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# TENSOR PRODUCTS OF CATEGORIES OF EQUIVARIANT PERVERSE SHEAVES

*by Volodymyr LYUBASHENKO*

**RESUME.** On démontre que le produit tensoriel introduit par Deligne des catégories de faisceaux pervers equivariants constructibles est encore une catégorie de ce type. Plus précisément, le produit des catégories construites pour une G-variété complexe algébrique X et pour une H-variété Y est une catégorie correspondante à la G x H-variété X x Y - produit des espaces constructibles.

## 1. Introduction

The main result of the author's paper [10] is that the geometrical external product of constructible perverse sheaves is their Deligne's tensor product. In the present paper we extend this result to equivariant perverse sheaves.

Let us recall the details. Given a perverse sheaf $F$ on a compactifiable pseudomanifold $X$ with stratification $\mathcal{X}$ and perversity $p : \mathcal{X} \to \mathbb{Z}$ and a perverse sheaf $E$ on a compactifiable pseudomanifold $Y$ with stratification $\mathcal{Y}$ and perversity $p : \mathcal{X} \to \mathbb{Z}$, we can construct their product $F \boxtimes E = \mathrm{pr}_X^* F \otimes \mathrm{pr}_Y^* E$ which is a perverse sheaf on $X \times Y$, equipped with stratification $\mathcal{X} \times \mathcal{Y}$ and perversity

$$p \dotplus q : \mathcal{X} \times \mathcal{Y} \to \mathbb{Z}, \qquad (p \dotplus q)(S \times T) = p(S) + q(T).$$

The $\mathbb{C}$-bilinear exact in each variable functor

$$\boxtimes : \mathrm{Perv}(X, \mathcal{X}, p) \times \mathrm{Perv}(Y, \mathcal{Y}, q) \to \mathrm{Perv}(X \times Y, \mathcal{X} \times \mathcal{Y}, p \dotplus q)$$

makes the target category into the Deligne tensor product [6] of categories-factors of the source category. In a precise sense $\boxtimes$ is universal between such $\mathbb{C}$-linear exact functors.

Our goal is to generalise this result to the following setup. Let a complex algebraic group $G$ (resp. $H$) act on a complex algebraic quasi-projective variety $X$ (resp. $Y$). View $X$ and $Y$ as constructible spaces $X_c$ and $Y_c$ equipping them with the filtering inductive system of all algebraic Whitney stratifications and with middle perversities. The corresponding categories $\mathrm{Perv}_G(X)$ and $\mathrm{Perv}_H(Y)$ of equivariant constructible perverse sheaves were defined by Bernstein and Lunts [3]. We will prove that there is the external product functor

$$\boxtimes : \mathrm{Perv}_G(X) \times \mathrm{Perv}_H(Y) \to \mathrm{Perv}_{G\times H}(X_c \times Y_c),$$
$$F \boxtimes E = \mathrm{pr}_1^* F \otimes \mathrm{pr}_2^* E,$$

which makes the target category into the Deligne tensor product of categories-factors of the source category.

The idea of proof is to reduce the result to non-equivariant one. By one of the definitions of Bernstein and Lunts [3] $\mathrm{Perv}_G(X)$ is an inverse limit of categories of the type $\mathrm{Perv}_c(Z)$, (non-equivariant) perverse sheaves on algebraic variety $Z$, viewed as a constructible space. In turn, $\mathrm{Perv}_c(Z)$ is an inductive filtering limit – a union of its full subcategories of the type $\mathrm{Perv}(Z, \mathcal{Z}, mp)$, where $mp$ is the middle perversity. Manipulating with these limits one gets the desired results.

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# 2. Preliminaries

## 2.1. Fibered categories

Let us recall the notion of fibered category. Let $J$ be an essentially small category. A *fibered category* $\mathcal{A}/J$ assigns a category $\mathcal{A}(X)$ to an object $X$ of $J$, assigns a functor $\mathcal{A}(f) : \mathcal{A}(Y) \to \mathcal{A}(X)$ to a morphism $f : X \to Y$ of $J$, so that $\mathcal{A}(1_X) = \mathrm{Id}_{\mathcal{A}(X)}$ (in a simplified version), to a pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $J$ assigns a natural isomorphism

$$\mathcal{A}(f,g) : \mathcal{A}(f)\mathcal{A}(g) \xrightarrow{\sim} \mathcal{A}(gf) : \mathcal{A}(Z) \to \mathcal{A}(X),$$

such that natural compatibility conditions hold: for a triple of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ we have an equation

$$\mathcal{A}(gf,h)\mathcal{A}(f,g) = \mathcal{A}(f,hg)[\mathcal{A}(f)\mathcal{A}(g,h)]$$

and $\mathcal{A}(f,1) = 1$, $\mathcal{A}(1,g) = 1$.

The inverse limit $\mathcal{A} = \varprojlim_{X \in J} \mathcal{A}(X)$ of the fibered category $\mathcal{A}/J$ is the following category. An object $K$ of $\mathcal{C}$ consists of functions

$$\operatorname{Ob} J \ni X \mapsto K(X) \in \operatorname{Ob} \mathcal{A}(X),$$
$$\operatorname{Mor} J \ni (f : X \to Y) \mapsto \big(K(f) : \mathcal{A}(f)K(Y) \xrightarrow{\sim} K(X)\big) \in \operatorname{Isomor} \mathcal{A}(X)$$

such that for any pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $J$ we have

$$K(f)[\mathcal{A}(f)K(g)] = K(gf)\mathcal{A}(f,g) : \mathcal{A}(f)\mathcal{A}(g)K(Z) \to \mathcal{A}(X), \tag{2.1}$$

and for any object $X$ of $J$ we have

$$K(1_X) = 1_{K(X)} : K(X) \to K(X). \tag{2.2}$$

A morphism $\phi : M \to N$ of $\mathcal{A}$ is a family of morphisms $\phi(X) : M(X) \to N(X) \in \mathcal{A}(X)$, $X \in \operatorname{Ob} J$, such that the following equation holds

$$\phi(X)M(f) = N(f)[\mathcal{A}(f)\phi(Y)] : \mathcal{A}(f)M(Y) \to N(X). \tag{2.3}$$

## 2.2. Equivariant derived categories

Our interest in inverse limit categories is motivated by the definition of the equivariant derived category as such a limit, given by Bernstein and Lunts [3]. Let $G$ be a complex algebraic group algebraically acting on a complex quasi-projective variety $X$. Consider $J = \operatorname{SRes}(X,G)$ – the category of smooth resolutions of $X$, whose objects are $G$-maps $p : P \to X$, which are smooth, that is, locally homeomorphic to the projection map $V \times \mathbb{C}^d \to V$ for a constant $d$ depending on $p$. The $G$-variety $P$ is supposed to be free, that is, the quotient map $q_P : P \to \overline{P} \stackrel{\text{def}}{=} G\backslash P$ is a locally trivial fibration with fibre $G$. Morphisms of $J$

are smooth $G$-maps $f : P \to R$ over $X$. To define a fibered category $\mathcal{D}/J$ set the category $\mathcal{D}(P)$ to $D^b_c(\overline{P})$, set the functor $\mathcal{D}(f)$ to $\overline{f}^*[d_f] : D^b_c(\overline{R}) \to D^b_c(\overline{P})$, where $\overline{f} : \overline{P} \to \overline{R}$ is the map of quotient spaces, and $d_f$ is the complex dimension of a fibre of $f$ or $\overline{f}$. The category $D^b_c(Z)$ is the union of strictly full subcategories $D^{b,c}_{\mathcal{Z}}(Z) \subset D^b(Z)$, consisting of $\mathcal{Z}$-cohomologically constructible complexes over all algebraic Whitney stratifications $\mathcal{Z}$. The isomorphisms $\mathcal{D}(f,g) : \overline{f}^*[d_f]\overline{g}^*[d_g] \to \overline{gf}^*[d_{gf}]$ are standard, clearly $d_{gf} = d_f + d_g$. And the equivariant derived category is defined as

$$D^{b,c}_G(X) = \varprojlim_{P \in J} D^b_c(\overline{P}).$$

In the original definition of Bernstein and Lunts [3] the shifts $[d_f]$ are not used. These definitions are clearly equivalent. The one with the shifts is convenient when dealing with perverse sheaves.

Let us discuss a $t$-category structure of $D^{b,c}_G(X)$. For a variety $Z$ with algebraic stratification $\mathcal{Z}$ define the middle perversity as the function $mp : \mathcal{Z} \to \mathbb{Z}$, $mp(S) = -\dim_{\mathbb{C}} S$, $\dim_{\mathbb{C}}$ is the half of the topological dimension. This perversity turns $D^b_c(\overline{P})$ into a $t$-category, union of full $t$-subcategories ${}^{mp}D^{b,c}_{\overline{\mathcal{P}}}(\overline{P})$ (see [1, Proposition 2.1.14]).

Denote by $T = T_X$ the trivial resolution $\mathrm{pr}_X : G \times X \to X$ with the diagonal action of $G$ in $G \times X$. The quotient space $\overline{T}$ is $X$ with the quotient map $q_T : T = G \times X \to X = \overline{T}$, $(g,x) \mapsto g^{-1}.x$. The following statement is essentially from [3].

**2.3. Proposition.** *The category $D^{b,c}_G(X)$ has the following $t$-structure*

$$\begin{aligned}
{}^{mp}D^{\leqslant 0}_G(X) &= \{K \in D^{b,c}_G(X) \mid K(T) \in {}^{mp}D^{\leqslant 0}_{b,c}(X)\} \\
&= \{K \in D^{b,c}_G(X) \mid \forall P \in \mathrm{SRes}(X,G) \quad K(P) \in {}^{mp}D^{\leqslant 0}_{b,c}(\overline{P})\}, \\
{}^{mp}D^{\geqslant 0}_G(X) &= \{K \in D^{b,c}_G(X) \mid K(T) \in {}^{mp}D^{\geqslant 0}_{b,c}(X)\} \\
&= \{K \in D^{b,c}_G(X) \mid \forall P \in \mathrm{SRes}(X,G) \quad K(P) \in {}^{mp}D^{\geqslant 0}_{b,c}(\overline{P})\}.
\end{aligned}$$

□

This proposition gives also a description of the heart $\mathrm{Perv}_G(X)$ of $D^{b,c}_G(X)$ – the category of equivariant perverse sheaves

$$\mathrm{Perv}_G(X) = \varprojlim_{P \in \mathrm{SRes}(X,G)} \mathrm{Perv}_c(\overline{P}, mp).$$

The corresponding fibered category $\mathcal{A}/J$ is defined as a such subcategory of $\mathcal{D}/J$ that $\mathcal{A}(P) = \mathrm{Perv}_c(\overline{P}, mp)$, which is the union of its full subcategories $\mathrm{Perv}(\overline{P}, \mathcal{S}, mp)$ over all algebraic Whitney stratifications $\mathcal{S}$ of $\overline{P}$; $\mathcal{A}(f) = \overline{f}^*[d_f]$ etc. All functors $\mathcal{D}_P : D_G^{b,c}(X) \to {}^{mp}D_c^b(\overline{P})$, $K \mapsto K(P)$ are $t$-exact, so all the functors $\mathcal{A}_P : \mathrm{Perv}_G(X) \to \mathrm{Perv}_c(\overline{P}, mp)$, $K \mapsto K(P)$ are exact. As in general case of fibered abelian categories, for $\phi : M \to N \in \mathrm{Perv}_G(X)$ its kernel and cokernel can be computed componentwise: $(\mathrm{Ker}\,\phi)(P) = \mathrm{Ker}\big(\phi(P) : M(P) \to N(P)\big)$, $(\mathrm{Coker}\,\phi)(P) = \mathrm{Coker}\big(\phi(P) : M(P) \to N(P)\big)$.

