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A CLASSIFICATION OF DEGREE *n* FUNCTORS, I

by B. JOHNSON and R. McCARTHY

RESUME. Utilisant une théorie de calcul pour des foncteurs de catégories pointées vers des catégories abéliennes qu'ils ont développée précédemment, les auteurs prouvent dans la Partie II de cet article que les foncteurs de degré *n* peuvent être classifiés en termes de modules sur une algèbre graduée différentielle $P_{n\times n}(C)$. Dans cette partie, ils développent les structures du calcul nécessaires pour prouver ceci et des résultats apparentés. Ils construisent aussi une filtration par rang pour des foncteurs de catégories pointées vers des catégories abéliennes et ils comparent les foncteurs de rang *n* et les foncteurs de degré *n*.

The Taylor series of a function is a tremendously important tool in analysis. A similar theory, the calculus of homotopy functors developed by Tom Goodwillie ([G1], [G2], [G3]), has recently been used to prove several important results in K-theory and homotopy theory. In [J-M3], we defined and established the basic properties for a theory of calculus for functors from pointed categories to abelian categories. Given a functor $F : \mathcal{C} \to Ch\mathcal{A}$ where $\mathcal{C}$ is a pointed category and $\mathcal{A}$ is a cocomplete abelian category, we showed that by using a particular cotriple one could construct a tower of functors and natural transformations (see figure 1). For each n, the functor P_nF is a degree n functor in the sense that its $n+1$st cross effect as defined by Eilenberg and Mac Lane ([E-M2]) is acyclic.

In this paper and its sequel [J-M4], we show that by using the models for P_n given in [J-M3], degree n functors can be classified in terms of modules over a differential graded algebra $P_{n\times n}(C)$. We also

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show that homogeneous degree n functors, i.e., degree n functors G for which $P_{n-1}G \simeq *$, can be classified in terms of modules over three different differential graded algebras. One of these classifications was inspired by Goodwillie's classification of homogeneous degree n functors of spaces ([G3]). These classifications extend a classification of linear functors proved in [J-M1]. As part of the development of these classifications we also show that all degree n functors arise naturally as functors on a particular category $P_n\mathcal{C}$, following a similar result for strictly degree n functors due to Pirashvili [P]. (A strictly degree n functor is one whose $n+1$st cross effect is isomorphic, rather than quasi-isomorphic, to 0.) † In addition, we develop a "rank" filtration of F, i.e., we look at approximations to F that agree with F on a specified collection of objects.

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
 & & P_{n+1}F \\
 & {\scriptstyle p_{n+1}F} \nearrow & \downarrow {\scriptstyle q_{n+1}F} \\
F & \xrightarrow{p_nF} & P_nF \\
 & {\scriptstyle p_{n-1}F} \searrow & \downarrow {\scriptstyle q_nF} \\
 & & P_{n-1}F \\
 & & \vdots \\
 & & \downarrow \\
 & & P_0F = F(*).
\end{array}
$$

figure 1

† For those familiar with [J-M2], degree n functors in this paper correspond to homologically degree n functors in [J-M2], and strictly degree n functors correspond to degree n functors in [J-M2].

The papers are organized as follows. This paper comprises sections 1, 2, and 3. Sections 4, 5, 6, and the appendix form [J-M4]. We begin in section 1 by reviewing the Taylor tower of [J-M3] and describing some natural transformations arising from the tower to be used in this work. We then start developing the framework needed to state and prove the classification results.

The sequence of results forming this framework were motivated and can be understood by considering a classification result for additive functors proved independently by Eilenberg and Watts:

Theorem ([E], [W]). *Let F be an additive, right continuous (preserves cokernels and filtered colimits) functor from the category of right R-modules to the category of right S-modules for some rings R and S. Let G be the functor given by*

$$G(-) = - \otimes_R F(R).$$

There is a natural transformation $\eta : G \to F$ that is an isomorphism on all R-modules. That is, additive, right continuous functors are characterized by $R - S$ bimodules $F(R)$.

To prove this result, one first establishes that $F(R)$ has the required bimodule structure and constructs the natural transformation η. The isomorphism is then proven in stages using various properties of the functors. The first stage is the observation that η is an isomorphism at R. Additivity of the functors then establishes the isomorphism at all finitely generated free R modules. From there, the fact that both functors preserve filtered colimits guarantees that η is an isomorphism on all free R modules. Finally, since every R module has a resolution by free R modules and the functors preserve cokernels, an isomorphism for all R modules is ensured. In essence, this proof depends upon two properties: the category of R modules has a generating object R and the functors behave well with respect to the operations needed to generate all R modules from R.

We will prove a similar result for degree n functors from a basepointed category $\mathcal{C}$ to $Ch\mathcal{A}$ for some abelian category $\mathcal{A}$. When considering this more general setting, one notices immediately that $\mathcal{C}$

lacks the generating object that was so useful for the classification of additive functors. This leads us to consider instead subcategories of $\mathcal{C}$ generated by objects C in $\mathcal{C}$. We call such subcategories "lines generated by C" and develop the notion of "functors defined along C" in parallel with the right continuous property used in the Eilenberg-Watts result. This material will be developed in section 2. The principal result will be the following.

Theorem 2.11. *Let $F, G : \mathcal{C} \to Ch\mathcal{A}$ be degree n functors defined along an object C in $\mathcal{C}$. A natural transformation $\eta : F \to G$ is an equivalence if and only if η is an equivalence at $\mathbf{n}_C = \vee_{i=1}^{n} C$.*

The theorem allows us to prove classification results by simply establishing equivalences at the object $\mathbf{n}_C$. In general, the class of functors that are determined by their value at $\mathbf{n}_C$ is strictly larger than the class of degree n functors defined along C. We refer to the functors that are determined by their value at $\mathbf{n}_C$ as rank n functors and explore the properties of such functors in section 3. In particular we show that any functor F from $\mathcal{C}$ to $Ch\mathcal{A}$ has a filtration of functors $\{\mathcal{L}_k F\}_{k\geq 0}$ of rank k, and show that degree n is a strictly stronger condition than rank n.

We will classify degree n functors defined along an object C by showing that any such functor F is equivalent to the functor

$$\perp_C^* (P_n F(\mathbf{n}_C) \hat{\otimes}_{P_{n\times n}(C)} P_n(C, -))$$

where $P_n(C, -) = P_n\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}_C, -)]$ and $P_{n\times n}(C)$ is the differential graded algebra $P_n\,\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}_C, \mathbf{n}_C)]$. (The symbol $\perp_C^*$ indicates the resolution of a functor along C and is defined in section 2.) Constructing such a functor requires that $P_n(C, -)$ be given certain differential graded algebra and module structures. The properties underlying these structures are developed in section 4, although the actual algebra and module structures are not specified until section 5. In section 4, we use the properties that must be established for the algebra and module structures to construct a category $P_n\mathcal{C}$ through which all degree n functors must factor. This extends a result due to Pirashvili ([P]) for strictly degree n functors.

In section 5 we state and prove our classification theorems using the results of the previous sections. We present four classification theorems, one for degree n functors defined along C, and three for homogeneous degree n functors defined along C:

1) degree n functors defined along C are classified by modules over the DGA $P_{n\times n}(C)$
2) homogeneous degree n functors defined along C are classified by modules over a DGA $D_{n\times n}(C)$, modules over a DGA $D_1(C)$, and modules over a wreath product $D_{1\times 1}(C)\int\Sigma_n$.

In section 6, we consider various natural operations developed in [J-M3] that change the degree of a functor and determine their effect on the classification results of section 5. In particular, we look at differentiation, the structure maps in the Taylor tower, composition, and the inclusions from degree n to higher degree functors and from homogeneous degree n to degree n functors. We also include an appendix explaining the relationship between the three different classifications of homogeneous degree n functors.

1. The Taylor tower

This section reviews the construction of the Taylor tower in [J-M3]. We also add some new natural transformations to the tower that will be needed for defining differential graded algebra and module structures later in this paper. Throughout this paper, unless otherwise indicated, we will let $\mathcal{C}$ be a pointed category (a category with an object $*$ that is both initial and final) with finite coproducts. We will let $\mathcal{A}$ be a cocomplete abelian category and F be a functor from $\mathcal{C}$ to $Ch\mathcal{A}$.

We start by recalling the ideas necessary to understand the degree of a functor. Key among these is the concept of cross effect.

Definition 1.1. [E-M2] *Let F be a functor from $\mathcal{C}$ to $Ch\mathcal{A}$. We say that F is reduced if $F(*)\cong 0$. The nth cross effect of F is the functor $cr_nF:\mathcal{C}^{\times n}\to Ch\mathcal{A}$ defined inductively for objects $M, M_1,\dots,M_n$ by*

$$cr_1F(M) \oplus F(*) \cong F(M)$$
$$cr_2F(M_1, M_2) \oplus cr_1F(M_1) \oplus cr_1F(M_2) \cong cr_1F(M_1 \vee M_2),$$

and in general,

$$cr_nF(M_1, .., M_n) \oplus cr_{n-1}F(M_1, M_3, .., M_n) \oplus cr_{n-1}F(M_2, M_3, .., M_n)$$

is equivalent to

$$cr_{n-1}F(M_1 \vee M_2, M_3, \ldots, M_n).$$

The nth cross effect of a functor F satisfies the following properties.

Proposition 1.2. *Let $F : \mathcal{C} \to Ch\mathcal{A}$, and $M_1, M_2, \ldots, M_n, M$ be objects in $\mathcal{C}$.*

1) *cr_nF is symmetric with respect to its n variables, i.e., for every $\sigma \in \Sigma_n$, the nth symmetric group, $cr_nF(M_1, \ldots, M_n) \cong cr_nF(M_{\sigma(1)}, \ldots, M_{\sigma(n)})$.*
2) *$cr_nF(M_1, \ldots, M_n) \cong 0$ if any $M_i = *$.*
3) *$F(M_1 \vee \cdots \vee M_n) \cong \bigoplus_{p=0}^{n} \left(\bigoplus_{j_1 < \ldots < j_p} cr_pF(M_{j_1}, \ldots, M_{j_p})\right)$ and*

$$F(\vee_{i=1}^{n} M) \cong \bigoplus_{p=0}^{n} \binom{n}{p} cr_pF(M)$$

where $cr_pF(M)$ denotes $cr_pF(M, \ldots, M)$.

