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THE BOOLEAN PRIME IDEAL THEOREM HOLDS IFF MAXIMAL OPEN FILTERS EXIST

by Y.T. RHINEGHOST

RESUME. L'auteur démontre qu'en ZF théorie des ensembles les propriétés suivantes sont équivalentes :

(a) Toute algèbre de Boole non triviale a un idéal premier.
(b) Tout espace topologique non vide a un filtre ouvert.

Theorem: *In* **ZF** *(i.e., Zermelo–Fraenkel set theory without the axiom of choice) the following conditions are equivalent:*

1. *The Boolean Prime Ideal Theorem.*
2. *For each non–empty topological space, its lattice of open sets contains a maximal filter.*
3. *For topological spaces, each open filter is contained in a maximal open filter.*

Proof: (1) $\Rightarrow$ (2) Let X be a topological space, let $\mathfrak{O}(X)$ be its lattice of open sets, and let $\mathfrak{P}(X)$ be the Boolean algebra of all subsets of X. By (1) the filter $\mathcal{F}$ of all subsets F of X, whose interior $\operatorname{int}(F)$ is dense in X, is contained in some prime filter $\mathfrak{G}$ of $\mathfrak{P}(X)$. Thus $\mathfrak{H} = \mathfrak{G} \cap \mathfrak{O}(X)$ is a prime filter in $\mathfrak{O}(X)$. For each $B \in \mathfrak{O}(X)$ let $B^* = \operatorname{int}(X \setminus B)$ be its pseudocomplement.

Then $B \cup B^*$ belongs to $\mathfrak{H}$. Thus either B or B^* belongs to $\mathfrak{H}$. This implies that $\mathfrak{H}$ is a maximal filter in $\mathfrak{O}(X)$.

(2) $\Rightarrow$ (3) Let X be a topological space, let $\mathfrak{O}(X)$ be its lattice of open sets, and let $\mathcal{F}$ be a filter in $\mathfrak{O}(X)$. Let Y be the topological space,

whose underlying set consists of all filters in $\mathfrak{O}(X)$ that refine $\mathcal{F}$, and whose topology $\mathfrak{O}(Y)$ consists of all subsets of Y that contain with any element $\mathfrak{G}$ all elements of Y that refine $\mathfrak{G}$. By (2) there exists a maximal filter $\mathfrak{M}$ in $\mathfrak{O}(Y)$. For each $B \in \mathfrak{O}(X)$ define $U(B) = \{\mathfrak{G} \in Y \mid B \in \mathfrak{G}\}$. Then $\mathfrak{U} = \{B \in \mathfrak{O}(X) \mid U(B) \in \mathfrak{M}\}$ is an element of Y. Maximality of $\mathfrak{U}$ will follow from the fact that for each $B \in \mathfrak{O}(X)$ either B or its pseudocomplement $B^* = \operatorname{int}(X \setminus B)$ belongs to $\mathfrak{U}$. To establish the latter fact, observe first that if $B \notin \mathfrak{U}$, i.e., if $U(B) \notin \mathfrak{M}$, maximality of $\mathfrak{M}$ implies that there exists some $M \in \mathfrak{M}$ with $M \cap U(B) = \emptyset$. Thus $B \notin \mathfrak{G}$ for each $\mathfrak{G} \in M$. This implies $B^* \in \mathfrak{G}$ for each $\mathfrak{G} \in M$, since $B^* \notin \mathfrak{G} \in M$ would imply that each member of $\mathfrak{G}$ would meet B, hence there would exist a refinement $\mathfrak{H}$ of $\mathfrak{G}$ — hence a member of $\mathfrak{H}$ of M — with $B \in \mathfrak{H}$; a contradiction. Consequently $M \subset U(B^*)$. This implies $U(B^*) \in \mathfrak{M}$, thus $B^* \in \mathfrak{U}$.

(3) $\Rightarrow$ (1) Condition (3), restricted to discrete spaces, is known to be equivalent to (1). □

Remarks: The equivalence of conditions (1) and (3) of the above theorem has been established independently in [2] and [3]. Moreover, in [3] the condition

4. In every pseudocomplemented bounded lattice, every filter is contained in a maximal filter

has been shown to be equivalent to (1) as well. This result, together with the above theorem, immediately implies that each of the following two conditions is equivalent to (1):

5. Every non–trivial frame contains a maximal filter.

6. In every frame, every filter is contained in a maximal filter.

Observe, for comparison (see [1]), that each of the following conditions is equivalent to the axiom of choice:

5* Every non–trivial frame contains a maximal ideal.

6* In every frame, every ideal is contained in a maximal ideal.

References

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