## 2.4. Tensor product of abelian $\mathbb{C}$-linear categories

In this section we consider only essentially small $\mathbb{C}$-linear abelian categories and $\mathbb{C}$-linear functors. Let $\mathcal{A}$, $\mathcal{B}$ be such categories. A category $\mathcal{C}$ equipped with an exact in each variable functor $\boxtimes^D : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is called *tensor product of* $\mathcal{A}$ *and* $\mathcal{B}$, if the induced functor

$$\mathrm{Hom}_{r.e.}^{\mathbb{C}-lin}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{r.e.}^{\mathbb{C}-bilin}(\mathcal{A} \times \mathcal{B}, \mathcal{D}), \qquad F \mapsto F \circ \boxtimes^D$$

between the categories of $\mathbb{C}$-(bi)linear right exact in each variable functors is an equivalence for each $\mathbb{C}$-linear abelian category $\mathcal{D}$ [6]. We will need the following classes of abelian categories for which tensor product exists. Categories equivalent to the category of finite dimensional modules $A$-mod over a finite dimensional associative unital algebra $A$ are called *bounded* [8]. (They are also called of Artin type in [11] or rigid-abelian in [9].) Tensor product of bounded categories exists and is bounded by Deligne's Proposition 5.11 [6]. Categories *with length* are those whose objects have finite length and Hom(-, -) are finite dimensional. Examples are categories of perverse sheaves $\mathrm{Perv}(X, \mathcal{X}, p)$ on a stratified space. Tensor product of categories with length exists and is a category with length by Deligne's Proposition 5.13 [6]. *Locally bounded* category $\mathcal{C}$ is a union of its strictly full abelian bounded subcategories with exact inclusion functor. A category with length is locally bounded, as follows from loc. cit. Proposition 2.14. Therefore, a category which is a union of its strictly full abelian subcategories with length is locally bounded. An example of such a category is $\mathrm{Perv}_c(Z, mp) = \cup_{\mathcal{Z}} \mathrm{Perv}(Z, \mathcal{Z}, mp)$, union over algebraic Whitney stratifications $\mathcal{Z}$. A related example of a locally bounded category is $\mathrm{Perv}_G(X)$. Indeed, the forgetful functor $\mathrm{For} = \mathcal{A}_T : \mathrm{Perv}_G(X) \to \mathrm{Perv}_c(X, mp)$, $K \mapsto K(T)$ is exact and faithful (since all $\mathcal{A}(f)$ are).

For a stratification $\mathfrak{X}$ of $X$ let

$$\mathrm{Perv}_G(X, \mathfrak{X}) = \mathrm{For}^{-1}(\mathrm{Perv}(X, \mathfrak{X}, mp))$$

be the preimage category. It is a strictly full abelian subcategory with length, since For is faithful and $\mathrm{Perv}(X, \mathfrak{X}, mp)$ is with length.

**2.5. Proposition.** *Tensor product of essentially small locally bounded categories exists and is locally bounded.*

*Proof.* For each object $M$ of a locally bounded category $\mathcal{B}$ there exists the smallest strictly full abelian subcategory $\mathcal{B}_M \hookrightarrow \mathcal{B}$ with exact inclusion, containing $M$. It is a category with length. Consider $\langle M \rangle \subset \mathcal{B}_M$ strictly full abelian subcategory consisting of subquotients of $M^n$, $n \in \mathbb{N}$. By minimality $\mathcal{B}_M = \langle M \rangle$, hence it is bounded [6, Proposition 2.14]. The subcategories $\mathcal{B}_M \subset \mathcal{B}$ form an inductive filtering system, since for all objects $M$, $N$ of $\mathcal{B}$ we have $\mathcal{B}_M \subset \mathcal{B}_{M\oplus N} \supset \mathcal{B}_N$. We presented $\mathcal{A}$ and $\mathcal{B}$ as inductive filtering limits, hence, tensor product of two locally bounded categories $\mathcal{A}$, $\mathcal{B}$ exists by loc. cit. Section 5.1, and

$$\varinjlim_{\substack{M\in \mathrm{Ob}\,\mathcal{A}\\ N\in \mathrm{Ob}\,\mathcal{B}}} (\mathcal{A}_M \boxtimes^D \mathcal{B}_N) \to \Big(\varinjlim_{M\in \mathrm{Ob}\,\mathcal{A}} \mathcal{A}_M\Big) \boxtimes^D \Big(\varinjlim_{N\in \mathrm{Ob}\,\mathcal{B}} \mathcal{B}_N\Big) = \mathcal{A} \boxtimes^D \mathcal{B}$$

is an equivalence. Using Lemma 2.6 we conclude that $\varinjlim_{M,N}(\mathcal{A}_M \boxtimes^D \mathcal{B}_N)$ is locally bounded. □

**2.6. Lemma.** *If* $\mathcal{A}_M \xhookrightarrow{E} \mathcal{A}_{M'}$ *and* $\mathcal{B}_N \xhookrightarrow{I} \mathcal{B}_{N'}$ *are exact full embeddings, then the functor* $\mathcal{A}_M \boxtimes^D \mathcal{B}_N \xrightarrow{E\boxtimes I} \mathcal{A}_{M'} \boxtimes^D \mathcal{B}_{N'}$ *is an exact full embedding as well.*

*Proof.* Exactness and faithfulness follow from [6, Proposition 5.14]. Surjectivity on morphisms follows from Lemma 2.7 applied to

$$\begin{array}{ccc}
\mathcal{A}_M \times \mathcal{B}_N & \xrightarrow{\boxtimes^D} & \mathcal{A}_M \boxtimes^D \mathcal{B}_N \\
{\scriptstyle E\times I}\downarrow & \searrow^{\mathcal{T}} \;\; \simeq & \downarrow{\scriptstyle E\boxtimes^D I = F} \\
 & = & \\
\mathcal{A}_{M'} \times \mathcal{B}_{N'} & \xrightarrow{\boxtimes^D} & \mathcal{A}_{M'} \boxtimes^D \mathcal{B}_{N'}
\end{array}$$

since for all $A_1, A_2 \in \operatorname{Ob} \mathcal{A}_M$, $B_1, B_2 \in \mathcal{B}_N$ the map induced by $T$

$$\begin{aligned}\operatorname{Hom}_{\mathcal{A}_M}(A_1, A_2) \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathcal{B}_N}(B_1, B_2) \\ \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}_{M'}}(EA_1, EA_2) \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathcal{B}_{N'}}(IB_1, IB_2) \\ \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}_{M'} \boxtimes^D \mathcal{B}_{N'}}(EA_1 \boxtimes^D IB_1, EA_2 \boxtimes^D IB_2)\end{aligned}$$

is an isomorphism. □

**2.7. Lemma.** *Let $\mathcal{A}$, $\mathcal{B}$ be locally bounded abelian categories, let $T : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a $\mathbb{C}$-bilinear functor exact in each variable. For all objects $A, A' \in \operatorname{Ob} \mathcal{A}$, $B, B' \in \operatorname{Ob} \mathcal{B}$ the map induced on morphisms by $T$ factorises as*

$$\begin{aligned}\operatorname{Hom}_{\mathcal{A}}(A, A') \times \operatorname{Hom}_{\mathcal{B}}(B, B') \to \operatorname{Hom}_{\mathcal{A}}(A, A') \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathcal{B}}(B, B') \\ \xrightarrow{\theta} \operatorname{Hom}_{\mathcal{C}}(T(A, B), T(A', B')).\end{aligned}$$

*Suppose that all $\theta$ are isomorphisms. Then the functor $F : \mathcal{A} \boxtimes^D \mathcal{B} \to \mathcal{C}$, such that*

$$T \simeq \left(\mathcal{A} \times \mathcal{B} \xrightarrow{\boxtimes^D} \mathcal{A} \boxtimes^D \mathcal{B} \xrightarrow{F} \mathcal{C}\right),$$

*is full and faithful.*

*Proof.* Let $V, W \in \mathcal{A} \boxtimes^D \mathcal{B}$. There exist $X \in \operatorname{Ob} \mathcal{A}$, $Y \in \operatorname{Ob} \mathcal{B}$ such that $V$ and $W$ are objects of $\mathcal{A}_X \boxtimes^D \mathcal{B}_Y$. Since $\mathcal{A}_X \simeq A_X$-mod, $\mathcal{B}_Y \simeq B_Y$-mod, we have $\mathcal{A}_X \boxtimes^D \mathcal{B}_Y \simeq A_X \otimes_{\mathbb{C}} B_Y$-mod. Hence, there exist $a_i : P' \to P \in \mathcal{A}_X$, $b_i : R' \to R \in \mathcal{B}_Y$, $c_j : Q \to Q' \in \mathcal{A}_X$, $d_j : S \to S' \in \mathcal{B}_Y$ and exact sequences in $\mathcal{A}_X \boxtimes^D \mathcal{B}_Y$

$$P' \boxtimes^D R' \xrightarrow{\sum a_i \boxtimes b_i} P \boxtimes^D R \to V \to 0,$$

$$0 \to W \to Q \boxtimes^D S \xrightarrow{\sum c_j \boxtimes^D d_j} Q' \boxtimes^D S'.$$

They are exact also in $\mathcal{A} \boxtimes^D \mathcal{B}$.

It is known that

$$\operatorname{Hom}_{\mathcal{A}}(A, A') \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathcal{B}}(B, B') \to \operatorname{Hom}_{\mathcal{A} \boxtimes^D \mathcal{B}}(A \boxtimes^D B, A' \boxtimes^D B')$$

is bijective. Hence, the maps induced by $F$

$$\mathrm{Hom}_{\mathcal{A}\boxtimes^D\mathcal{B}}(A\boxtimes^D B, A'\boxtimes^D B') \to \mathrm{Hom}_{\mathcal{C}}(T(A,B), T(A',B')) \\ \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(F(A\boxtimes^D B), F(A'\boxtimes^D B'))$$

are isomorphisms. So in the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_{\mathcal{A}\overset{D}{\boxtimes}\mathcal{B}}(V, A'\overset{D}{\boxtimes}B') & \longrightarrow & \mathrm{Hom}_{\mathcal{A}\overset{D}{\boxtimes}\mathcal{B}}(P\overset{D}{\boxtimes}R, A'\overset{D}{\boxtimes}B') & \longrightarrow & \mathrm{Hom}_{\mathcal{A}\overset{D}{\boxtimes}\mathcal{B}}(P'\overset{D}{\boxtimes}R', A'\overset{D}{\boxtimes}B') \\ & \downarrow & & \downarrow\wr & & \downarrow\wr \\ 0 \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(FV, F(A'\overset{D}{\boxtimes}B')) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(T(P,R), T(A',B')) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(T(P',R'), T(A',B')) \end{array}$$

two vertical maps are bijective, therefore the left is bijective. Hence, in the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_{\mathcal{A}\boxtimes^D\mathcal{B}}(V,W) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}\boxtimes^D\mathcal{B}}(V, Q\boxtimes^D S) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}\boxtimes^D\mathcal{B}}(V, Q'\boxtimes^D S') \\ & \downarrow & & \downarrow\wr & & \downarrow\wr \\ 0 \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(FV, FW) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(FV, F(Q\boxtimes^D S)) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(FV, F(Q'\boxtimes^D S')) \end{array}$$

the middle and the right maps are bijective. Hence, so is the left map. □

# 3. The external tensor product functor

## 3.1. Main result

We define the external tensor product functor as follows

$$\boxtimes : D^b_G(X_c) \times D^b_H(Y_c) \xrightarrow{\mathrm{pr}_1^* \times \mathrm{pr}_2^*} D^b_{G\times H}(X_c \times Y_c) \times D^b_{G\times H}(X_c \times Y_c) \to \\ \xrightarrow{\otimes} D^b_{G\times H}(X_c \times Y_c).$$