Considering cr_nF as a functor of a single variable, i.e.,

$$cr_nF(C) := cr_nF(C, \ldots, C)$$

one can show that $cr_p(cr_qF) \cong cr_q(cr_pF)$ for all p and q. For details, see remark 1.3 in [J-M3]. For examples of cross effects and alternative definitions of cross effects, see section 2 of [J-M2].

The nth cross effect functor can also be realized as a right adjoint to the diagonal functor from functors of a single variable to functors of n variables. This adjoint pair, described below, is fundamental to the Taylor tower construction.

Remark 1.3. (THE ADJOINT PAIR (Δ^*, cr_{n+1})) Let $Func_*(\mathcal{C}^{\times n+1}, \mathcal{A})$ be the category of functors of $n+1$ variables from $\mathcal{C}$ to $\mathcal{A}$ that are reduced in each variable separately. Let Δ^* be the functor from $Func_*(\mathcal{C}^{\times n+1}, \mathcal{A})$ to $Func_*(\mathcal{C}, \mathcal{A})$ obtained by composing a functor with the diagonal functor from $\mathcal{C}$ to $\mathcal{C}^{\times n+1}$. The $(n+1)st$ cross effect is the right adjoint to Δ^*. The natural isomorphism

$$\mathrm{Hom}_{Func_*(\mathcal{C},\mathcal{A})}(F \circ \Delta, G) \cong \mathrm{Hom}_{Func_*(\mathcal{C}^{\times n+1},\mathcal{A})}(F, cr_{n+1}G) \quad (1.4)$$

is defined as follows. For objects $M_1, \ldots, M_{n+1}$ in $\mathcal{C}$, let i from $F(M_1, \ldots, M_{n+1})$ to $cr_{n+1}(F \circ \Delta)(M_1, \ldots, M_{n+1})$ be the composite

$$\begin{array}{ccc} F(M_1, \ldots, M_{n+1}) & \xrightarrow{F(i_1, \ldots, i_{n+1})} & F(\vee_{i=1}^{n+1} M_i, \ldots, \vee_{i=1}^{n+1} M_i) \\ & & \downarrow \pi \\ & & cr_{n+1}(F \circ \Delta)(M_1, \ldots, M_{n+1}) \end{array}$$

where $i_1, \ldots, i_{n+1}$ are the natural inclusions and π is the projection of $F \circ \Delta(\vee_{i=1}^{n+1} M_i)$ onto its summand $cr_{n+1}(F \circ \Delta)(M_1, \ldots, M_{n+1})$. Then, the isomorphism (1.4) takes a natural transformation $\beta : F \circ \Delta \to G$ to the composite

$$F \xrightarrow{i} cr_{n+1}(F \circ \Delta) \xrightarrow{cr_{n+1}\beta} cr_{n+1}G.$$

And, for an object M in $\mathcal{C}$, a natural transformation $\alpha : F \to cr_{n+1}G$ is sent to the natural composite

$$F(M, \ldots, M) \xrightarrow{\alpha} cr_{n+1}G(M, \ldots, M) \xrightarrow{inc} G(\vee_{i=1}^{n+1} M) \xrightarrow{G(+)} G(M)$$

where inc denotes the inclusion of $cr_{n+1}G(M, \ldots, M)$ as a direct summand of $G(\vee_{i=1}^{n+1} M)$ and $+$ is the fold map. We set p_n = cokernel $[\Delta^* \circ cr_{n+1} \xrightarrow{\text{coadj}} id]$ where *coadj* denotes the counit of the adjunction.

Cross effects are used to define the degree of a functor.

Definition 1.5. *A functor F from $\mathcal{C}$ to $Ch\mathcal{A}$ is degree n if $cr_{n+1}F$ is acyclic. We say that F is strictly degree n if $cr_{n+1}F$ is isomorphic, rather than quasi-isomorphic, to* 0.

Note that $cr_{n+1}F \simeq 0$ if and only if $cr_kF \simeq 0$ for all $k \geq n+1$ or, equivalently, if cr_nF is a degree 1 functor in each of its n variables separately. If a functor is degree n, then we consider it to be degree k for all $k \geq n$ as well. When one composes a degree n and a degree m functor, the result is a functor of degree $n \cdot m$, as proved in [J-M3], 1.5.

The adjoint pair (Δ^*, cr_{n+1}) of remark 1.3 produces the cotriple used to create the Taylor tower. We next review the cotriple properties that will be needed to describe the tower.

Definition 1.6. *A cotriple (or comonad) ($\perp$, ϵ, δ) in a category $\mathcal{A}$ is a functor $\perp$: $\mathcal{A} \to \mathcal{A}$ together with natural transformations $\epsilon : \perp \Rightarrow \mathrm{id}_{\mathcal{A}}$ and $\delta : \perp \Rightarrow \perp\perp$ such that the following diagrams commute:*

$$
\begin{array}{ccc}
\perp & \xrightarrow{\delta} & \perp(\perp) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_{\perp}} \\
\perp(\perp) & \xrightarrow{\perp\delta} & \perp(\perp\perp) = \perp\perp(\perp)
\end{array}
\qquad
\begin{array}{ccccc}
 & {=}\swarrow & \perp & \searrow{=} & \\
 & & \downarrow{\scriptstyle \delta} & & \\
\perp & \xleftarrow{\perp\epsilon} & \perp(\perp) & \xrightarrow{\epsilon_{\perp}} & \perp .
\end{array}
$$

Cotriples often arise from adjoint pairs.

Example 1.7. Let (F, U) be a pair of adjoint functors and $\perp = FU$. Let ϵ be a counit and η be a unit for the adjoint pair. Let η_U be the natural transformation that for an object B is given by $\eta_{U(B)}$: $U(B) \to UF(U(B))$. Then $(\perp, \epsilon, F(\eta_U))$ is a cotriple. In particular, the adjoint pair of remark 1.3 yields the cotriple $\Delta^* \circ cr_{n+1}$.

Cotriples yield simplicial objects in the following manner.

Remark 1.8. Let $(\perp, \epsilon, \delta)$ be a cotriple in $\mathcal{A}$ and let A be an object in $\mathcal{A}$. Then $\perp^{*+1} A$ is the following simplicial object in $\mathcal{A}$:

$$[n] \mapsto \perp^{(n+1)} A = \overbrace{\perp \cdots \perp}^{n+1 \text{ times}} A$$

$$d_i = \perp^{(i)} \epsilon \perp^{(n-i)} : \perp^{(n+1)} A \to \perp^{(n)} A$$
$$s_i = \perp^{(i)} \delta \perp^{(n-i)} : \perp^{(n+1)} A \to \perp^{(n+2)} A.$$

Observe that $\perp^{*+1}$ is augmented over $\mathrm{id}_{\mathcal{A}}$ by ϵ. In particular, if we consider $(\mathrm{id}_{\mathcal{A}}, \mathrm{id}, \mathrm{id})$ as the trivial cotriple, then ϵ gives a natural simplicial map from $\perp^{*+1}$ to id^{*+1} where id^{*+1} is the trivial simplicial $\mathcal{A}$-object.

Using $\perp^{*+1} A$ and the augmentation one can construct the following chain complex when $\mathcal{A}$ is an abelian category.

Definition 1.9. *Let $(\perp, \epsilon, \delta)$ be a cotriple on an abelian category $\mathcal{A}$ and let A be an object in $\mathcal{A}$. Then $C_*^{\perp}(A)$ is the chain complex with*

$$C_*^{\perp}(A) = \begin{cases} A & \text{if } n = 0 \\ \perp^n A & \text{if } n > 0 \end{cases}$$

and $\partial_n : C_n^{\perp}(A) \to C_{n-1}^{\perp}(A)$ is defined by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i.$$

The chain complex $C_*^{\perp}(A)$ is the mapping cone of the composition

$$C(\perp^{*+1} A) \xrightarrow{\epsilon} C(\mathrm{id}^{*+1} A) \xrightarrow{\simeq} N(\mathrm{id}^{*+1} A) = A$$

where $C(\perp^{*+1} A)$ and $C(\mathrm{id}^{*+1} A)$ are the chain complexes associated to $\perp^{*+1} A$ and $\mathrm{id}^{*+1} A$, respectively, and $N(\mathrm{id}^{*+1} A)$ is the normalized chain complex associated to $\mathrm{id}^{*+1}A$. (See [We], §8.2-3 for definitions of C and N.)

The Taylor tower is defined as follows.

Definition 1.10. *Let $\perp_{n+1} = \Delta^* \circ cr_{n+1}$ be the cotriple on the category $Func_*(\mathcal{C}, Ch\mathcal{A})$ obtained from the adjoint pair (Δ^*, cr_{n+1}) of remark 1.3. We define P_n to be the functor from $Func_*(\mathcal{C}, Ch\mathcal{A})$ to $Func_*(\mathcal{C}, Ch\mathcal{A})$ given by*

$$P_n = \text{Mapping Cone } [N(\perp_{n+1}^{*+1}) \xrightarrow{\epsilon} N(\mathrm{id}^{*+1}) = \mathrm{id}]$$

where N is the associated normalized chain complex of the simplicial object. We let $p_n : \mathrm{id} \to P_n$ be the natural transformation obtained from the mapping cone.

Note that for a functor F in $Func_*(\mathcal{C}, Ch\mathcal{A})$, P_nF is naturally chain homotopy equivalent to the chain complex $C_*^{\perp_{n+1}}(F)$. Both of these models will be important for subsequent results. We will use them interchangeably.

We extend the definition of P_nF to all functors F from $\mathcal{C}$ to $Ch\mathcal{A}$ as follows. Every functor F is naturally isomorphic to $F(*) \oplus cr_1F$ where cr_1F is reduced. We define

$$P_nF = \text{Cone } [N(\perp_{n+1}^{*+1} cr_1F) \xrightarrow{\epsilon} N(\mathrm{id}^{*+1} cr_1F) = cr_1F \xrightarrow{inc} F].$$

Furthermore, we define

$$D_1F = P_1(cr_1F).$$

Remark. Recall that we use p_nF to denote H_0P_nF. By the above,

$$p_nF = \mathrm{coker}(P_nF \xrightarrow{\epsilon} F),$$

where, by remark 1.3, the natural transformation ϵ is determined by the composite

$$cr_{n+1}F(M, \ldots, M) \xrightarrow{inc} F(\vee_{i=1}^{n} M) \xrightarrow{+} F(M).$$

The functor p_n is left adjoint to the forgetful functor from degree n functors to all functors and is also known as the Passi functor.

