) holds because it reduces to (19), and therefore f is homotopic to zero.

Conversely, let $f : X_\varphi \to L_{\mathcal{C}}(A, n)$ be a map homotopic to ζ. Let $f_n(x) = (\varphi_n(x), c(x))$. Then c is a n-cocycle and we have to prove that it is a coboundary, e.i. that there exits $c' \in C^{n-1}_\varphi(X, A)$ such that $\delta c' = c$. Let us define

$$c'(y) = \sum_{k=0}^{n-1} (-1)^k c^k(y).$$

It is immediate to verify that in terms of c', (18) just reads $\delta c' = c$. □

A local coefficient system on a simplicial set X is a $\mathcal{C}$-module $A : \mathcal{C} \to$ **Ab** where $\mathcal{C}$ is the fundamental groupoid of X, $\mathcal{C} = \Pi_1(X)$. According to Theorem 4.5, the n^{th} generalized Eilenberg-Mac Lane space of X with coefficients in A, $L_X(A, n)$, (as an object in **Sset**/ $\mathrm{Ner}(\Pi_1(X))$ via the canonical projection ℓ) represents the singular cohomology of $\langle X, \eta_X \rangle$ with coefficients in A in the sense that

Corollary 4.6. *There is a natural bijection,*

$$H^n(X, A) \;\cong\; [X, L_X(A, n)],$$

between the elements of the n^{th} singular cohomology group of X with local coefficients in A and the homotopy classes of maps from $\langle X, \eta_X \rangle$ to $L_X(A, n)$ in **Sset**/ $\mathrm{Ner}(\Pi_1(X))$.

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