Here $\mathrm{pr}_1$ refers to the map of groups $G \times H \to G$ and to the map of varieties $X \times Y \to X$. The inverse image functor $\mathrm{pr}_1^*$ is denoted $Q^*_{\mathrm{pr}_1}$ by Bernstein and Lunts. Let us describe this functor in details following [3, Definition 6.5]. Denote $I = \mathrm{SRes}(X,G)$, $J = \mathrm{SRes}(Y,H)$. Then $D_{G\times H}(X_c \times Y_c)$ is equivalent to the inverse limit

$$\varprojlim_{(P,R)\in I\times J} D^b(\overline{P}_c \times \overline{R}_c)$$

by loc. cit. Proposition 2.4.4 (note that $(G \times H)\backslash(P \times R) = (G\backslash P) \times (H\backslash R) \equiv \overline{P} \times \overline{R}$). So in the following by $D^b_{G\times H}(X_c \times Y_c)$ we mean this inverse limit. Let $K \in D_G(X)$, $I \ni P \mapsto K(P) \in D^b_c(\overline{P})$ and $L \in D_H(Y)$, $J \ni R \mapsto L(R) \in D^b_c(\overline{R})$. Define $\mathrm{pr}_1^* K \in D_{G\times H}(X_c \times Y_c)$ via a map of resolutions, compatible with the projection of varieties

$$\begin{array}{ccc} (P \times R, G \times H) & \xrightarrow{\mathrm{pr}_1} & (P, G) \\ \downarrow & & \downarrow \\ (X \times Y, G \times H) & \xrightarrow{\mathrm{pr}_1} & (X, G) \end{array}$$

namely,

$$(\mathrm{pr}_1^* K)(P \times R) = \overline{\mathrm{pr}}_1^*(K(P)) \in D^b(\overline{P}_c \times \overline{R}_c).$$

The structure isomorphisms $(\mathrm{pr}_1^* K)(P \to P', R \to R')$ are easy to construct. Similarly, $\mathrm{pr}_2^* L \in D_{G\times H}(X_c \times Y_c)$ is defined as

$$(\mathrm{pr}_2^* L)(P \times R) = \overline{\mathrm{pr}}_2^*(L(R)) \in D^b(\overline{P}_c \times \overline{R}_c).$$

Thus,

$$\begin{aligned}(K \boxtimes L)(P \times R) &= (\mathrm{pr}_1^* K)(P \times R) \otimes (\mathrm{pr}_2^* L)(P \times R) \\ &= \mathrm{pr}_{\overline{P}}^*(K(P)) \otimes \mathrm{pr}_{\overline{R}}^*(L(R)) \\ &= K(P) \boxtimes L(R) \in D^b(\overline{P}_c \times \overline{R}_c).\end{aligned}$$

Since $mp \dotplus mp = mp$, the functor

$$\boxtimes : D^{b,c}_G(X) \times D^{b,c}_H(Y) \to D^b_{G\times H}(X_c \times Y_c)$$

is $t$-exact by [10, Proposition 2.10]. The restriction of $\boxtimes$ to perverse sheaves gives a $\mathbb{C}$-bilinear exact in each variable functor

$$\boxtimes : \mathrm{Perv}_G(X) \times \mathrm{Perv}_H(Y) \to \mathrm{Perv}_{G\times H}(X_c \times Y_c).$$

We want to prove

**Main Theorem.** *This functor makes* $\mathrm{Perv}_{G\times H}(X_c \times Y_c)$ *into the tensor product of categories* $\mathrm{Perv}_G(X)$ *and* $\mathrm{Perv}_H(Y)$.

Since we already know that the tensor product of categories exists, we can decompose the functor $\boxtimes$ as follows

$$\boxtimes \simeq \big(\mathrm{Perv}_G(X) \times \mathrm{Perv}_H(Y) \xrightarrow{\boxtimes^D} \mathrm{Perv}_G(X)\boxtimes^D \mathrm{Perv}_H(Y) \to \\ \xrightarrow{F} \mathrm{Perv}_{G\times H}(X_c \times Y_c)\big).$$

Our goal is to prove that $F$ is an equivalence. To achieve it we show first that $\boxtimes^D$ commutes with finite inverse limits. Then we reduce the involved inverse limits to finite ones.

## 3.2. Properties of inverse limit categories

Let $\mathcal{A}/J$ be a fibered category and let $\mathcal{A} = \varprojlim_{X\in J}\mathcal{A}(X)$ be its inverse limit. We have canonical functors $\mathcal{A}_X : \mathcal{A} \to \mathcal{A}(X)$, $K \mapsto K(X)$ for each object $X$ of $J$ and canonical natural isomorphisms

$$\mathcal{A}_f : \mathcal{A}(f)\mathcal{A}_Y \xrightarrow{\sim} \mathcal{A}_X : \mathcal{A} \to \mathcal{A}(X),$$
$$(\mathcal{A}_f)_K = K(f) : \mathcal{A}(f)K(Y) \xrightarrow{\sim} K(X) \in \mathcal{A}(X),$$

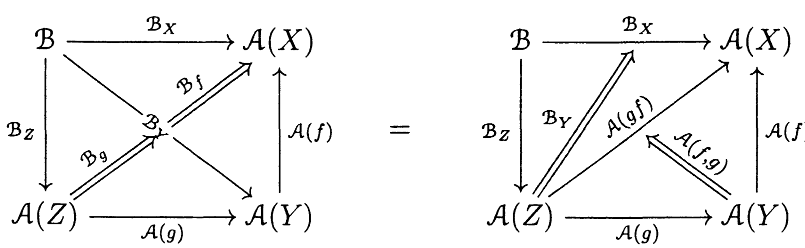
for each $f : X \to Y \in J$ which satisfy equations (2.1) and (2.2). Thus for any functor $F : \mathcal{B} \to \mathcal{A}$ we have composite functors

$$\mathcal{B}_X = \big(\mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{\mathcal{A}_X} \mathcal{A}(X)\big)$$

and functorial isomorphisms

$$\mathcal{B}_f = \mathcal{A}_f \circ F : \mathcal{A}(f)\mathcal{B}_Y = \mathcal{A}(f)\mathcal{A}_Y F \to \mathcal{A}_X F = \mathcal{B}_X \tag{3.1}$$

such that for $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have



(3.2)

and $\mathcal{B}_{\mathrm{id}} = \mathrm{id}$.

Vice versa, for data consisting of a category $\mathcal{B}$, a functor $\mathcal{B}_X : \mathcal{B} \to \mathcal{A}(X)$ for each object $X$ of $J$, a functorial isomorphism $B_f : \mathcal{A}(f)\mathcal{B}_Y \xrightarrow{\sim} \mathcal{B}_X : \mathcal{B} \to \mathcal{A}(X)$ for each morphism $f : X \to Y$ of $J$ such that (3.2) holds and $\mathcal{B}_{\mathrm{id}} = \mathrm{id}$, there exists a unique functor $F : \mathcal{B} \to \mathcal{A}$ such that $\mathcal{B}_X = \mathcal{A}_X \circ F$ and (3.1) holds. This is a useful way to encode functors into inverse limits.

Furthermore, assume that $\mathcal{B}$ and all $\mathcal{A}(X)$ are abelian categories and $\mathcal{A}(f)$ are exact functors. Then $F$ is exact if and only if all $\mathcal{B}_X$ are exact. Similar conclusion about $\mathbb{C}$-linearity.

The above considerations extend to morphisms $\lambda : F \to F' : \mathcal{B} \to \mathcal{A}$. They give rise to functorial morphisms

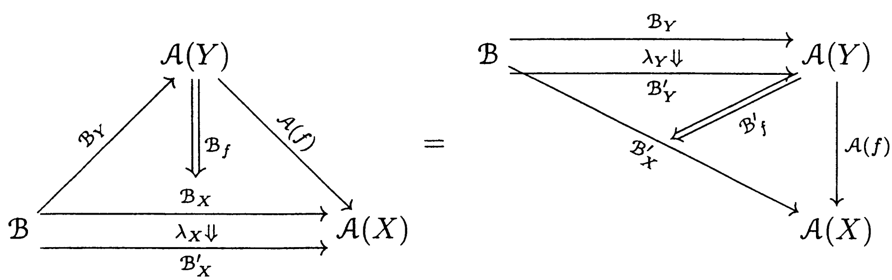
$$\lambda_X = \Big(\mathcal{B} \underset{F'}{\overset{F}{\underset{\Downarrow\lambda}{\rightrightarrows}}} \mathcal{A} \xrightarrow{\mathcal{A}_X} \mathcal{A}(X)\Big) \tag{3.3}$$

between the functors

$$\mathcal{B}_X = \big(\mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{\mathcal{A}_X} \mathcal{A}(X)\big), \tag{3.4}$$

$$\mathcal{B}'_X = \big(\mathcal{B} \xrightarrow{F'} \mathcal{A} \xrightarrow{\mathcal{A}_X} \mathcal{A}(X)\big). \tag{3.5}$$

They satisfy the equations




(3.6)

Vice versa, a system of functorial morphisms $\lambda_X$ satisfying (3.6) determines a unique morphism $\lambda : F \to F'$ between functors corresponding to $\mathcal{B}_X$, $\mathcal{B}'_X$ via (3.5), and (3.3) holds. Equation (3.6) is interpreted as morphism condition (2.3).

**3.3. Lemma.** *Let $\mathcal{A}/J$ and $\mathcal{B}/J$ be filtered categories with locally bounded abelian $\mathbb{C}$-linear categories $\mathcal{A}(X)$, $\mathcal{B}(Y)$ and exact $\mathbb{C}$-linear functors $\mathcal{A}(f)$, $\mathcal{B}(g)$. Then the external tensor products $\mathcal{A}(X)\boxtimes^D\mathcal{B}(Y)$, $\mathcal{A}(f)\boxtimes^D\mathcal{B}(g)$, $\mathcal{A}(f',f'')\boxtimes^D\mathcal{B}(g',g'')$ form a filtered category $\mathcal{A}\boxtimes^D\mathcal{B}/(I\times J)$.* □

**3.4. Lemma.** *Let $\mathcal{E}/(I\times J)$ be a fibered category. The categories $\mathcal{C}(Y)=\varprojlim_{X\in I}\mathcal{E}(X,Y)$, the functors $\mathcal{C}(g:Y'\to Y''):\mathcal{C}(Y'')\to\mathcal{C}(Y')$ constructed as $\varprojlim_{X\in I}\mathcal{E}(X,g)$ and canonically constructed functorial isomorphisms $\mathcal{C}(g',g'')$ form a fibered category $\mathcal{C}/J$. The restriction functors*

$$R_Y:\ \varprojlim_{(X,Y)\in I\times J}\mathcal{E}(X,Y)\to\varprojlim_{X\in I}\mathcal{E}(X,Y)=\mathcal{C}(Y)$$

*together with canonical isomorphisms induce a functor $\Phi$ which closes the following commutative diagram of functors.*

$$\begin{array}{ccc}
\varprojlim\limits_{(X,Y)\in I\times J}\mathcal{E}(X,Y) & \xrightarrow{\quad\Phi\quad} & \varprojlim\limits_{Y\in J}\Big(\varprojlim\limits_{X\in I}\mathcal{E}(X,Y)\Big) \\
 & \searrow^{R_Y}\quad = & \big\downarrow \mathcal{C}_Y \\
 & & \mathcal{C}(Y)
\end{array}$$

*The functor $\Phi$ is an isomorphism of categories (bijective on objects and morphisms).* □

The proof is based on the criterion of Section 3.2.