The following properties of P_nF are proved in section 2 of [J-M3].

Remarks 1.11.

1) The functor P_nF is degree n.
2) If F is degree n then $p_n : F \to P_nF$ is a quasi-isomorphism.
3) The pair (P_n, p_n) is universal up to natural quasi-isomorphism with respect to degree n functors with natural transformations from F.
4) If $F'' \to F \to F'$ is a short exact sequence of functors (that is, short exact upon evaluation at any given object) then $P_n(F'') \to P_n(F) \to P_n(F')$ is a short exact sequence of chain complexes.
5) The functor P_n preserves quasi-isomorphisms of functors.

The natural transformations that make up the Taylor tower of F are defined in the next remark.

Remark 1.12. Given a natural transformation of cotriples η from $\perp_{n+1}$ to $\perp_n$, we can define a natural transformation from P_n to P_{n-1} by using the chain map given by $\eta([k]) = \perp_n^{k-1} (\eta_{\perp_{n+1}}) \circ \perp_n^{k-2} (\eta_{\perp_{n+1}^2}) \circ \ldots \circ \eta_{\perp_{n+1}^k}$. Then $q_n : P_n \to P_{n-1}$ is the natural transformation q_n determined by the natural composite:

$$\begin{array}{ccc} cr_{n+1}F(M, M \ldots, M) & \xrightarrow{inc} & cr_nF(M \vee M, M, \ldots, M) \\ & & \big\downarrow {\scriptstyle cr_nF(+,id,\ldots,id)} \\ & & cr_nF(M, \ldots, M). \end{array}$$

Theorem 1.13. *Given a functor F from $\mathcal{C}$ to $Ch\mathcal{A}$, there is a natural tower of functors:*

$$\begin{array}{ccccccc} & & & F & & & \\ & p_{n+1} \swarrow & & \big\downarrow {\scriptstyle p_n} & & \searrow p_{n-1} & \\ \cdots \to & P_{n+1}F & \xrightarrow{q_{n+1}} & P_nF & \xrightarrow{q_n} & P_{n-1}F & \to \cdots \to P_0F = F(*). \end{array}$$

The pairs (P_n, p_n) are universal (up to natural quasi-isomorphism) with respect to maps from F to degree n functors.

We use $P_\infty F$ to denote the homotopy inverse limit of this tower. Conditions under which the tower converges to $P_\infty F$ are discussed in [J-M3], §4.

Definition 1.14. *For a functor F from $\mathcal{C}$ to $Ch\mathcal{A}$, we define D_nF : $\mathcal{C} \longrightarrow Ch\mathcal{A}$ to be the homotopy fiber*

$$D_nF = \text{fiber}(P_nF \xrightarrow{q_nF} P_{n-1}F).$$

That is, D_nF is the chain complex obtained by shifting the mapping cone of q_nF down one degree. The functor D_nF is degree n since both P_nF and $P_{n-1}F$ are degree n and cr_{n+1} is exact. We say that a functor F is homogeneous degree n if it is degree n and $P_{n-1}F \simeq *$.

The natural transformations $q_n : \perp_{n+1} \to \perp_n$ can also be used to construct natural transformations $q : P_t \to P_n$, $\Sigma(n,t) : P_tP_n \to P_n$ and $\Sigma(n,t) : P_nP_t \to P_n$ for $t \geq n$. These natural transformations will be used in sections 4, 5, and 6.

Definition 1.15. *For $t > n$, let $q : \perp_{t+1} \to \perp_{n+1}$ be the natural transformation of cotriples $q = q_{n+1} \circ q_{n+2} \circ \ldots \circ q_t : \perp_{t+1} \to \perp_{n+1}$ and for $t = n$, let $q = \text{id}_{\perp_{n+1}}$. We will also use q to denote the natural transformations $q : P_t \to P_n$ induced by these maps of cotriples.*

Definition 1.16. *Let $t \geq n$. The functor P_tP_nF is the total complex of the bicomplex given by $(p,r) \mapsto \perp_{t+1}^p (\perp_{n+1}^r F)$. Let $\Sigma(t,n)$: $P_tP_nF \to P_nF$ be the map that in degree s is given by*

$$\begin{array}{ccc} \oplus_{p+r=s} \perp_{t+1}^p (\perp_{n+1}^r F) & \xrightarrow{\oplus(q[p])(\text{id})} & \oplus_{p+r=s} \perp_{n+1}^p (\perp_{n+1}^r F) \\ & & \| \\ \perp^s F & \xleftarrow{+} & \oplus_{p+r=s} \perp^s F. \end{array}$$

The map $\Sigma(n,t) : P_nP_tF \to P_nF$ is defined using $\oplus(\text{id})(q[r])$.

It is straightforward to show that $\Sigma(t,n)$ and $\Sigma(n,t)$ are natural chain maps, and that the diagrams below commute for $u \geq t \geq n$.

$$\begin{array}{ccccc} P_tF & \xrightarrow{P_t(p_nF)} & P_tP_nF & \xleftarrow{p_t(P_nF)} & P_nF \\ & q \searrow & \Big\downarrow \Sigma(t,n) & \swarrow = & \\ & & P_nF & & \end{array}$$

$$\begin{array}{ccccc} P_nF & \xrightarrow{P_n(p_tF)} & P_nP_tF & \xleftarrow{p_n(P_tF)} & P_tF \\ & = \searrow & \Big\downarrow \Sigma(n,t) & \swarrow q & \\ & & P_nF & & \end{array}$$

$$(1.17)\qquad \begin{array}{ccc} P_uP_tP_n & \xrightarrow{P_u(\Sigma(t,n))} & P_uP_n \\ \Big\downarrow \Sigma(u,t)P_n & & \Big\downarrow \Sigma(u,n) \\ P_tP_n & \xrightarrow{\Sigma(t,n)} & P_n \end{array} \qquad \begin{array}{ccc} P_nP_tP_u & \xrightarrow{P_n(\Sigma(t,u))} & P_nP_t \\ \Big\downarrow \Sigma(n,t)P_u & & \Big\downarrow \Sigma(n,t) \\ P_nP_u & \xrightarrow{\Sigma(n,u)} & P_n. \end{array}$$

Using the natural transformations $\Sigma(t,n)$ and $\Sigma(n,t)$ of definition 1.16, we can define natural chain maps $\Sigma(t,n)_D : P_tD_n \to D_n$ and $\Sigma(n,t) : D_nP_t \to D_n$.

Definition 1.18. *For $t \geq n$, $\Sigma(t,n)_D : P_tD_n \to D_n$ is the mapping cone of the map of bicomplexes given by*

$$\begin{pmatrix} \Sigma(t,n) \\ \\ \Sigma(t,n-1) \end{pmatrix} : \begin{pmatrix} P_tP_n \\ \Big\downarrow P_tq_n \\ P_tP_{n-1} \end{pmatrix} \to \begin{pmatrix} P_n \\ \Big\downarrow q_n \\ P_{n-1} \end{pmatrix},$$

and $\Sigma_D(n,t) : D_nP_t \to D_n$ is the mapping cone of the map of bicomplexes given by

$$\begin{pmatrix} \Sigma(n,t) \\ \\ \Sigma(n-1,t) \end{pmatrix} : \begin{pmatrix} P_nP_t \\ \Big\downarrow q_nP_t \\ P_{n-1}P_t \end{pmatrix} \to \begin{pmatrix} P_n \\ \Big\downarrow q_n \\ P_{n-1} \end{pmatrix}.$$

It follows from the definitions that the following diagrams commute:

$$\begin{array}{ccccc} P_tP_nF & \xleftarrow{P_t(d_n)} & P_tD_nF & \xleftarrow{p_t(D_nF)} & D_nF \\ \downarrow{\scriptstyle \Sigma(t,n)} & & \downarrow{\scriptstyle \Sigma(t,n)_D} & \swarrow = & \\ P_nF & \xleftarrow{d_n} & D_nF & & \end{array}$$

$$\begin{array}{ccccc} D_nF & \xrightarrow{D_n(p_tF)} & D_nP_tF & \xrightarrow{d_n(P_tF)} & P_nP_tF \\ & = \searrow & \downarrow{\scriptstyle \Sigma_D(n,t)} & & \downarrow{\scriptstyle \Sigma(n,t)} \\ & & D_nF & \xrightarrow{d_n} & P_nF. \end{array} \tag{1.19}$$

It is straightforward to check that the maps $\Sigma(t,n), \Sigma(n,t), \Sigma(t,n)_D$ and $\Sigma_D(n,t)$ are associative with respect to one another.

We can also define natural chain maps from D_nD_t to D_n. However, for $t > n$, we have

$$\begin{aligned} D_tD_n &= \text{hfiber}(D_tP_n \to D_{t-1}P_n) \\ &\simeq \text{hfiber}(P_n \to P_n) \\ &\simeq *. \end{aligned}$$

Similarly, $D_nD_t \simeq *$, and so we focus our attention on defining a chain map $\Sigma_D(n) : D_nD_n \to D_n$. To do so, note that D_nD_n is the total complex of the square of complexes given by

$$\begin{array}{ccc} P_nP_n & \xrightarrow{P_n(q_n)} & P_nP_{n-1} \\ \downarrow{\scriptstyle q_n(P_n)} & & \downarrow{\scriptstyle q_n(P_{n-1})} \\ P_{n-1}P_n & \xrightarrow{P_{n-1}(q_n)} & P_{n-1}P_{n-1}. \end{array}$$

If we consider Tot of the squares of complexes in figure 2 we see that the lower square of complexes can be mapped in two equivalent ways to D_n. Namely, we have (1.21)

$$\begin{array}{ccc} P_n & \longrightarrow & 0 \\ \downarrow{\scriptstyle q_n} & & \downarrow \\ P_{n-1} & \longrightarrow & 0 \end{array} \xleftarrow{\simeq} \begin{array}{ccc} P_n & \xrightarrow{q_n} & P_{n-1} \\ \downarrow{\scriptstyle q_n} & & \| \\ P_{n-1} & = & P_{n-1} \end{array} \xrightarrow{\simeq} \begin{array}{ccc} P_n & \xrightarrow{q_n} & P_{n-1} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}.$$

$$\begin{array}{ccc}
P_nP_n & \xrightarrow{P_n(q_n)} & P_nP_{n-1} \\
\Big\downarrow{\scriptstyle q_n(P_n)} & & \Big\downarrow{\scriptstyle q_n(P_{n-1})} \\
P_{n-1}P_n & \xrightarrow{P_{n-1}(q_n)} & P_{n-1}P_{n-1}.
\end{array}$$

$$\Big\downarrow \begin{pmatrix} \Sigma(n,n) & \Sigma(n,n-1) \\ \Sigma(n-1,n) & \Sigma(n-1,n-1) \end{pmatrix}$$