# 4. Commutation of tensor product of abelian categories with inverse limits

Let $\mathcal{A}/I$, $\mathcal{B}/J$ be fibered categories with $\mathbb{C}$-linear abelian locally bounded fibres and $\mathbb{C}$-linear exact faithful functors $\mathcal{A}(f)$, $\mathcal{B}(g)$.

**4.1. Proposition.** *There exist canonically defined functors*

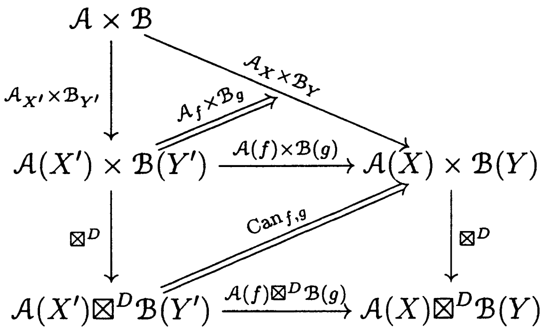
$$\boxtimes:\varprojlim_{X\in I}\mathcal{A}(X)\times\varprojlim_{Y\in J}\mathcal{B}(Y)\to\varprojlim_{(X,Y)\in I\times J}\mathcal{A}(X)\boxtimes^D\mathcal{B}(Y),\tag{4.1}$$

$$\tilde{\boxtimes} : \Big(\varprojlim_{X\in I} \mathcal{A}(X)\Big) \boxtimes^D \Big(\varprojlim_{Y\in J} \mathcal{B}(Y)\Big) \to \varprojlim_{(X,Y)\in I\times J} \big(\mathcal{A}(X)\boxtimes^D \mathcal{B}(Y)\big). \qquad (4.2)$$

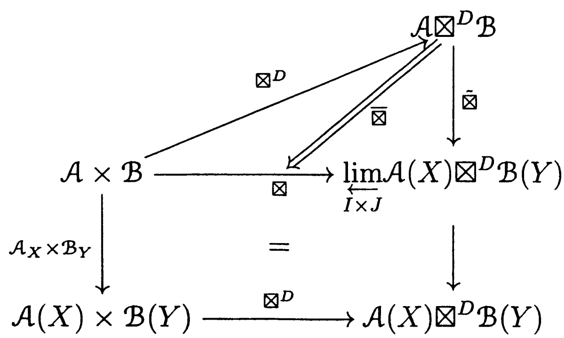
*Proof.* Denote $\mathcal{A} = \varprojlim_{X\in I} \mathcal{A}(X)$ and $\mathcal{B} = \varprojlim_{Y\in J} \mathcal{B}(Y)$. The system of functors

$$\mathcal{A}\times\mathcal{B} \xrightarrow{\mathcal{A}_X\times\mathcal{B}_Y} \mathcal{A}(X)\times\mathcal{B}(Y) \xrightarrow{\boxtimes^D} \mathcal{A}(X)\boxtimes^D\mathcal{B}(Y)$$

together with isomorphisms




for each pair of morphisms $f : X \to X' \in I$, $g : Y \to Y' \in J$ defines uniquely functor (4.1) via criterion of Section 3.2. Functor (4.2) is obtained from it, unique up to an isomorphism. See the following scheme.




(4.3)

□

We are going to exhibit some cases in which $\tilde{\boxtimes}$ is an equivalence.

## 4.2. A bounded particular case

Now we consider a particular case: $I$ is a one-morphism category, and $\mathcal{A}$ is bounded.

**Proposition.** *If $\mathcal{A}$ is bounded, then*

$$\tilde{\boxtimes} : \mathcal{A}\boxtimes^D \varprojlim_{Y\in J} \mathcal{B}(Y) \to \varprojlim_{Y\in J} \big(\mathcal{A}\boxtimes^D \mathcal{B}(Y)\big)$$

*is an equivalence.*

*Proof.* We may assume that $\mathcal{A} = A$-mod for a finite dimensional associative unital $\mathbb{C}$-algebra $A$. For a $\mathbb{C}$-linear abelian category $\mathcal{D}$ denote ${}_A\mathcal{D}$ the category of objects $M$ of $\mathcal{D}$ equipped with an action of $A$ – algebra homomorphism $A \to \operatorname{End}_{\mathcal{D}} M$. The morphisms of ${}_A\mathcal{D}$ are those commuting with the action of $A$. The functor

$$\otimes : \mathcal{A} \times \mathcal{D} \to {}_A\mathcal{D}, \qquad (N, M) \mapsto N \otimes M$$

turns ${}_A\mathcal{D}$ into the tensor product category $\mathcal{A}\boxtimes^D\mathcal{D}$ [6, Proposition 5.11]. Here the object $N \otimes M \in \mathcal{D}$ might be described as follows [7]. Let $(v_1, \dots, v_n)$ be a chosen $\mathbb{C}$-basis of $N$, then $N \otimes M = M^n$ with canonical embeddings $\iota_j : M \to M^n$ and canonical projections $\pi^j : M^n \to M$. Let $(e_1, \dots, e_a)$ be a basis of $A$. The structure constants $n_{ij}^k$ of the $A$-module $N$, $a_i.v_j = \sum_k n_{ij}^k v_k$, determine the action of $A$ in $N \otimes M$ via

$$a_i = \sum_{k,j} n_{ij}^k \iota_k \circ \pi^j.$$

This formula imitates the one for $\mathbb{C}$-vector spaces

$$a_i.(v_j \otimes m) = \sum_k n_{ij}^k v_k \otimes m.$$

Consider the beginning of the bar-resolution of $M \in \operatorname{Ob} {}_A\mathcal{D}$

$$A \otimes_{\mathbb{C}} A \otimes M \to A \otimes M \to M \to 0,$$

the maps imitate the differentials in $\mathbb{C}$-vect

$$a \otimes b \otimes m \mapsto ab \otimes m - a \otimes b.m, \qquad a \otimes m \mapsto a.m.$$

Using the structure constants $a_{ij}^k$ of the algebra $A$ we can write this exact sequence in ${}_A\mathcal{D}$ as

$$M^{a^2} \xrightarrow{\sum(a_{ij}^k \iota_k - \iota_i e_j)\pi^{ij}} M^a \xrightarrow{\sum e_i \pi^i} M \to 0,$$

where $\pi^{ij} : M^{a^2} \to M$ are the canonical projections, $1 \leqslant i, j \leqslant a$. Another way to write this sequence is

$$\oplus_{i=1}^a A \otimes M \xrightarrow{\sum_{i=1}^a (R(e_i)\otimes 1 - 1 \otimes e_i)\pi^i} A \otimes M \to M \to 0, \qquad (4.4)$$

where $R(e_i) : A \to A$, $a \mapsto ae_i$ are endomorphisms of the left regular module.

Denote $\mathcal{B} = \varprojlim_{Y \in J} \mathcal{B}(Y)$. Diagram (4.3) which determines $\tilde{\boxtimes}$ takes the form

$$\begin{array}{ccc}
 & & {}_A\mathcal{B} \\
 & \nearrow^{\otimes} \quad \swarrow^{\overline{\boxtimes}} & \downarrow \tilde{\boxtimes} \\
\mathcal{A} \times \varprojlim_{Y \in J} \mathcal{B}(Y) & \xrightarrow[\boxtimes]{} & \varprojlim_{Y \in J} {}_A\mathcal{B}(Y) \\
{\scriptstyle 1 \times \mathcal{B}_Y} \downarrow & = & \downarrow {\scriptstyle \mathcal{C}_Y} \\
\mathcal{A} \times \mathcal{B}(Y) & \xrightarrow{\otimes} & {}_A\mathcal{B}(Y)
\end{array}$$

Let $M$ be an object of $\mathcal{C} = \varprojlim_{Y \in J} {}_A\mathcal{B}(Y)$. Forgetting the action of $A$ we can view $M$ as an object of $\mathcal{B} = \varprojlim_{Y \in J} \mathcal{B}(Y)$. Tensoring it with $A$ we get an object $A \boxtimes M$ of $\mathcal{C}$, whose components are $(A \boxtimes M)(Y) = A \otimes M(Y)$, $(A \boxtimes M)(f) = A \otimes M(f)$. Replacing in (4.4) $M$ with $M(Y)$ we get exact sequences in ${}_A\mathcal{B}(Y)$. Furthermore, the isomorphism $\big(1 \otimes \mathcal{B}(g)\big)(N \otimes L) \simeq N \otimes \mathcal{B}(g)L$ gives for each $g : Y \to Z \in J$ a commutative diagram

$$\begin{array}{ccccccc}
\bigoplus_{i=1}^{a} A \otimes \mathcal{B}(g)M(Z) & \xrightarrow{\sum_i (R(e_i)\otimes 1 - 1 \otimes e_i)} & A \otimes \mathcal{B}(g)M(Z) & \longrightarrow & \mathcal{B}(g)M(Z) & \longrightarrow & 0 \\
\downarrow {\scriptstyle \oplus 1 \otimes M(g)} & & \downarrow {\scriptstyle 1 \otimes M(g)} & & \downarrow {\scriptstyle M(g)} & & \\
\oplus_{i=1}^a A \otimes M(Y) & \xrightarrow{\sum_i (R(e_i)\otimes 1 - 1 \otimes e_i)} & A \otimes M(Y) & \longrightarrow & M(Y) & \longrightarrow & 0
\end{array}$$

with exact rows. Therefore, there is an exact sequence in $\mathcal{C}$

$$\oplus_{i=1}^{a} A \boxtimes M \xrightarrow{\sum_i R(e_i)\boxtimes 1 - 1\boxtimes e_i} A \boxtimes M \xrightarrow{\text{action}} M \to 0.$$

The object $A \boxtimes M \simeq \tilde{\boxtimes}(A \otimes M)$ is in essential image of $\tilde{\boxtimes}$, thus, the first morphism is essentially in the image of $\tilde{\boxtimes}$. Hence, $M$ is essentially in $\tilde{\boxtimes}({}_A\mathcal{B})$.

If $\phi : L \to M$ is in $\mathcal{C}$ we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\oplus_{i=1}^{a} A \boxtimes L & \xrightarrow{\sum_i R(e_i)\boxtimes 1 - 1\boxtimes e_i} & A \boxtimes L & \xrightarrow{\text{action}} & L & \longrightarrow & 0 \\
\downarrow{\scriptstyle \oplus 1\boxtimes\phi} & & \downarrow{\scriptstyle 1\boxtimes\phi} & & \downarrow{\scriptstyle \phi} & & \\
\oplus_{i=1}^{a} A \boxtimes M & \xrightarrow{\sum_i R(e_i)\boxtimes 1 - 1\boxtimes e_i} & A \boxtimes M & \xrightarrow{\text{action}} & M & \longrightarrow & 0
\end{array}$$

Its left square lies essentially in $\tilde{\boxtimes}({}_A\mathcal{B})$, hence so is the right vertical arrow $\phi$. We proved that $\tilde{\boxtimes}$ is essentially surjective on objects and surjective on morphisms.

We claim that for arbitrary $A$-modules $P, Q \in A$-mod and arbitrary objects $M$, $N$ of $\mathcal{B}$ the map

$$\mathrm{Hom}_A(P, Q) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{B}}(M, N) \to \mathrm{Hom}_{\mathcal{C}}(P \otimes M, Q \otimes N)$$

is injective. Indeed, let $\sum_k \phi_k \otimes \psi_k$ be given from the source, where $(\phi_k)$ is a $\mathbb{C}$-basis of $\mathrm{Hom}_A(P, Q)$, $\psi_k : M \to N \in \mathcal{B}$. Assume that this sum is mapped to 0, then for each $Y \in J$

$$\sum_k \phi_k \otimes \psi_k(Y) = 0 : P \otimes M(Y) \to Q \otimes N(Y).$$

Complementing $(\phi_k)$ to a basis of $\mathrm{Hom}_{\mathbb{C}}(P, Q)$ and passing to the standard basis we deduce that all $\psi_k(Y) = 0$. Therefore, $\psi_k = 0$ for all $k$. It remains to apply Lemma 2.7 to deduce that $\tilde{\boxtimes}$ is full and faithful. □

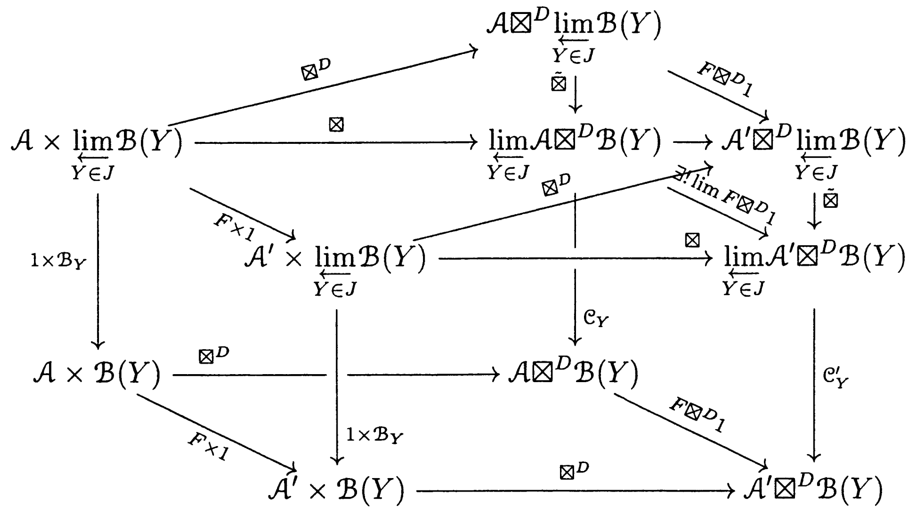
## 4.3. A locally bounded particular case

**Proposition.** *Let $\mathcal{A}$ be a locally bounded category. Then*

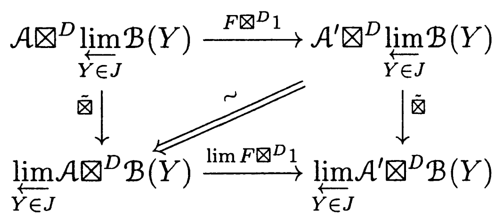
$$\tilde{\boxtimes} : \mathcal{A} \boxtimes^D \varprojlim_{Y\in J} \mathcal{B}(Y) \to \varprojlim_{Y\in J} (\mathcal{A} \boxtimes^D \mathcal{B}(Y))$$

*is full and faithful. If $J$ has finite set of objects, then $\tilde{\boxtimes}$ is an equivalence.*

*Proof.* Consider an exact $\mathbb{C}$-linear functor $F : \mathcal{A} \to \mathcal{A}'$ between locally bounded categories. It combines diagrams (4.3) for $\mathcal{A}$ and $\mathcal{A}'$ into the following diagram.



In particular, there is an isomorphism



and the lower functor is constructed uniquely.

Given a filtered inductive limit $\mathcal{A} = \varinjlim_{i\in I} \mathcal{A}_i$, we may apply the above diagram to each exact functor $F_{ii'} : \mathcal{A}_i \to \mathcal{A}_{i'}$ from the inductive system. Also we may apply it to the canonical functors $\mathrm{Can}_i^{\mathcal{A}} : \mathcal{A}_i \to$

$\varinjlim_{i\in I}\mathcal{A}_i$. Taking the inductive limit we get the following diagram.

$$\begin{array}{ccccc}
\mathcal{A}_i\boxtimes^D\varprojlim_{Y\in J}\mathcal{B}(Y) & \xrightarrow{\mathrm{Can}_i} & \varinjlim_{i\in I}\Big(\mathcal{A}_i\boxtimes^D\varprojlim_{Y\in J}\mathcal{B}(Y)\Big) & \xrightarrow{\sim} & \mathcal{A}\boxtimes^D\varprojlim_{Y\in J}\mathcal{B}(Y) \\
\tilde{\boxtimes}\downarrow & \simeq & \lim\tilde{\boxtimes}\downarrow & \simeq & \downarrow\tilde{\boxtimes} \\
\varprojlim_{Y\in J}\mathcal{A}_i\boxtimes^D\mathcal{B}(Y) & \xrightarrow{\mathrm{Can}'_i} & \varinjlim_{i\in I}\Big(\varprojlim_{Y\in J}\mathcal{A}_i\boxtimes^D\mathcal{B}(Y)\Big) & \xrightarrow{\Phi} & \varprojlim_{Y\in J}\mathcal{A}\boxtimes^D\mathcal{B}(Y)
\end{array}$$

$$\varprojlim_{Y\in J}\mathcal{A}_i\boxtimes^D\mathcal{B}(Y) \xrightarrow{\lim\mathrm{Can}_i^{\mathcal{A}}\boxtimes^D 1} \varprojlim_{Y\in J}\mathcal{A}\boxtimes^D\mathcal{B}(Y)$$

Now for a given locally bounded $\mathcal{A}$ we consider its presentation as an inductive limit $\varinjlim_{i\in I}\mathcal{A}_i$ of bounded $\mathcal{A}_i$ such that all $F_{ii'}:\mathcal{A}_i\to\mathcal{A}_{i'}$ are exact, full and faithful. It follows from Lemma 2.6 that $F_{ii'}\boxtimes^D 1$, $\varprojlim_J F_{ii'}\boxtimes^D 1$, $\mathrm{Can}_i^{\mathcal{A}}$, $\mathrm{Can}_i$, $\mathrm{Can}'_i$, $\varprojlim\mathrm{Can}_i^{\mathcal{A}}\boxtimes^D 1$ are also exact, full and faithful. We deduce that $\Phi$ is exact, full and faithful as well. From Proposition 4.2 we know that the leftmost $\tilde{\boxtimes}$ is an equivalence, so the middle $\varinjlim_{i\in I}\tilde{\boxtimes}$ is an equivalence. The right top horizontal functor is an equivalence, hence, the rightmost $\tilde{\boxtimes}$ is full and faithful.

Now assume that $\mathrm{Ob}\,J$ is finite. We have to show that $\Phi$ is essentially surjective on objects. Let $K$ be an object of $\varprojlim_{Y\in J}\mathcal{A}\boxtimes^D\mathcal{B}(Y)$. For each $Y\in\mathrm{Ob}\,J$ there is an $i\in I$ such that $K(Y)\in\mathcal{A}_i\boxtimes^D\mathcal{B}(Y)\hookrightarrow\mathcal{A}\boxtimes^D\mathcal{B}(Y)$. Take $i'\in I$ bigger than all such $i(Y)$. Then $K(Y)\in\mathcal{A}_{i'}\boxtimes^D\mathcal{B}(Y)$ determine an object $K'$ of $\varprojlim_{Y\in J}\mathcal{A}_{i'}\boxtimes^D\mathcal{B}(Y)$ such that $\Phi(\mathrm{Can}'_i K')\simeq K$. We deduce that the rightmost $\tilde{\boxtimes}$ is essentially surjective on objects as well. □

## 4.4. The general case

**Proposition.** *For all $I$, $J$ the functor $\tilde{\boxtimes}$ from* (4.2) *is full and faithful. If* $\mathrm{Ob}\,I$ *and* $\mathrm{Ob}\,J$ *are finite, the functor $\tilde{\boxtimes}$ is an equivalence.*

*Proof.* The idea is to reduce the result to the particular case considered in Proposition 4.3. In diagram in Fig. 1 all functors are already

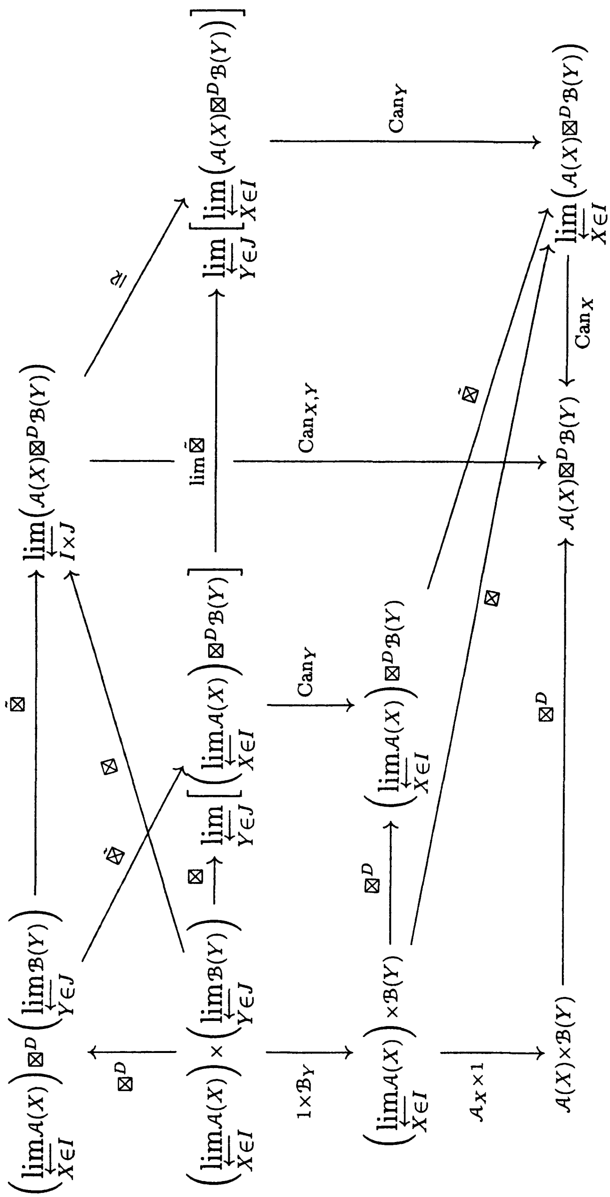


Figure 1: Repeated vs. double inverse limits

constructed, except $\varprojlim_{Y\in J}\tilde{\boxtimes}$. To get it one varies $Y$ and checks compatibility conditions (3.2). Consider both directed paths of 4 arrows in this diagram starting with $\left(\varprojlim_{X\in I}\mathcal{A}(X)\right)\times\left(\varprojlim_{Y\in J}\mathcal{B}(Y)\right)$ and ending with

$$\varprojlim_{Y\in J}\left[\varprojlim_{X\in I}\left(\mathcal{A}(X)\boxtimes^D\mathcal{B}(Y)\right)\right]\xrightarrow{\mathrm{Can}_Y}\varprojlim_{X\in I}\left(\mathcal{A}(X)\boxtimes^D\mathcal{B}(Y)\right)\to$$
$$\xrightarrow{\mathrm{Can}_X}\mathcal{A}(X)\boxtimes^D\mathcal{B}(Y).$$

The both composite functors are isomorphic. Moreover, the isomorphisms satisfy the compatibility conditions when $X$ varies, hence, the subpaths of 3 arrows ending with $\varprojlim_{X\in I}\left(\mathcal{A}(X)\boxtimes^D\mathcal{B}(Y)\right)$ are isomorphic. Again, the isomorphisms satisfy the compatibility conditions when $Y$ varies, hence, the subpaths of length 2 ending with

$$\varprojlim_{Y\in J}\left[\varprojlim_{X\in I}\left(\mathcal{A}(X)\boxtimes^D\mathcal{B}(Y)\right)\right]$$

give isomorphic bilinear functors. By the universal property of $\boxtimes^D$ we get the top face isomorphism repeated below.

$$\begin{array}{ccc}
\left(\varprojlim_{X\in I}\mathcal{A}(X)\right)\boxtimes^D\left(\varprojlim_{Y\in J}\mathcal{B}(Y)\right) & \xrightarrow{\tilde{\boxtimes}} & \varprojlim_{I\times J}\left(\mathcal{A}(X)\boxtimes^D\mathcal{B}(Y)\right) \\
\downarrow{\scriptstyle\tilde{\boxtimes}^D} & \simeq & \downarrow{\scriptstyle\cong} \\
\varprojlim_{Y\in J}\left[\left(\varprojlim_{X\in I}\mathcal{A}(X)\right)\boxtimes^D\mathcal{B}(Y)\right] & \xrightarrow{\lim\tilde{\boxtimes}} & \varprojlim_{Y\in J}\left[\varprojlim_{X\in I}\left(\mathcal{A}(X)\boxtimes^D\mathcal{B}(Y)\right)\right]
\end{array}$$

The right vertical functor is an isomorphism by Lemma 3.4. The left vertical and the bottom functors are full and faithful by Proposition 4.3, hence, so is the top functor.