(1.20)
$$\begin{array}{ccc}
P_n & \xrightarrow{q_n} & P_{n-1} \\
\Big\downarrow{\scriptstyle q_n} & & \| \\
P_{n-1} & = & P_{n-1}.
\end{array}$$

figure 2

To produce the desired map from D_nD_n to D_n, we may use either of the maps in (1.21), provided that we are consistent in our choice. We define $\Sigma_D(n) : D_nD_n \to D_n$ to be the natural chain map obtained by composing the map of (1.20) with the left equivalence of (1.21). From this definition, it follows readily that the diagrams below commute:

(1.22)
$$\begin{array}{ccccc}
P_nD_nF & \xleftarrow{d_n(D_nF)} & D_nD_nF & \xrightarrow{D_n(d_n)} & D_nP_nF \\
& {\scriptstyle \Sigma(n,n)_D} \searrow & \Big\downarrow{\scriptstyle \Sigma_D(n)} & \swarrow {\scriptstyle \Sigma_D(n,n)} & \\
& & D_nF & &
\end{array}$$

$$\begin{array}{ccc}
D_nD_nD_n & \xrightarrow{D_n(\Sigma_D(n))} & D_nD_n \\
\Big\downarrow{\scriptstyle \Sigma_D(n)D_n} & & \Big\downarrow{\scriptstyle \Sigma_D(n)} \\
D_nD_n & \xrightarrow{\Sigma_D(n)} & D_n.
\end{array}$$

2. Functors on a line

A linear function f from $\mathbb{R}$ to $\mathbb{R}$ is completely determined by its values at two points. For example, if $f(0) = 0$ and the value of $f(1)$ is known, then the fact that $f(x) = xf(1)$ enables us to find $f(x)$ for all x. Being able to determine $f(x)$ in this way depends both on a property of f and properties of $\mathbb{R}$.

In the introduction, we discussed a similar result proved by Eilenberg and Watts that showed that an additive, right continuous functor from R-modules to S-modules was determined by its value at R. The proof relied on the facts that R generated the category of R-modules, and that the functor F preserved the operations used to generate R-modules from R, namely F preserved sums, filtered colimits, and cokernels. In order to extend this result to degree n functors in section 5, we use this section to develop the analogs of these properties for a basepointed category $\mathcal{C}$ and functors from $\mathcal{C}$ to $Ch\mathcal{A}$.

In working with the category $\mathcal{C}$ instead of a module category, we may no longer have a single object that can be used to "generate" the domain category and classify functors. Instead we will work with collections of objects generated by a single object in $\mathcal{C}$ and functors that behave well with respect to the operations that generate these collections from C. We will refer to such collections as "lines" and such functors as "functors defined along a line". A key result of this section will be to show that any degree n functor defined along C is determined by its value at the object $\vee_{i=1}^{n} C$.

To understand what we mean by a "line" in the category $\mathcal{C}$, first consider the real number line and the category of R-modules for a ring R. The real number line can be generated by the set of natural numbers $\mathbb{N}$ through standard operations such as addition, subtraction, multiplication, division, and completion. Similarly, we can consider the category of R-modules being generated, up to quasi-isomorphism, by the finitely generated R-modules, R, $R^{\oplus 2}$, $R^{\oplus 3}, \ldots$, via free resolutions and colimits. In considering the category of R-modules as a "line", the finitely generated R-modules, R, $R^{\oplus 2}$, $R^{\oplus 3}, \ldots$, play the role of the natural numbers, and two points on the line are considered to be close together if there is a sequence of highly connected chain maps (possibly going in different directions) that relate them.

For an arbitrary basepointed category with finite coproducts $\mathcal{C}$ and an object C in that category, we will consider the line generated by C. On this line, the role of the natural numbers or finitely generated free C objects will be played by the objects $\mathbf{1}_C, \mathbf{2}_C, \mathbf{3}_C, \ldots$ defined below.

Definition 2.1. *Let C be an object in $\mathcal{C}$ and $n \geq 0$ be an integer. Let* Sets_* *denote the category whose objects are basepointed sets and whose morphisms are basepoint-preserving set maps, and $*$ denote the basepoint for an object in* Sets_*. *Then* $\mathbf{n}$ *is the object in* Sets_* *given by*

$$\mathbf{n} = \{* = 0, 1, 2, 3, \dots, n\}$$

and $\mathbf{n}_C$ *is the object in $\mathcal{C}$ defined by*

$$\mathbf{n}_C = \bigvee_{\mathbf{n}\setminus\{*\}} C.$$

We must also develop the idea of resolutions along C. To do so, we must place some additional conditions on $\mathcal{C}$. As in previous sections, $\mathcal{C}$ will be a basepointed category, but in this section we require that it have arbitrary coproducts rather than finite coproducts. Furthermore, we will assume that for any object C in $\mathcal{C}$ the functor $\mathrm{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} \longrightarrow \mathrm{Sets}_*$ has a left adjoint $C \wedge - : \mathrm{Sets}_* \to \mathcal{C}$ that is natural in C. Hence, for any objects C and C' in $\mathcal{C}$ and based set U, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(C \wedge U, C') \cong \mathrm{Hom}_{\mathrm{Sets}_*}(U, \mathrm{Hom}_{\mathcal{C}}(C, C')).$$

Moreover, for any based set U, $C \wedge U \cong \bigvee_{U\setminus\{*\}} C$, since

$$\mathrm{Hom}_{\mathcal{C}}(\bigvee_{U\setminus\{*\}} C, C') \cong \prod_{U\setminus\{*\}} \mathrm{Hom}_{\mathcal{C}}(C, C') \cong \mathrm{Hom}_{\mathrm{Sets}_*}(U, \mathrm{Hom}_{\mathcal{C}}(C, C')).$$

Thus, $\mathbf{n}_C \doteq C \wedge \mathbf{n}$.

To resolve objects with respect to an object C in $\mathcal{C}$, we use the cotriple associated to the adjoint pair $(C \wedge -, \mathrm{Hom}_{\mathcal{C}}(C, -))$.

Definition 2.2. *Let C and X be objects in $\mathcal{C}$. We let $\perp_C = (C \wedge -) \circ \mathrm{Hom}_{\mathcal{C}}(C, -)$ denote the cotriple associated to the adjoint pair $(C \wedge -, \mathrm{Hom}_{\mathcal{C}}(C, -))$. The resolution of X along C is the simplicial object $\perp_C^* X$ associated to the cotriple $\perp_C$.*

By the line generated by C, we will mean the collection of objects generated from C by means of the functor $C \wedge -$ and resolutions along C. Unlike the line generated by R for the category of R-modules, we lack a means of determining whether or not objects are close together on the line generated by C since $\mathcal{C}$ is not necessarily abelian. However, since we are interested in functors from $\mathcal{C}$ to an abelian category $Ch\mathcal{A}$, we will be most concerned about the image of the line generated by C under such functors. For that reason we define resolutions of functors along C.

Definition 2.3. *For a functor $F : \mathcal{C} \longrightarrow Ch\mathcal{A}$, the canonical C-resolution of F is the simplicial functor $\perp_C^* F$ whose value at an object X in $\mathcal{C}$ is obtained by applying F degreewise to $\perp_C^* X$. That is, $\perp_C^* F(X) = F(\perp_C^* X)$.*

We note that $\perp_C^* F$ can also be viewed as the simplicial object arising from a cotriple on the category $\text{Func}(\mathcal{C}, Ch\mathcal{A})$. The cotriple in this case, which we will also denote $\perp_C$, is the one associated to the adjoint pair $(\text{Hom}_{\mathcal{C}}(C,-)^*, (C \wedge -)^*)$ where $\text{Hom}_{\mathcal{C}}(C,-)^* : \text{Func}(\text{Sets}_*, Ch\mathcal{A}) \to \text{Func}(\mathcal{C}, Ch\mathcal{A})$ and $(C \wedge -)^* : \text{Func}(\mathcal{C}, Ch\mathcal{A}) \to \text{Func}(\text{Sets}_*, Ch\mathcal{A})$.

Recall from remark 1.8 that a simplicial object arising from a cotriple is augmented. In this case, for an object X in $\mathcal{C}$, the augmentation map from $\perp_C F(X) = F(C \wedge \text{Hom}_{\mathcal{C}}(C, X))$ to $F(X)$ is $F(ev)$ where ev is the evaluation map. For certain objects this augmentation has a section and is an equivalence.

Lemma 2.4. *Let C be an object in $\mathcal{C}$, F be a functor from $\mathcal{C}$ to $Ch\mathcal{A}$, and U be an object in* Sets_*. *Then*

$$\perp_C^* F(C \wedge U) \xrightarrow{\simeq} F(C \wedge U).$$

Proof. From the adjoint pair $(C \wedge -, \text{Hom}_{\mathcal{C}}(C,-))$, the unit for the adjunction gives us a morphism $U \xrightarrow{\eta} \text{Hom}_{\mathcal{C}}(C, C \wedge U)$. This yields the map

$$\text{id}_C \wedge \eta : C \wedge U \longrightarrow C \wedge \text{Hom}_{\mathcal{C}}(C, C \wedge U)$$

which is a natural section to $ev_{C\wedge U}$. Thus, $\perp_C F(C\wedge U) \stackrel{F(ev_{C\wedge U})}{\longrightarrow} F(C\wedge U)$ has a natural section and it follows from proposition 2.5 of [J-M3] that

$$\perp_C^* F(C\wedge U) \xrightarrow{\simeq} F(C\wedge U)$$

is a simplicial homotopy equivalence.

Thus we see that F and $\perp_C^* F$ agree on the subcategory of $\mathcal{C}$ generated by objects of the form $C\wedge U$ for $U\in \text{Sets}_*$. Moreover, if G is a functor defined on this subcategory of $\mathcal{C}$, then $\perp_C^* G$ is naturally a functor from all of $\mathcal{C}$ to $Ch\mathcal{A}$.

The canonical C-resolution of F agrees with standard free resolutions as follows.

Examples 2.5.

1) Let $\mathcal{C} = \text{Sets}_*$ and consider the object $S^0 = \{*, 1\}$ in Sets_*. The functor $S^0 \wedge -$ is an equivalence and the identity functor, $\text{id}_{\text{Sets}_*}$, is its right adjoint. Thus, $\perp_{S^0} F = F$ and $\perp_{S^0}^* F$ is the trivial simplicial functor associated to F. It follows that the adjunction map is a simplicial homotopy equivalence from $\perp_{S^0}^* F$ to F.

2) Let $\mathcal{M}_R$ be the category of right R-modules for a ring R. The functor $\text{Hom}_{\mathcal{M}_R}(R, -)$ is equivalent to the forgetful functor, i.e., the functor that takes an R-module to its underlying set. The left adjoint of the forgetful functor is the reduced free R-module functor $\widetilde{R}[-] : \text{Sets}_* \longrightarrow \mathcal{M}_R$ that takes a based set X to the R-module $R[X]/R[*]$. It follows that $\perp_R F(-) = F(\widetilde{R}[-])$. Thus for an R-module M, $\perp_R^* F(M) = F(\perp_R^* M)$ is the simplicial object obtained by applying F degreewise to the canonical free R-module resolution of M.

3) Let k be a fixed commutative ring and Comm_k be the category of augmented commutative k-algebras (with units). The basepoint in this category is k. As a functor from Comm_k to Sets_*, $\text{Hom}_{\text{Comm}_k}(k, -)$ is the forgetful functor and its left adjoint is $k[-]$ where for a set X, $k[X]$ is the free commutative k-algebra generated by X. Thus $\perp_k F(-) = F(k[-])$ and for a commuta-

tive augmented k-algebra A, $\perp_k^* F(A) = F(\perp_k^* A)$ is the simplicial object obtained by applying F degreewise to the usual free commutative k-algebra resolution of A.

4) Let $\mathcal{A}$ and $\mathcal{C}$ be the category of abelian groups. For $n \in \mathbb{N}$, consider $\mathbb{Z}/n\mathbb{Z} \wedge -$ and its right adjoint $\mathrm{Hom}_{\mathcal{C}}(\mathbb{Z}/n\mathbb{Z}, -)$. For an abelian group A, let $A_n = \mathrm{Hom}_{\mathcal{C}}(\mathbb{Z}/n\mathbb{Z}, A)$. Then $\perp_{\mathbb{Z}/n\mathbb{Z}} (A) = \widetilde{\mathbb{Z}/n\mathbb{Z}}[A_n]$ and $\perp^*_{\mathbb{Z}/n\mathbb{Z}} \mathrm{id}(A) \xrightarrow{\simeq} A_n$. It follows that for any functor $F : \mathcal{C} \to \mathcal{A}$, $\perp^*_{\mathbb{Z}/n\mathbb{Z}} F(X) \longrightarrow F(X)$ is an equivalence whenever X is n-torsion.

In lemma 2.4, we saw that $\perp_C^* F$ agrees with F on objects of the form $C \wedge U$. We now wish to consider functors that are completely determined by their behavior on the line generated by C. To that end we define two properties of functors related to C. The first property guarantees that values of the functor at $C \wedge U$ for any basepointed set U can be expressed in terms of the objects $\mathbf{0}_C, \mathbf{1}_C, \mathbf{2}_C, \ldots, \mathbf{n}_C, \ldots$. This is analogous to the condition that functors preserve filtered colimits in the additive functor classification of Eilenberg and Watts.

Definition 2.6. *A functor F from $\mathcal{C}$ to $Ch\mathcal{A}$ satisfies the limit axiom at $C \in \mathcal{C}$ if for $U \in \mathrm{Sets}_*$,*

$$\mathrm{colim}_{\{X \subseteq U \ | \ |X| < \infty\}} F(C \wedge X) \xrightarrow{\simeq} F(C \wedge U)$$

where the structure maps of the colimit are induced by inclusions. This is equivalent to,

$$\mathrm{colim}_{\{X \subseteq U \ | \ |X| < \infty\}} H_k F(C \wedge X) \xrightarrow{\simeq} H_k(F \wedge U)$$

for all $k \in \mathbb{Z}$ since homology commutes with filtered direct limits.

Example 2.7. Every degree n functor satisfies the limit axiom along C for all objects C in $\mathcal{C}$. To see this, recall that H_k of a degree n functor is a strictly degree n functor. For any strictly degree n functor F, it follows from proposition 1.2 of [J-M3] that

$$F(C \wedge U) \cong \bigoplus_{\substack{Y \subseteq U \\ |Y| \leq n}} cr_{|Y|} F(C, \ldots, C) \cong \operatorname{colim}_{\{X \subseteq U \mid |X| < \infty\}} F(C \wedge X).$$

The next condition guarantees that a functor is completely determined by its behavior on objects of the form $C \wedge U$ and objects resolved along C, i.e., a functor is determined by its behavior on the line generated by C. This condition is analogous to the condition of right continuity in the classification of additive functors.

Definition 2.8. *A functor $F : \mathcal{C} \to Ch\mathcal{A}$ that satisfies the limit axiom along C is defined along C if $\perp_C^* F \xrightarrow{\simeq} F$. If F is defined along C, we will use $F(\mathbf{n})$ to denote $F(\mathbf{n}_C)$.*

As a consequence of the above definitions, we have the following.

Lemma 2.9. *Let C be an object in $\mathcal{C}$.*

1) *For any functor $F : \mathcal{C} \to Ch\mathcal{A}$, the functor $\perp_C^* F$ is defined along C.*
2) *Let $G, G' : \mathcal{C} \to Ch\mathcal{A}$ be functors that satisfy the limit axiom along C and $\eta : G \longrightarrow G'$ be a natural transformation. The natural transformation $\perp_C^* \eta : \perp_C^* G \to \perp_C^* G'$ is an equivalence if and only if $\eta_{\mathbf{n}_C}$ is an equivalence for all n.*
3) *For any functor $F : \mathcal{C} \to Ch\mathcal{A}$, F is degree n if and only if $\perp_C^* F$ is degree n for all objects C in $\mathcal{C}$.*
4) *For any functor $F : \mathcal{C} \to Ch\mathcal{A}$, F is homogeneous degree n if and only if $\perp_C^* F$ is homogeneous degree n for all objects C in $\mathcal{C}$.*

Proof.

1) By lemma 2.4, $\perp_C^* F(C \wedge U) \simeq F(C \wedge U)$ for all sets U. It follows from the definition of $\perp_C$ that $\perp_C^{(k)} \perp_C^* F \simeq \perp_C^{(k)} F$ for all $k \geq 1$ and so $\perp_C^* \perp_C^* F \simeq \perp_C^* F$. That $\perp_C^* F$ satisfies the limit axiom follows from the fact that F does. Hence, $\perp_C^* F$ is defined along C.

2) Suppose $\eta_{\mathbf{n}_C}$ is an equivalence for all n. Since G and G' satisfy the limit axiom along C, it follows that $\eta_{C \wedge U}$ is an equivalence for all basepointed sets U. Then, again as in part 1), we have $\perp_C^* \eta : \perp_C^* G \xrightarrow{\simeq} \perp_C^* G'$. Conversely, $\perp_C^* G(\mathbf{n}_C) \simeq \perp_C^* G'(\mathbf{n}_C)$ implies $G(\mathbf{n}_C) \simeq G'(\mathbf{n}_C)$ for all n by lemma 2.4.

3) Let k be a natural number and C be an object in $\mathcal{C}$. Setting $U = \mathbf{k}$ in lemma 2.4, we see that $\perp_C^* F(\bigvee_{i=1}^k C) \xrightarrow{\simeq} F(\bigvee_{i=1}^k C)$. It follows from the definition of cross effects that $cr_k \perp_C^* F(C, \ldots, C)$ is equivalent to $cr_k F(C, \ldots, C)$. Hence, if $\perp_C^* F$ is degree n for all objects C, then $cr_{n+1} F(C, \ldots, C) \simeq 0$ for all C. One can show (see the proof of 2.11 in [J-M3]) that this is enough to ensure that $cr_{n+1} F(C_1, \ldots, C_{n+1}) \simeq 0$ for all choices of $C_1, \ldots, C_{n+1}$ in $\mathcal{C}$ and so F is degree n.

   Conversely, if F is degree n, then we know that for any object C in $\mathcal{C}$, $cr_k \perp_C^* F(C, \ldots, C) \simeq 0$ for all $k > n$. To conclude that $cr_{n+1} \perp_C^* F(C_1, C_2, \ldots, C_{n+1}) \simeq 0$ for any objects $C_1, C_2, \ldots, C_{n+1}$ in $\mathcal{C}$, it will suffice to show that

$$cr_{n+1} \perp_C^* F(C', C', \ldots, C') \simeq 0 \text{ for any } C' \in \mathcal{C}.$$

   However, since $\perp_C^* F$ is defined along C, the diagonals of its cross effects are as well, and so by part 2) it suffices to show that $cr_{n+1} \perp_C^* F(\mathbf{n}_C, \ldots, \mathbf{n}_C) \simeq 0$. But $cr_{n+1} \perp_C^* F(\mathbf{n}_C, \ldots, \mathbf{n}_C)$ is a direct sum of terms of the form $cr_k \perp_C^* F(C, \ldots, C)$ where $k > n$ and hence $cr_{n+1} \perp_C^* F(\mathbf{n}_C, \ldots, \mathbf{n}_C) \simeq 0$. Therefore, $\perp_C^* F$ is degree n for all objects C in $\mathcal{C}$.

4) The proof is similar to that of 3).

Using lemma 2.9, we can prove that degree n functors defined along C are determined by their values at $\mathbf{n}_C$. This follows immediately from the next result.

Lemma 2.10. *For $\eta : F \longrightarrow F'$ a natural transformation between degree n functors, the following are equivalent:*

1) $\perp_C^* \eta : \perp_C^* F \longrightarrow \perp_C^* F'$ *is an equivalence.*

2) $\eta_{\mathbf{n}_C} : F(\mathbf{n}_C) \xrightarrow{\simeq} F'(\mathbf{n}_C)$ *is an equivalence.*

3) $cr_k\eta(C) : cr_kF(C) \xrightarrow{\simeq} cr_kF'(C)$ *is an equivalence for all* $k \leq n$*.*

Proof. Conditions 2 and 3 are equivalent by the definition of cross effects. By lemma 2.9.2 and example 2.7, condition 1 implies condition 2. If $cr_k\eta(C)$ is an equivalence for all $0 \leq k \leq n$, then $\eta_{\mathbf{t}_C}$ is an equivalence for all t since F and F' are degree n. Hence, by lemma 2.9.2 and example 2.7 we see that condition 3 implies condition 1.