If $\mathrm{Ob}\, I$ and $\mathrm{Ob}\, J$ are finite, then the left and the bottom functors are equivalences, hence, so is the top functor. □

## 5. Reduction to finite case

Consider an interval $N=\{x\in\mathbb{Z}\mid a\leqslant x\leqslant b\}\subset\mathbb{Z}$. Denote

$$D^N(Z)=\{K\in D(Z)\mid k\notin N\Rightarrow H^k(K)=0\}.$$

For a stratified space $(Z, \mathcal{Z})$ the full subcategory of $\mathrm{Perv}(Z, \mathcal{Z}, p)$

$$\mathrm{Perv}^N(Z, \mathcal{Z}, p) = \mathrm{Perv}(Z, \mathcal{Z}, p) \cap D^N(Z)$$

is additive, $\mathbb{C}$-linear, closed under direct summands and extensions. Indeed, $0 \to A \to B \to C \to 0$ in $\mathrm{Perv}(Z, \mathcal{Z}, p)$ extends to a distinguished triangle $A \longrightarrow B \longrightarrow C \xrightarrow{(1)}$ , and $A, C \in D^N(Z)$ implies $B \in D^N(Z)$ by the long cohomology sequence. Similarly, for an algebraic variety $Z$

$$\mathrm{Perv}_c^N(Z, mp) = \mathrm{Perv}_c(Z, mp) \cap D^N(Z)$$

is additive, $\mathbb{C}$-linear, closed under direct summands and extensions.

For a finite family $\mathcal{S}$ of isomorphism classes of simple objects of the category $\mathrm{Perv}(Z, \mathcal{Z}, p)$ there is an interval $N \subset \mathbb{Z}$ such that all $S \in \mathcal{S}$ are in $D^N(Z)$. Denote $/\mathcal{S}/$ the strictly full subcategory of $\mathrm{Perv}(Z, \mathcal{Z}, p)$ consisting of objects whose simple subquotients are from $\mathcal{S}$. Then $/\mathcal{S}/ \subset \mathrm{Perv}^N(Z, \mathcal{Z}, p)$ since all objects of $/\mathcal{S}/$ are obtained by repeated extension from $\mathcal{S}$. This shows that $\mathrm{Perv}(Z, \mathcal{Z}, p)$ is a union of its abelian subcategories, contained in $\mathrm{Perv}^N(Z, \mathcal{Z}, p)$ for some $N \subset \mathbb{Z}$. Indeed, for an object $K \in \mathrm{Perv}(Z, \mathcal{Z}, p)$ denote by $\mathcal{S}$ the set of its simple subquotients, then $K \in /\mathcal{S}/$.

Now let us study the full subcategory

$$\mathrm{Perv}_G^N(X) = \{K \in \mathrm{Perv}_G(X) \mid K(T) \in \mathrm{Perv}_c^{N+\dim_{\mathbb{C}} G}(X)\}.$$

It coincides with

$$\{K \in \mathrm{Perv}_G(X) \mid \forall (p : P \to X) \in \mathrm{SRes}(X, G) \quad K(p) \in \mathrm{Perv}_c^{N+d_p}(\overline{P})\}.$$

If $X$ is $G$-free, then $\mathrm{Perv}_G^N(X) \simeq \mathrm{Perv}_c^N(G\backslash X)$.

One of the important ideas of Bernstein and Lunts [3] is that the category $D_G^N(X)$ can be presented as an inverse limit of the diagram with two arrows only. Let $(p : P \to X) \in \mathrm{SRes}(X, G) = I$ be a smooth $n$-acyclic resolution [3, Definition 1.9.1] for $n \geqslant |N| = \max\{k \in N\} - \min\{k \in N\}$. Consider a subcategory $I(P)$ of $I$ consisting of three objects and two morphisms besides identities:

$$T \longleftarrow T \times_X P \longrightarrow P.$$

Here $T$ is the trivial resolution and the morphisms are the projections. The $G$-quotient of this diagram is

$$X \longleftarrow P \longrightarrow \overline{P}.$$

**5.1. Lemma.** *The restriction functor*

$$\mathrm{Perv}_G^N(X) = \varprojlim_{(r:R\to X)\in I} \mathrm{Perv}_c^{N+d_r}(\overline{R}) \longrightarrow \varprojlim_{(r:R\to X)\in I(P)} \mathrm{Perv}_c^{N+d_r}(\overline{R})$$

*is an equivalence.*

*Proof.* Let us construct a quasi-inverse functor. A quasi-inverse functor to the restriction functor

$$\varprojlim_{(r:R\to X)\in I} D_c^{N+d_r}(\overline{R}) \longrightarrow \varprojlim_{(r:R\to X)\in I(P)} D_c^{N+d_r}(\overline{R})$$

is constructed in [3, Remark 2.4.3]. We will show that it restricts to perverse sheaves.

Let $s : S \to X$ be a smooth $G$-resolution. Following the scheme of Bernstein and Lunts we define

$$\mathrm{Perv}_G^N(X, S) = \varprojlim_{(r:R\to X)\in I(S)} \mathrm{Perv}_c^{N+d_r}(\overline{R}).$$

The projection maps

$$\begin{array}{ccccc}
T & \longleftarrow & T \times_X P \times_X S & \longrightarrow & P \times_X S \\
\Big\| & & \Big\downarrow{\scriptstyle \mathrm{pr}_{13}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_2} \\
T & \longleftarrow & T \times_X S & \longrightarrow & S
\end{array}$$

induce a functor between the inverse limits:

$$\mathrm{pr}_S^* : \mathrm{Perv}_G^N(X, S) \to \mathrm{Perv}_G^N(X, P \times_X S).$$

Similarly, there is a functor

$$\mathrm{pr}_P^* : \mathrm{Perv}_G^N(X, P) \to \mathrm{Perv}_G^N(X, P \times_X S).$$

The above functor $\mathrm{pr}_S^*$ is a restriction of the functor $\mathrm{Pr}_S^* : D_G^N(X,S) \to D_G^N(X, P \times_X S)$, which is an equivalence by [3, Corollary 2.2.2]. Notice that $\mathrm{pr}_S^*$ is essentially surjective on objects. Indeed, let $K \in D_G^N(X,S)$ be such that $\mathrm{pr}_S^* K \in \mathrm{Perv}_G^N(X, P \times_X S)$. The properties $\overline{\mathrm{pr}_2}^* K(S) \in \mathrm{Perv}_c^{N+d_s}(\overline{P \times_X S})$, $\overline{\mathrm{pr}_{13}}^* K(T \times_X S) \in \mathrm{Perv}_c^{N+d_t+d_s}(\overline{T \times_X P \times_X S})$ imply $K(S) \in \mathrm{Perv}_c^{N+d_s}(\overline{S})$, $K(T \times_X S) \in \mathrm{Perv}_c^{N+d_t+d_s}(\overline{T \times_X S})$ by the proof of Proposition 2.3. Hence, $K \in \mathrm{Perv}_G^N(X,S)$. Since $\mathrm{Perv}_G^N(X,S) \subset D_G^N(X,S)$ and $\mathrm{Perv}_G^N(X, P \times_X S) \subset D_G^N(X, P \times_X S)$ are strictly full subcategories, we deduce that $\mathrm{pr}_S^*$ is an equivalence.

As shown in [3] the composite functor

$$\mathrm{Perv}_G^N(X,P) \xrightarrow{\mathrm{pr}_P^*} \mathrm{Perv}_G^N(X, P \times_X S) \xrightarrow{\mathrm{pr}_S^{*\,-1}}$$
$$\to \mathrm{Perv}_G^N(X,S) \xrightarrow{\mathrm{Can}_S} \mathrm{Perv}_c^{N+d_s}(\overline{S}), \qquad V \mapsto \tilde{V}(S)$$

defines an object $\tilde{V}$ of $D_G^N(X)$ (the isomorphisms $\tilde{V}(f : S \to S')$ being easy to construct). On the other hand, $\tilde{V}(S) \in \mathrm{Perv}_c(\overline{S})$, hence, $\tilde{V} \in \mathrm{Perv}_G^N(X)$. Thus, the quasi-inverse functors

$$D_G^N(X,P) \overset{\mathrm{res}}{\longleftrightarrow} D_G^N(X), \qquad V \mapsto \tilde{V}$$

restrict to functors between the strictly full subcategories $\mathrm{Perv}_G^N(X,P)$ and $\mathrm{Perv}_G^N(X)$, defining an equivalence of these. $\square$

### 5.2. Essential surjectivity of $\tilde{\boxtimes}$ on objects

We know that

$$\tilde{\boxtimes} : \Big(\varprojlim_{P \in I} \mathrm{Perv}_c(\overline{P})\Big) \boxtimes^D \Big(\varprojlim_{R \in J} \mathrm{Perv}_c(\overline{R})\Big) \to \varprojlim_{(P,R) \in I \times J} \mathrm{Perv}(\overline{P}_c \times \overline{R}_c)$$

is full and faithful. Let us prove that it is essentially surjective on objects. Take an object $K$ of the target category $\mathrm{Perv}_{G \times H}(X_c \times Y_c)$. The perverse sheaf $K(T) \in \mathrm{Perv}(X_c \times Y_c, mp) = \mathrm{Perv}_c(X, mp) \boxtimes^D \mathrm{Perv}_c(Y, mp)$ is contained in some subcategory $\mathrm{Perv}(X, \mathcal{X}, mp) \boxtimes^D \mathrm{Perv}(Y, \mathcal{Y}, mp)$. Moreover, there exist strictly full bounded subcategories $\mathcal{A} \subset \mathrm{Perv}(X, \mathcal{X}, mp)$, $\mathcal{B} \subset \mathrm{Perv}(Y, \mathcal{Y}, mp)$ such that $K(T) \in \mathcal{A} \boxtimes^D \mathcal{B}$. Let $\underline{A} \in \mathcal{A}$, $\underline{B} \in \mathcal{B}$ be projective generators. Let $\mathcal{S}(\underline{A})$, $\mathcal{S}(\underline{B})$ be the sets of simple factors of the