Theorem 2.11 *If F and G are degree n functors defined along C, then a natural transformation $\eta : F \to G$ is an equivalence if and only if $\eta_{\mathbf{n}_C} : F(\mathbf{n}_C) \to G(\mathbf{n}_C)$ is an equivalence.*

Proof. By the lemma $\perp_C^* \eta$ is an equivalence if and only if $\perp_C^* \eta_{\mathbf{n}_C}$ is an equivalence. Since F and G are defined along C, $\perp_C^* \eta$ is equivalent to η and the result follows.

Definition 2.12. (section 5 of [J-M3]) *Let F be a functor from $\mathcal{C}$ to $\mathcal{A}$, and X and Y objects in $\mathcal{C}$. We set* $F_Y(X) = \ker(F(Y \vee X) \to F(Y))$ *(equivalent in our setting to the expression $f(y+v) - f(y)$) and define the differential of F to be the bifunctor*

$$\nabla_X F(Y) = \nabla F(X; Y) = (D_1 F_Y)(X).$$

The differential $\nabla_X F(Y)$ plays the role of the derivative of F at Y in the direction of X.

From definition 2.12 we can define the derivative of a functor in the "direction" of a given object X at the point Y. However, if we consider functions $f : \mathbb{R} \to \mathbb{R}$, we need not consider derivatives in different directions - we need only differentiate in the direction of the real line $\mathbb{R}$. Moreover, the derivative determines the function up to a constant. Similarly, for functors defined along the line determined by C, it suffices to study their derivatives in the direction of C. We define the derivative of a functor in the direction of the line generated by C below and show that when a functor is defined along C, this derivative along C determines the functor up to a constant term.

Definition 2.13. *Let $F : \mathcal{C} \to Ch\mathcal{A}$ be a functor and C an object in $\mathcal{C}$. We define $\frac{d}{dC}F$, the derivative of F along C to be:*

$$\begin{aligned}\frac{d}{dC}F(Y) = \nabla_C F(Y) = \nabla F(\mathbf{1};Y) &= D_1[\frac{F(-\vee Y)}{F(Y)}](\mathbf{1}_C)\\ &= D_1[cr_2F(-,Y) \oplus cr_1F(-)](\mathbf{1}_C)\\ &= D_1[cr_2F(-,Y)](\mathbf{1}) \oplus D_1F(\mathbf{1}).\end{aligned}$$

Similarly, the nth derivative of F along C is given by

$$\frac{d^n}{dC^n}F(Y) = \nabla^n F(C,\ldots,C;Y) = \nabla^n F(\mathbf{1},\ldots,\mathbf{1};Y).$$

Remarks 2.14.

a) By corollaries 5.10 and 5.11 of [J-M3], $\frac{d^n}{dC^n}F$ is a Σ_n–equivariant functor such that $\frac{d^n}{dC^n}F(*) = D_nF(\mathbf{1})_{h\Sigma_n}$.

b) If F is degree n, then by proposition 5.4 of [J-M3], $\frac{d}{dC}F$ is degree $n-1$. Moreover, by proposition 5.18 of [J-M3], we have

$$\frac{d}{dC}(P_nF) \simeq P_{n-1}(\frac{d}{dC}F).$$

In the next proposition we see that the differential for a functor defined along C is completely determined by the derivative of F in the direction of C.

Proposition 2.15. *If $\eta : F \longrightarrow G$ is a natural transformation of functors defined along C then $\frac{d}{dC}\eta$ is an equivalence if and only if $\nabla\eta$ is an equivalence of bifunctors.*

Proof. If $\nabla\eta$ is an equivalence then $\frac{d}{dC}\eta = \nabla\eta(C, -)$ is an equivalence. Conversely, note that $\nabla\eta(\star, Y)$ is linear in its first variable for any object Y. Hence, since $\frac{d}{dC}\eta(Y) = \nabla\eta(C, Y)$ is an equivalence, $\nabla\eta(\mathbf{n}_C, Y)$ is an equivalence for any n. By lemma 2.9, it follows that $\perp_C^* \nabla\eta(\star, Y)$ is an equivalence. But F and G are defined along C, and so $\nabla F(\star, Y)$ and $\nabla G(\star, Y)$ are also defined along C in $\star$. This follows from the definition of ∇, and the fact that the cross effects of F and G are defined along C. Thus, since $\nabla F(\star, Y)$ and $\nabla G(\star, Y)$ are defined along C and $\perp_C^* \nabla\eta(\star, Y)$ is an equivalence, $\nabla\eta(\star, Y)$ must also be an equivalence.

Finally, we see that a functor defined along C is determined, up to a constant, by its derivative in the direction of C. Recall from [J-M3] that $P_\infty F$ is the homotopy inverse limit of the Taylor tower for F.

Theorem 2.16. *If $\eta : F \longrightarrow G$ is a natural transformation of functors defined along C such that $\frac{d}{dC}\eta$ is an equivalence then*

$$\begin{array}{ccc} P_\infty F & \xrightarrow{P_\infty \eta} & P_\infty G \\ \downarrow p & & \downarrow p \\ F(*) & \xrightarrow{\eta_*} & G(*) \end{array}$$

is Cartesian.

Proof. The result follows from propositions 5.3 of [J-M3] and 2.15.

3. Filtration by rank

In the previous section, we saw that if a functor is degree n and is defined along the line generated by an object C, then it is determined by its value at $\mathbf{n}_C$. The converse of this statement is not true. In this section, we will see that a functor determined by its value at $\mathbf{n}_C$ and defined along C (which we will call a rank n functor) is not necessarily degree n. We will produce for a functor F a universal sequence of functors $\{\mathcal{L}_i F\}_{i\geq 0}$ such that each $\mathcal{L}_i F$ has rank i. We will study the relationship between this rank filtration of F, and the Taylor tower and derivative constructions of section 5 of [J-M3]. We will be working throughout this section with functors from $\mathcal{C}$ to $Ch\mathcal{A}$ where $\mathcal{C}$ is a basepointed category with coproducts, and $\mathcal{A}$ is a cocomplete full and faithful subcategory of the category of left modules over a ring A.

Our first task is to produce, in a systematic fashion, functors that are determined by their values at the points $\mathbf{0}_C, \mathbf{1}_C, \ldots, \mathbf{n}_C$. We do so by means of left Kan extensions, defined as follows.

Definition 3.1. *Let $\mathcal{D}$ be a small, full, and faithful subcategory of $\mathcal{C}$, and let G be a functor from $\mathcal{D}$ to $\mathcal{A}$. The* left Kan extension of G *is the simplicial functor $L_{\mathcal{D}}G : \mathcal{C} \to$ Simp $Ch\mathcal{A}$ that in degree n is given by setting $L_{\mathcal{D}}G([n])(-)$ to be:*

$$\bigoplus_{D_0,\ldots,D_n \in \mathrm{Obj}\mathcal{D}} G(D_0) \wedge [\mathrm{Hom}_{\mathcal{C}}(D_0, D_1) \times \cdots \times \mathrm{Hom}_{\mathcal{C}}(D_n, -)]_+.$$

(For a set X, X_+ is the associated pointed set with basepoint $+$.) For an element, $(x; \alpha_1, \ldots, \alpha_n; \beta)$, of a direct summand of $L_{\mathcal{D}}G([n])(-)$, the face and degeneracy maps are defined as follows.

$$d_i(x;\alpha_1,\ldots,\alpha_n;\beta) = \begin{cases} (G(\alpha_1)(x);\alpha_2,\ldots,\alpha_n;\beta) & \text{if } i=0 \\ (x;\alpha_1,..,\alpha_{i+1}\circ\alpha_i,..,\alpha_n;\beta) & \text{if } 1 \leq i \leq n-1 \\ (x;\alpha_1,\ldots,\alpha_{n-1};\beta\circ\alpha_n) & \text{if } i=n \end{cases}$$

$$s_j(x;\alpha_1,\ldots,\alpha_n;\beta) = (x;\alpha_1,\ldots,id_{D_j},\ldots,\alpha_n;\beta).$$

For a functor F from $\mathcal{D}$ to $Ch\mathcal{A}$, we will use $L_{\mathcal{D}}F$ to denote $L_{\mathcal{D}}(F|_{\mathcal{D}})$.

For any functor F from $\mathcal{D}$ to $\mathcal{C}hA$, the simplicial functor $L_{\mathcal{D}}F$ is augmented over F. That is, there is a natural transformation $L_{\mathcal{D}}F \longrightarrow F$ given by sending $(x;\alpha_1,\ldots,\alpha_n;\beta)$ to $F(\beta\circ\alpha_n\circ\ldots\circ\alpha_1)(x)$. For an object D in $\mathcal{D}$, this augmentation map is a simplicial homotopy equivalence by sending $x \in F(D)$ to $(x; id_D,\ldots,id_D;id_D)$. Similarly, if $\mathcal{D}'$ is a faithful and full subcategory of $\mathcal{D}$ and F is a functor from $\mathcal{D}'$ to $\mathcal{C}h\mathcal{A}$, then $L_{\mathcal{D}}(L_{\mathcal{D}'}F) \longrightarrow L_{\mathcal{D}'}F$ has an inverse defined by mapping $L_{\mathcal{D}'}F$ to $L_{\mathcal{D}}(L_{\mathcal{D}'}F)[0]$ by means of the map that sends

$$(x;\alpha_1,\ldots,\alpha_n;\beta) \in F(D_0') \wedge [\mathrm{Hom}_{\mathcal{C}}(D_0',D_1') \times \ldots]_+ \subseteq (L_{\mathcal{D}'}F)[n](\star)$$

to $((x;\alpha_1,\ldots,\alpha_n;\mathrm{id}_{\mathcal{D}_n'});\beta)$ in

$$\begin{gathered}(F(D_0') \wedge [\mathrm{Hom}_{\mathcal{C}}(D_0',D_1') \times \ldots]_+) \wedge (\mathrm{Hom}_{\mathcal{C}}(D_n',\star))_+ \\ \cap\!| \\ L_{\mathcal{D}}((L_{\mathcal{D}'}F[n])[0])(\star).\end{gathered}$$

Thus, on $\mathcal{C}$,

$$L_{\mathcal{D}}(L_{\mathcal{D}'}F) \xrightarrow{\simeq} L_{\mathcal{D}'}F. \tag{3.2}$$

We are interested in constructing left Kan extensions using the following subcategories of $\mathcal{C}$.