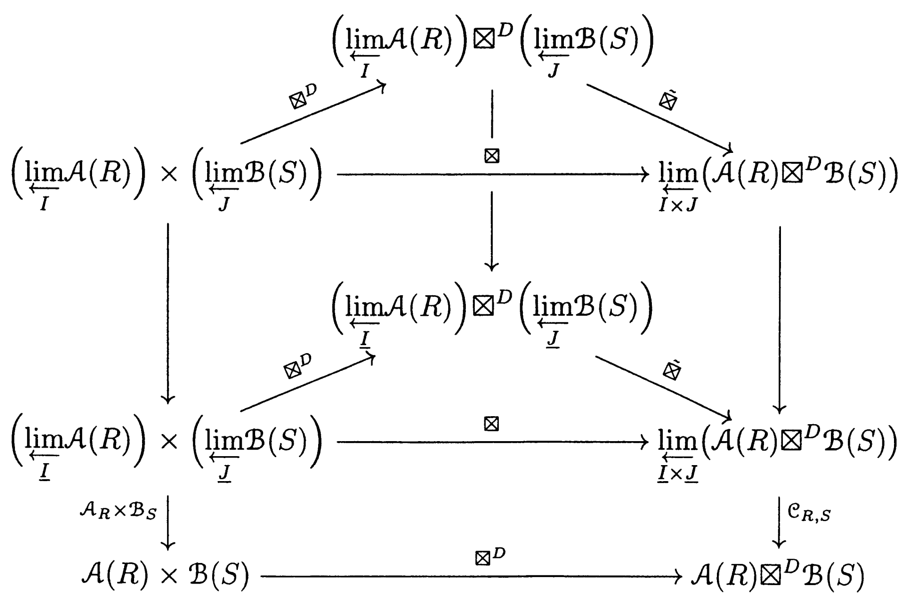
Jordan-Hölder series of $\underline{A}$, $\underline{B}$ in categories with length $\operatorname{Perv}(X, \mathcal{X}, mp)$, $\operatorname{Perv}(Y, \mathcal{Y}, mp)$. Denote $\mathcal{C} = /\mathcal{S}(\underline{A})/ \subset \operatorname{Perv}(X, \mathcal{X}, mp)$, $\mathcal{D} = /\mathcal{S}(\underline{B})/ \subset \operatorname{Perv}(Y, \mathcal{Y}, mp)$ the full subcategories consisting of objects, whose simple subquotients are in the lists $\mathcal{S}(\underline{A})$, $\mathcal{S}(\underline{B})$. We have $\mathcal{A} \subset \mathcal{C}$, $\mathcal{B} \subset \mathcal{D}$ and $K(T) \in \mathcal{C} \boxtimes^D \mathcal{D}$. There exist intervals $N', N'' \subset \mathbb{Z}$ such that $\mathcal{C} \subset D^{N'+d_G}(X)$, $\mathcal{D} \subset D^{N''+d_H}(Y)$. Then $\mathcal{C} \boxtimes^D \mathcal{D} \subset D^N(X \times Y)$ for $N = N' + N''$. Let $n = |N| = |N'| + |N''|$. Let $P \in \operatorname{SRes}(X, G)$, $Q \in \operatorname{SRes}(Y, H)$ be $n$-acyclic resolutions. Then $P \times Q \in \operatorname{SRes}(X \times Y, G \times H)$ is also an $n$-acyclic resolution. Consider the subcategories

$$\underline{I} = I(P) = \{T_X \leftarrow T_X \times_X P \rightarrow P\} \subset I,$$
$$\underline{J} = J(Q) = \{T_Y \leftarrow T_Y \times_Y Q \rightarrow Q\} \subset J.$$

**5.3. Lemma.** *There is an isomorphism*

$$\begin{array}{ccc}
\varprojlim_{R \in I} \operatorname{Perv}_c(\overline{R}) \boxtimes^D \varprojlim_{S \in J} \operatorname{Perv}_c(\overline{S}) & \xrightarrow{\tilde{\boxtimes}} & \varprojlim_{I \times J} \operatorname{Perv}(\overline{R}_c \times \overline{S}_c) \\
\Big\downarrow {\scriptstyle \operatorname{res} \boxtimes^D \operatorname{res}} & \simeq & \Big\downarrow {\scriptstyle \operatorname{res}} \\
\varprojlim_{R \in I(P)} \operatorname{Perv}_c(\overline{R}) \boxtimes^D \varprojlim_{S \in J(Q)} \operatorname{Perv}_c(\overline{S}) & \xrightarrow{\tilde{\boxtimes}} & \varprojlim_{I(P) \times J(Q)} \operatorname{Perv}(\overline{R}_c \times \overline{S}_c)
\end{array} \tag{5.1}$$

*Proof.* In a slightly more general context we have the following diagram.



Oriented paths beginning with $\big(\varprojlim_I \mathcal{A}(R)\big) \times \big(\varprojlim_J \mathcal{B}(S)\big)$ and ending at the right bottom corner give isomorphic functors. Section 3.2 implies the same statement for paths which begin at the same place and end up with $\varprojlim_{\underline{I}\times\underline{J}} \big(\mathcal{A}(R)\boxtimes^D\mathcal{B}(S)\big)$. Properties of tensor product imply this statement for paths beginning at the top and ending at the same place as above. □

Now look at the object

$$\operatorname{res} K \in \varprojlim_{I(P)\times J(Q)} \operatorname{Perv}(\overline{R}_c \times \overline{S}_c) \simeq \varprojlim_{I(P)\times J(Q)} \operatorname{Perv}_c(\overline{R})\boxtimes^D \operatorname{Perv}_c(\overline{S}).$$

We have

$$K(T_X \times T_Y) \in \mathcal{C}\boxtimes^D\mathcal{D},$$
$$K(T_X \times (T_Y \times_Y Q) \simeq (1 \boxtimes \overline{\mathrm{pr}_1}^*)(K(T_X \times T_Y)) \in \mathcal{C}\boxtimes^D \operatorname{Perv}_c(Q),$$
$$K(T_X \times Q) \in \operatorname{Perv}_c(X)\boxtimes^D \operatorname{Perv}_c(\overline{Q}), \tag{5.2}$$
$$(1 \boxtimes \overline{\mathrm{pr}_2}^*)(K(T_X \times Q)) \simeq K(T_X \times (T_Y \times_Y Q)) \in \mathcal{C}\boxtimes^D \operatorname{Perv}_c(Q). \tag{5.3}$$

**5.4. Lemma.** *Conditions* (5.2), (5.3) *imply*

$$K(T_X \times Q) \in \mathcal{C}\boxtimes^D \operatorname{Perv}_c(\overline{Q}).$$

*Proof.* First of all, let us prove that

$$K(T_X \times Q) \in \operatorname{Perv}(X, \mathcal{X}, mp)\boxtimes^D \operatorname{Perv}_c(\overline{Q}). \tag{5.4}$$

Let

$$K(T_X \times Q) \in \operatorname{Perv}(X, \mathcal{S}, mp)\boxtimes^D \operatorname{Perv}(\overline{Q}, \overline{\mathcal{Q}}, mp).$$

Denote $\mathcal{Q}$ the stratification $\mathrm{pr}_2^{-1}(\overline{\mathcal{Q}})$ of $Q$. Then

$$\begin{gathered}(1 \boxtimes \mathrm{pr}_2^*)K(T_X \times Q) \in \\ \operatorname{Perv}(X, \mathcal{S}, mp)\boxtimes^D \operatorname{Perv}(Q, \mathcal{Q}, mp)\cap\operatorname{Perv}(X, \mathcal{X}, mp)\boxtimes^D \operatorname{Perv}(Q, \mathcal{T}, mp)\end{gathered}$$

by (5.3). We may assume that $\mathcal{S}$ is finer than $\mathcal{X}$, and $\mathcal{T}$ is finer than $\mathcal{Q}$. Notice that the intersection of subcategories $\operatorname{Perv}(X \times Q, \mathcal{S}\times\mathcal{Q}, mp)$ and

$\mathrm{Perv}(X \times Q, \mathcal{X} \times \mathcal{T}, mp)$ coincides with $\mathrm{Perv}(X \times Q, \mathcal{X} \times \mathcal{Q}, mp)$. Actually, this is a statement about cohomologically constructible complexes and it reduces to a similar statement for sheaves, proven in Lemma A.2.

Now from

$$K(T_X \times Q) \in \mathrm{Perv}(X \times \overline{Q}, \mathcal{S} \times \overline{\mathcal{Q}}, mp) \subset D^{b,c}_{\mathcal{S}\times\overline{\mathcal{Q}}}(X \times \overline{Q}),$$
$$(1 \times \mathrm{pr}_2)^* K(T_X \times Q) \in \mathrm{Perv}(X \times Q, \mathcal{X} \times \mathcal{Q}, mp) \subset D^{b,c}_{\mathcal{X}\times\mathcal{Q}}(X \times Q)$$

we have to deduce that

$$K(T_X \times Q) \in D^{b,c}_{\mathcal{X}\times\overline{\mathcal{Q}}}(X \times \overline{Q}).$$

By [1] it would imply

$$K(T_X \times Q) \in \mathrm{Perv}(X \times \overline{Q}, \mathcal{X} \times \overline{\mathcal{Q}}, mp). \tag{5.5}$$

It suffices to work with cohomology sheaves of these complexes. We have to prove that if $M \in \mathcal{S}h(X \times \overline{Q})$ is $\mathcal{S} \times \overline{Q}$-constructible and $(1 \times \overline{\mathrm{pr}_2})^* M \in \mathcal{S}h(X \times Q)$ is $\mathcal{X} \times Q$-constructible, then $M$ is $\mathcal{X} \times \overline{Q}$-constructible.

Let $\cup_\alpha \overline{Q}_\alpha = \overline{Q}$ be an open cover of $\overline{Q}$ such that the quotient map $q_Q = \overline{\mathrm{pr}_2} : Q \to \overline{Q}$ admits a local section $r_\alpha : \overline{Q}_\alpha \hookrightarrow Q$. Denoting $Q_\alpha = q_Q^{-1}(\overline{Q}_\alpha)$, $q_\alpha = q_Q\big|_{Q_\alpha} : Q_\alpha \to \overline{Q}_\alpha$, we have $q_\alpha \circ r_\alpha = \mathrm{id}$. Let $\overline{\mathcal{Q}}_\alpha =$"$\overline{Q}_\alpha \cap \overline{\mathcal{Q}}$" be the induced stratification of $\overline{Q}_\alpha$, then

$$M\big|_{X\times\overline{Q}_\alpha} \simeq (1 \times r_\alpha)^* [(1 \times q)^* M]\big|_{X \times Q_\alpha}$$

is $\mathcal{X} \times \overline{\mathcal{Q}}_\alpha$-constructible. Gluing the stratifications we deduce that $M$ is $\mathcal{X} \times \overline{\mathcal{Q}}$-constructible, hence, (5.5) and (5.4) hold.

Recall that $\mathcal{C} = /\mathcal{S}(\underline{A})/ \subset \mathrm{Perv}(X, \mathcal{X}, mp)$ is closed under extensions. Its list of simple objects is $\mathcal{S}(\underline{A})$. Simple objects of the category $\mathrm{Perv}(X, \mathcal{X}, mp) \boxtimes^D \mathrm{Perv}(\overline{Q}, \overline{\mathcal{Q}}, mp)$ are of the form $S \boxtimes T$, where $S$ is simple in $\mathrm{Perv}(X, \mathcal{X}, mp)$, and $T$ is simple in $\mathrm{Perv}(\overline{Q}, \overline{\mathcal{Q}}, mp)$. In particular, any simple subquotient of $K(T_X \times Q)$ in $\mathrm{Perv}(X, \mathcal{X}, mp) \boxtimes^D \mathrm{Perv}(\overline{Q}, \overline{\mathcal{Q}}, mp)$ is of the form $S \boxtimes T$. An arbitrary simple subquotient of $(1 \boxtimes q_Q^*) K(T_X \times Q)$ is of the form $S \boxtimes R$, where $S$ is simple in $\mathrm{Perv}(X, \mathcal{X}, mp)$, and $R$ is a simple subquotient of $q_Q^* T \neq 0$ in $\mathrm{Perv}(Q, \mathcal{Q}, mp)$. Since $(1 \boxtimes q_Q^*) K(T_X \times Q) \in \mathcal{C} \boxtimes^D \mathrm{Perv}_c(Q)$, we see that for all simple subquotients $S \boxtimes T$ of $K(T_X \times Q)$ the object $S$ is in $\mathcal{S}(\underline{A}) \subset \mathcal{C}$.