Definition 3.3. *Let C be an object in $\mathcal{C}$.*

1) *$\mathcal{S}_C$ is the full subcategory of $\mathcal{C}$ generated by objects of the form $C \wedge U$ for $U \in$ Sets$_*$. Note that $\mathcal{S}_C$ is not small.*
2) *$\mathcal{N}_C$ is the full subcategory of $\mathcal{S}_C$ generated by the set of objects*

$$\mathrm{Obj}(\mathcal{N}_C) = \{C \wedge \mathbf{n} \mid n \in \mathbb{Z}, n \geq 0\}.$$

3) *For $0 \leq k \leq n$, $\mathcal{N}_C[k,n]$ is the full subcategory of $\mathcal{S}_C$ generated by the set of objects*

$$\mathrm{Obj}(\mathcal{N}_C[k,n]) = \{C \wedge \mathbf{t} \mid k \leq t \leq n\}.$$

Definition 3.4. *For a functor F from $\mathcal{C}$ to $Ch\mathcal{A}$ and an object C in $\mathcal{C}$, we denote the left Kan extensions of F along the categories $\mathcal{N}_C[k,n]$ and $\mathcal{N}_C$ as follows:*

$$L_n^C F = L_{\mathcal{N}_C[n,n]}F,$$
$$L_{[k,n]}^C F = L_{\mathcal{N}_C[k,n]}F,$$
$$L^C F = L_{\mathcal{N}_C}F.$$

Note that $L_n^C F, L_{[k,n]}^C F$, and $L^C F$ are functors from $\mathcal{S}_C$ to the category of simplicial objects in $Ch\mathcal{A}$. Equivalently, by applying the normalization functor N and taking the total complex (using direct sums) of the resulting bicomplex, we can consider $L_n^C F, L_{[k,n]}^C F$, and $L^C F$ as functors from $\mathcal{S}_C$ to $Ch\mathcal{A}$. We will omit the superscript of C when the context is clear.

By the comments following definition 3.1, we know that $L_{[k,n]}F$ and F agree on $\mathcal{N}_C[k,n]$. In the next lemma, we see that left Kan extensions along $\mathcal{N}_C[k,n]$ are determined by their values at $\mathbf{n}_C$.

Lemma 3.5. *Let C be an object in $\mathcal{C}$, F and F' be functors from $\mathcal{C}$ to $Ch\mathcal{A}$, and $\eta : F \longrightarrow F'$ be a natural transformation. For $0 \le k \le n$, the following are equivalent:*

1) $L_{[k,n]}\eta : L_{[k,n]}F \longrightarrow L_{[k,n]}F'$ is an equivalence.

2) $\eta_{\mathbf{n}_C} : F(\mathbf{n}_C) \longrightarrow F'(\mathbf{n}_C)$ is an equivalence.

3) $cr_k\eta(C) : cr_kF(C) \longrightarrow cr_kF'(C)$ is an equivalence for $k \le n$.

Proof. Conditions 2 and 3 are equivalent by the definition of cross effects. Condition 3 implies that $\eta_{\mathbf{t}_C}$ is an equivalence for all $0 \le t \le n$. As a result, $L_{[k,n]}\eta$ is an equivalence since it is a map of simplicial chain complexes that is an equivalence in each simplicial dimension. Condition 1 implies condition 2 since $\mathbf{n}_C$ is an object in $\mathcal{N}_C[k,n]$.

With lemma 3.5, we can show that extending along $\mathcal{N}_C[n,n]$ is equivalent to extending along $\mathcal{N}_C[k,n]$.

Lemma 3.6. *Let F be a functor from $\mathcal{C}$ to $Ch\mathcal{A}$ and C be an object in $\mathcal{C}$. For all $0 \leq k \leq n$, the natural map $\eta : L_nF \longrightarrow L_{[k,n]}F$ induced by the inclusion $\mathcal{N}_{[n,n]} \subseteq \mathcal{N}_{[k,n]}$ is an equivalence of functors on $\mathcal{S}_C$.*

Proof. Consider the commuting diagram:

$$\begin{array}{ccc} L_{[k,n]}(L_nF) & \xrightarrow{L_{[k,n]}(\eta)} & L_{[k,n]}(L_{[k,n]}F) \\ \downarrow \simeq & & \downarrow \simeq \\ L_nF & \xrightarrow{\eta} & L_{[k,n]}F. \end{array}$$

The vertical maps in the diagram are equivalences on $\mathcal{S}_C$ by (3.2). Since $\mathbf{n}_C$ is an object in $\mathcal{N}_C[k,n]$ and $\mathcal{N}_C[n,n]$, it follows that

$$L_nF(\mathbf{n}_C) \simeq F(\mathbf{n}_C) \simeq L_{[k,n]}F(\mathbf{n}_C).$$

Hence, the top horizontal map in the diagram is an equivalence by lemma 3.5 and the result follows.

Although L_nF and $L_{[k,n]}F$ are determined by their values at $\mathbf{n}_C$, they do not necessarily satisfy the limit axiom along C, and hence may not be defined along C. The reason for this is that the functors $\mathrm{Hom}_{\mathcal{C}}(D, \star)$ used in the construction of the left Kan extensions may not satisfy the limit axiom along C. However, if the functors $\mathrm{Hom}_{\mathcal{C}}(D, \star)$ satisfy the limit axiom, then L_nF and LF satisfy the limit axiom for all functors F. This is because $X \wedge -$ commutes with filtered direct limits for all objects X in $\mathcal{C}$, as do $\bigoplus$ and $Tot^{\oplus}$. To remedy the problem in general, we apply the following.

Definition 3.7. *For a functor G from $\mathcal{S}_C$ to $Ch\mathcal{A}$ and an object C in $\mathcal{C}$, we define $G^\infty : \mathcal{S}_C \to Ch\mathcal{A}$ by*

$$G^\infty(C \wedge U) := \mathrm{colim}_{\{X \subseteq U \ | \ |X| < \infty\}} G|_{\mathcal{N}_C}(C \wedge X).$$

There is a natural transformation from G^∞ to G that is an equivalence when the functors are restricted to $\mathcal{N}_C$. More generally, the natural transformation is an equivalence if and only if G satisfies the limit axiom along C.

Our last step in constructing rank n functors from the left Kan extensions is to expand the domain of these functors to all of $\mathcal{C}$ and, at the same time, guarantee that the resulting functors are defined along C. We do so by using the resolutions of the functors along C.

Definition 3.8. *For a functor F from $\mathcal{C}$ to $Ch\mathcal{A}$ and an object C in $\mathcal{C}$, the functors $\mathcal{L}_n^C F$, $\mathcal{L}_{[k,n]}^C F$, and $\mathcal{L}^C F$ from $\mathcal{C}$ to $Ch\mathcal{A}$ are defined as follows:*

$$\mathcal{L}_n^C F = \perp_C^* ([L_{\mathcal{N}_C[n,n]}F]^\infty),$$
$$\mathcal{L}_{[k,n]}^C F = \perp_C^* ([L_{\mathcal{N}_C[k,n]}F]^\infty),$$
$$\mathcal{L}^C F = \perp_C^* ([L_{\mathcal{N}_C}F]^\infty).$$

We will omit the superscript of C when the context is clear.

We use $\mathcal{L}_n^C$ and $\mathcal{L}_{[k,n]}^C$ to produce a filtration of a functor as follows.

Definition 3.9. *Let F be a functor from $\mathcal{C}$ to $Ch\mathcal{A}$ and C be an object in $\mathcal{C}$.*

1) The functors $\mathcal{L}_n F$ fit into a filtered system of functors:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \\
\mathcal{L}_{[0,n+1]}F & \xleftarrow{\simeq} & \mathcal{L}_{n+1}F \\
\uparrow & & \\
\mathcal{L}_{[0,n]}F & \xleftarrow{\simeq} & \mathcal{L}_{n}F \\
\uparrow & & \\
\mathcal{L}_{[0,n-1]}F & \xleftarrow{\simeq} & \mathcal{L}_{n-1}F \\
\uparrow & & \\
\vdots & & \vdots
\end{array}$$

where the natural transformations are induced by inclusions of categories and the natural transformations $\mathcal{L}_n F \longrightarrow \mathcal{L}_{[0,n]} F$ are equivalences by lemma 3.6. We will refer to this system as the rank filtration of F along C.

2) *There are natural transformations $\mathcal{L}_n F \longrightarrow F$ and $\mathcal{L}F \longrightarrow F$ induced by the augmentations of the left Kan extensions over $\mathcal{N}_C[n,n]$ and $\mathcal{N}_C$, respectively. If $\mathcal{L}_n F \longrightarrow F$ is an equivalence, we say that F is a* rank n functor along C. *If $\mathcal{L}F \longrightarrow F$ is an equivalence, we say that F is* polynomial along C.

We will drop the phrase "along C" when the context is clear. Note that F is polynomial if and only if F is equivalent to the colimit of the rank filtration of F.

Remark 3.10. The functor $\mathcal{L}_n F$ is always a rank n functor. To see this note that from the definition of $\mathcal{L}_n$, we have $\mathcal{L}_n F(\mathbf{n}_C) \simeq F(\mathbf{n}_C)$. By lemma 3.5, $L_n \mathcal{L}_n F \simeq L_n F$, and so $\mathcal{L}_n(\mathcal{L}_n F) \simeq \mathcal{L}_n F$.

Note that rank n functors are completely determined by their values at $\mathbf{n}_C$ in that for such a functor F, $F(\mathbf{n}_C)$ is the only value used in the construction of $\mathcal{L}_n F$. More generally, we have the following.

Lemma 3.11. *A functor F from $\mathcal{C}$ to $Ch\mathcal{A}$ is defined along an object C in $\mathcal{C}$ if and only if it is polynomial along C.*

Proof. Suppose F is defined along C. Since $LF \longrightarrow F$ is an equivalence at $\mathbf{t}_C$ for all t and F satisfies the limit axiom along C, we have $(LF)^\infty \xrightarrow{\simeq} F^\infty \xrightarrow{\simeq} F$. Then, since F is defined along C, we have

$$\mathcal{L}F = \perp_C^* ((LF)^\infty) \simeq \perp_C^* F \simeq F$$

and so F is polynomial. The converse follows from the fact that $\mathcal{L}F$ is defined along C.

The rank filtration of F is universal in the following way.

Proposition 3.12. *Let F be a functor from $\mathcal{C}$ to $Ch\mathcal{A}$ and C be an object in $\mathcal{C}$. The functor $\mathcal{L}_n F$ is universal (up to natural quasi-isomorphism) among rank n functors with natural transformations to F.*

Proof. The proof is similar to the proof of lemma 2.11 of [J-M3].

We now wish to consider the relationship between rank and degree for a functor. In particular, we will show that every degree n functor has rank n whereas a rank n functor need not have finite degree. We do so by means of the next proposition.

Proposition 3.13. *If $F : \mathcal{C} \to Ch\mathcal{A}$ is polynomial (respectively, rank k) along an object C in $\mathcal{C}$, then $P_n F$ is polynomial (respectively, rank k) along C for all $n \geq 0$.*

Proof. We will prove the proposition in the case that F is polynomial. The finite rank case can be proved in a similar fashion by using $\mathcal{L}_k$ in place of $\mathcal{L}$.

Note that it suffices to show

$$\mathcal{L}F \xrightarrow{\simeq} F \Rightarrow \mathcal{L}(\perp_{n+1} F) \xrightarrow{\simeq} \perp_{n+1} F$$

since it will follow that the map $\mathcal{L}(\perp_{n+1}^* F) \longrightarrow \perp_{n+1}^* F$ is a map of simplicial chain complexes that is an equivalence in each simplicial dimension. Recall from proposition 1.2.3 that for any object X in $\mathcal{C}$ there is a natural isomorphism of functors:

$$F(\vee_{i=1}^n X) \cong \bigoplus_{p=0}^{n} \bigoplus_{t=0}^{\binom{n}{p}} \perp_p F(X).$$

It follows from the definition that $\mathcal{L}$ preserves such direct sum decompositions. Hence, the result follows from the commutative diagram:

$$\begin{array}{ccc}
\mathcal{L}\left(\bigoplus_{p=0}^{n}\bigoplus_{t=0}^{\binom{n}{p}} \perp_p F(X)\right) & \cong & \bigoplus_{p=0}^{n}\bigoplus_{t=0}^{\binom{n}{p}} \mathcal{L}(\perp_p F)(X) \\
\downarrow \simeq & & \downarrow \simeq \\
F(\vee_{i=1}^{n} X) & \cong & \bigoplus_{p=0}^{n}\bigoplus_{t=0}^{\binom{n}{p}} \perp_p F(X).
\end{array}$$

Remark 3.14. Using a proof similar to that of proposition 3.13, one can also show that if $LF \simeq F$, then $LP_nF \simeq P_nF$. Similarly, $L_kF \simeq F$ implies that $L_kP_nF \simeq P_nF$.

As a consequence of proposition 3.13, we can show that $\mathcal{L}$ and P_n commute for polynomial functors.

Corollary 3.15. *Let $F : \mathcal{C} \to Ch\mathcal{A}$ and C be an object in $\mathcal{C}$. If F is polynomial, then $\mathcal{L}P_nF$ is naturally equivalent to $P_n\mathcal{L}F$ for all $n \geq 0$. If F is rank k, then $\mathcal{L}_kP_nF$ is naturally equivalent to $P_n\mathcal{L}_kF$ for all $n \geq 0$.*

Proof. We prove the result for polynomial F. The rank k case is similar. Consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathcal{L}P_nF & & \\
 & \swarrow p_n(\mathcal{L}P_n) & & \searrow & \\
P_n\mathcal{L}P_nF & \longrightarrow & P_nP_nF & \xrightarrow{\Sigma(n,n)} & P_nF \\
 & P_n\mathcal{L}(p_n) \nwarrow & & \nearrow & \\
 & & P_n\mathcal{L}F. & &
\end{array}$$

Using proposition 3.13, definition 1.16, and the fact that P_n preserves quasi-isomorphisms, one can show that all of the maps above are equivalences. The result follows.

Corollary 3.16. *Every degree n functor $F : \mathcal{C} \to Ch\mathcal{A}$ defined along an object C in $\mathcal{C}$ has rank n along C. Moreover, if F is a homogeneous degree n functor, then F has rank 1 along C.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{L}_nF & \longrightarrow & F \\ \big\downarrow{\scriptstyle \mathcal{L}_n p_n} & & \big\downarrow{\scriptstyle p_n} \\ \mathcal{L}_nP_nF & \longrightarrow & P_nF. \end{array}$$

The two vertical maps are equivalences because F is degree n and $\mathcal{L}_n$ preserves equivalences. To see that $\mathcal{L}_nP_nF \longrightarrow P_nF$ is an equivalence, note that since F is defined along C and $F \xrightarrow{\simeq} P_nF$, it follows that P_nF is defined along C. Hence by lemma 3.11, P_nF is polynomial. Then by corollary 3.15, $\mathcal{L}_nP_nF \simeq P_n\mathcal{L}_nF$, and so $\mathcal{L}_nP_nF$ is degree n. Thus, the lower horizontal map is a map between degree n functors that are defined along C. Moreover, it is an equivalence at $\mathbf{n}_C$ and so, by theorem 2.11 is an equivalence on $\mathcal{C}$. Hence, $\mathcal{L}_nF \xrightarrow{\simeq} F$ and F is rank n along C.

Now, suppose F is a homogeneous degree n functor. Then $F \simeq D_nF$ and so by proposition 3.10 of [J-M3], $F \simeq \perp_n F_{h\Sigma_n}$ where $\perp_n F$ is the diagonal of the functor of n variables, cr_nF. To show that F is a rank 1 functor it suffices to establish that $\perp_n F$ is rank 1 since $(\ )_{h\Sigma_n}$ preserves equivalences and hence preserves the rank of a functor. But, cr_nF is degree 1 in each of its variables since F is degree n and so by the above, cr_nF is a rank 1 functor in each of its variables. Then, letting $\mathcal{L}_1^{(n)}(cr_nF)$ denote the functor obtained by applying $\mathcal{L}_1$ to each variable of cr_nF separately, we see that $\mathcal{L}_1^{(n)}(cr_nF) \simeq cr_nF$. Moreover, $\mathcal{L}_1(\perp_n F) = \text{Diag } \mathcal{L}_1^{(n)}(cr_nF)$ where Diag $\mathcal{L}_1^{(n)}(cr_nF)$ is the diagonal of the n-multisimplicial object

$$[p_1, p_2, \ldots, p_n] \mapsto \mathcal{L}_1^1[p_1]\mathcal{L}_1^2[p_2] \ldots \mathcal{L}_1^n[p_n]cr_nF.$$

Here $\mathcal{L}_1^j[p_j]$ indicates the degree p_j term in $\mathcal{L}_1$ applied to the jth variable of cr_nF. By the Eilenberg-Zilber theorem Diag $\mathcal{L}_1^{(n)}cr_nF \simeq \mathcal{L}_1^{(n)}cr_nF \simeq cr_nF$ and the result follows.

The converse of corollary 3.16 is not true, i.e., $\mathcal{L}_nF$ does not always have finite degree. In particular, consider the following example.

Example 3.17. Let $\mathcal{C}$ and $\mathcal{A}$ be the category of abelian groups and let $C = \mathbb{Z}$. For any functor $F : \mathcal{C} \to \mathcal{A}$, and any object A in $\mathcal{C}$, it follows from definitions 3.1 and 3.3 that

$$H_0(L_1F)(A) \cong F(\mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[A].$$

The functor $F(\mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}]} -$ is a degree one functor, but the functor $\mathbb{Z}[-]$ has infinite degree. Hence $H_0(L_1F)$ and L_1F do not in general have finite degree. It follows that $\mathcal{L}_1F$ may not have finite degree although it is a rank one functor.

We can also use proposition 3.13 to show that derivatives along C preserve the rank of a functor.

Corollary 3.18. *Let C be an object in $\mathcal{C}$ and let $F : \mathcal{C} \to Ch\mathcal{A}$ be defined along C.*

1) *If F is polynomial (respectively, rank k), then $\frac{d}{dC}F$ is polynomial (respectively, rank k).*
2) *If $LF \xrightarrow{\simeq} F$ (respectively, $L_kF \xrightarrow{\simeq} F$), then $L(\frac{d}{dC}F) \xrightarrow{\simeq} \frac{d}{dC}F$ (respectively, $L_k(\frac{d}{dC}F) \xrightarrow{\simeq} \frac{d}{dC}F$).*

Proof. We will prove the proposition for $\mathcal{L}$. The other cases are similar. From proposition 3.13, we know that $\mathcal{L}P_nF \simeq P_nF$. It is straightforward to show that $\mathcal{L}$ preserves fibers, so that $\mathcal{L}D_nF \simeq D_nF$. Now, recall that $\frac{d}{dC}F(Y) = D_1[cr_2F(-,Y)](C) \oplus D_1F(C)$. By the proof of 3.13, $\mathcal{L}cr_2F(-,Y) \simeq cr_2F(-,Y)$ for all Y. It follows by the above that

$$\begin{aligned}\mathcal{L}(\frac{d}{dC}F)(Y) &\simeq \mathcal{L}([D_1cr_2F(-,Y)](C)) \oplus \mathcal{L}(D_1F(C))\\ &\simeq D_1[cr_2F(-,Y)](C) \oplus D_1F(C)\\ &\simeq \frac{d}{dC}F.\end{aligned}$$

Remark 3.19. We will be interested later in applying 3.18.2 to the functor $\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}_{\mathcal{C}}, \star)] : \mathcal{C} \to \mathbb{Z}-\mathrm{Mod}$. In particular, the augmentation $L_n\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}_{\mathcal{C}}, \star)] \to \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}_{\mathcal{C}}, \star)]$ has an inverse by sending $k[\alpha]$ for $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{n}_{\mathcal{C}}, \star)$ and $k \in \mathbb{Z}$ to $(k[\mathrm{id}_{\mathbf{n}_{\mathcal{C}}}]; \mathrm{id}_{\mathbf{n}_{\mathcal{C}}}, \ldots, \mathrm{id}_{\mathbf{n}_{\mathcal{C}}}; \alpha)$. Hence, by corollary 3.18, we have

$$L_n\left(\frac{d}{d\mathcal{C}}\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}_{\mathcal{C}}, \star)]\right) \xrightarrow{\simeq} \frac{d}{d\mathcal{C}}\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}_{\mathcal{C}}, \star)].$$

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