Consider now the strictly full subcategory $\mathcal{E}$ of the tensor product category $\operatorname{Perv}(X, \mathcal{X}, mp) \boxtimes^D \operatorname{Perv}(\overline{Q}, \overline{\mathcal{Q}}, mp)$ – extension closure of the set of simple objects

$$\{S \boxtimes T \mid S \in \mathcal{S}(\underline{A}), T \text{ simple in } \operatorname{Perv}(\overline{Q}, \overline{\mathcal{Q}}, mp)\}. \tag{5.6}$$

We have $K(T_X \times Q) \in \operatorname{Ob}\mathcal{E}$ and $\mathcal{C} \boxtimes \operatorname{Perv}(\overline{Q}, \overline{\mathcal{Q}}, mp) \subset \mathcal{E}$. These two categories have the same list of simple objects (5.6). The inclusion functors induce mappings of Yoneda Ext groups

$$^Y\mathrm{Ext}^k_{\mathcal{C}}(L, L') \to {}^Y\mathrm{Ext}^k_{\operatorname{Perv}(X,\mathcal{X})}(L, L'),$$
$$^Y\mathrm{Ext}^k_{\mathcal{E}}(L \boxtimes M, L' \boxtimes M') \to {}^Y\mathrm{Ext}^k_{\operatorname{Perv}(X,\mathcal{X})\boxtimes\operatorname{Perv}(\overline{Q},\overline{\mathcal{Q}})}(L \boxtimes M, L' \boxtimes M').$$

Since the subcategories are closed under extensions, the above mappings are bijective for $k = 0, 1$ and injective for $k = 2$ [2, Lemma 3.2.3]. Using Theorem XI.3.1 of Cartan and Eilenberg [5] we see that the composite mapping

$$^Y\mathrm{Ext}^k_{\mathcal{C}\boxtimes\operatorname{Perv}(\overline{Q},\overline{\mathcal{Q}})}(L \boxtimes M, L' \boxtimes M') \xrightarrow{\alpha} {}^Y\mathrm{Ext}^k_{\mathcal{E}}(L \boxtimes M, L' \boxtimes M')$$
$$\xrightarrow{\beta} {}^Y\mathrm{Ext}^k_{\operatorname{Perv}(X,\mathcal{X})\boxtimes\operatorname{Perv}(\overline{Q},\overline{\mathcal{Q}})}(L \boxtimes M, L' \boxtimes M')$$

equals to

$$\bigoplus_{i+j=k} {}^Y\mathrm{Ext}^i_{\mathcal{C}}(L, L') \otimes_{\mathbb{C}} {}^Y\mathrm{Ext}^j_{\operatorname{Perv}(\overline{Q},\overline{\mathcal{Q}})}(M, M')$$
$$\xrightarrow{\gamma} \bigoplus_{i+j=k} {}^Y\mathrm{Ext}^i_{\operatorname{Perv}(X,\mathcal{X})}(L, L') \otimes_{\mathbb{C}} {}^Y\mathrm{Ext}^j_{\operatorname{Perv}(\overline{Q},\overline{\mathcal{Q}})}(M, M').$$

Mappings $\beta$, $\gamma$ are bijective for $k = 0, 1$ and injective for $k = 2$. Hence, $\alpha$ has the same property. By [10, Lemma 5.4] we deduce that the embedding $\mathcal{C}\boxtimes^D \operatorname{Perv}(\overline{Q}, \overline{\mathcal{Q}}, mp) \hookrightarrow \mathcal{E}$ is an equivalence. The lemma is proven. □

Symmetrical result also holds:

$$K(P \times T_Y) \in \operatorname{Perv}_c(\overline{P})\boxtimes^D \mathcal{D}$$

as a corollary. So the object res $K$ lies in the inverse limit of the diagram, which represents a fibered category over $I(P) \times J(Q)$ – a subcategory of $\mathrm{Perv}_c \boxtimes^D \mathrm{Perv}_c / I(P) \times J(Q)$.

$$\begin{array}{ccccc}
\mathcal{C} \boxtimes \mathcal{D} & \xrightarrow{1\boxtimes q_1^*} & \mathcal{C} \boxtimes \mathrm{Perv}_c(Q) & \xleftarrow{1\boxtimes q_2^*} & \mathcal{C} \boxtimes \mathrm{Perv}_c(\overline{Q}) \\
\downarrow{\scriptstyle p_1^*\boxtimes 1} & & \downarrow{\scriptstyle p_1^*\boxtimes 1} & & \downarrow{\scriptstyle p_1^*\boxtimes 1} \\
\mathrm{Perv}_c(P) \boxtimes \mathcal{D} & \xrightarrow{1\boxtimes q_1^*} & \mathrm{Perv}_c(P) \boxtimes \mathrm{Perv}_c(Q) & \xleftarrow{1\boxtimes q_2^*} & \mathrm{Perv}_c(P) \boxtimes \mathrm{Perv}_c(\overline{Q}) \\
\uparrow{\scriptstyle p_2^*\boxtimes 1} & & \uparrow{\scriptstyle p_2^*\boxtimes 1} & & \uparrow{\scriptstyle p_2^*\boxtimes 1} \\
\mathrm{Perv}_c(\overline{P}) \boxtimes \mathcal{D} & \xrightarrow{1\boxtimes q_1^*} & \mathrm{Perv}_c(\overline{P}) \boxtimes \mathrm{Perv}_c(Q) & \xleftarrow{1\boxtimes q_2^*} & \mathrm{Perv}_c(\overline{P}) \boxtimes \mathrm{Perv}_c(\overline{Q})
\end{array} \tag{5.7}$$

This fibered category is a tensor product of a fibered subcategory of $\mathrm{Perv}_c / I(P)$

$$\mathcal{A}/I(P) : \mathcal{C} \xrightarrow{p_1^*} \mathrm{Perv}_c(P) \xleftarrow{p_2^*} \mathrm{Perv}_c(\overline{P})$$

and of a fibered subcategory of $\mathrm{Perv}_c / J(Q)$

$$\mathcal{B}/J(Q) : \mathcal{D} \xrightarrow{q_1^*} \mathrm{Perv}_c(Q) \xleftarrow{q_2^*} \mathrm{Perv}_c(\overline{Q}).$$

Since the bottom $\tilde{\boxtimes}$ in diagram (5.1) is an equivalence, the object res $K$ comes from an object

$$K' \in \Big( \varprojlim_{R\in I(P)} \mathcal{A}(R) \Big) \boxtimes^D \Big( \varprojlim_{S\in J(Q)} \mathcal{B}(S) \Big) \subset$$
$$\subset \Big( \varprojlim_{R\in I(P)} \mathrm{Perv}_c(\overline{R}) \Big) \boxtimes^D \Big( \varprojlim_{S\in J(Q)} \mathrm{Perv}_c(\overline{S}) \Big).$$

Hence, there is an exact sequence in $\big(\varprojlim_{R\in I(P)} \mathcal{A}(R)\big) \boxtimes^D \big(\varprojlim_{S\in J(Q)} \mathcal{B}(S)\big)$

$$V \boxtimes W \xrightarrow{\sum_k t_k \boxtimes s_k} L \boxtimes M \to K' \to 0.$$

As $\mathcal{A}(T_X) = \mathcal{C} \subset D^{N'+d_G}(X)$ (resp. $\mathcal{B}(T_Y) = \mathcal{D} \subset D^{N''+d_H}(Y)$) the objects $V$, $L$ and morphisms $t_k$ (resp. $W$, $M$, $s_k$) define objects $\tilde{V}$, $\tilde{L}$

and morphisms $\tilde{t_k}$ of $\mathrm{Perv}_G^{N'}(X)$ (resp. $\tilde{W}$, $\tilde{M}$, $\tilde{s_k}$ of $\mathrm{Perv}_H^{N''}(Y)$) via Lemma 5.1. So we have the cokernel $\tilde{K} \in \mathrm{Perv}_G(X) \boxtimes \mathrm{Perv}_H(Y)$

$$\tilde{V} \boxtimes \tilde{W} \xrightarrow{\sum \tilde{t_k} \boxtimes \tilde{s_k}} \tilde{L} \boxtimes \tilde{M} \to \tilde{K} \to 0$$

such that $(\mathrm{res} \boxtimes \mathrm{res})\tilde{K} \equiv K'$. Since $\tilde{V}(T_X), \tilde{L}(T_X) \in \mathcal{C}$ and $\tilde{W}(T_Y)$, $\tilde{M}(T_Y) \in \mathcal{D}$, we have $\tilde{\boxtimes}(\tilde{V} \boxtimes \tilde{W})(T_X \times T_Y), \tilde{\boxtimes}(\tilde{L} \boxtimes \tilde{M})(T_X \times T_Y) \in D^{N+d_G+d_H}(X \times Y)$. Therefore, $(\tilde{\boxtimes}\tilde{K})(T_X \times T_Y) \in D^{N+d_G+d_H}(X \times Y)$. Notice that $\mathrm{res}(\tilde{\boxtimes}\tilde{K}) \simeq \tilde{\boxtimes}(\mathrm{res} \boxtimes \mathrm{res})\tilde{K} \simeq \mathrm{res}\, K$ and $K(T_X \times T_Y) \in D^{N+d_G+d_H}(X \times Y)$ as well. Lemma 5.1 implies that there is no more than one object of $\mathrm{Perv}_{G\times H}^N(X_c \times Y_c)$ with given restriction to the diagonal of diagram (5.7). Thus $\tilde{\boxtimes}\tilde{K} \simeq K$. This implies that

$$\tilde{\boxtimes} : \varprojlim_{R\in I} \mathrm{Perv}_c(\overline{R}) \boxtimes^D \varprojlim_{S\in J} \mathrm{Perv}_c(\overline{S}) \longrightarrow \varprojlim_{(R,S)\in I\times J} \mathrm{Perv}(\overline{R}_c \times \overline{S}_c)$$

is an equivalence. It means that

$$\boxtimes : \mathrm{Perv}_G(X) \times \mathrm{Perv}_H(Y) \to \mathrm{Perv}_{G\times H}(X_c \times Y_c)$$

makes the target category into tensor product of source categories, and $\boxtimes$ is a concrete realisation of $\boxtimes^D$.

## A. Some properties of constructible sheaves

Let $X_1 \subset X$, $Q_1 \subset Q$ be closed submanifolds of complex manifolds. Denote the open complements $X_0 = X - X_1$, $Q_0 = Q - Q_1$. We get stratifications $\mathcal{X} = \{X_0, X_1\}$, $\mathcal{Q} = \{Q_0, Q_1\}$.

**A.1. Lemma.** *If a sheaf $S$ on $X \times Q$ is $\mathcal{X} \times \{Q\}$-constructible and $\{X\} \times \mathcal{Q}$-constructible, then it is locally constant on $X \times Q$.*

*Proof.* The statement is local, so we may assume that $X_1 = \mathbb{C}^a$ (resp. $Q_1 = \mathbb{C}^c$) is a linear subspace of $X = \mathbb{C}^b$ (resp. of $Q = \mathbb{C}^d$). The restrictions $S\big|_{X_0\times Q}$ and $S\big|_{X\times Q_0}$ are locally constant. Hence, $S\big|_W$ is locally constant for a connected open set $W = X_0 \times Q \cup X \times Q_0 = X \times Q - X_1 \times Q_1$. Since $\pi_1(W)$ is trivial, $S\big|_W$ is a constant sheaf. The restrictions $S\big|_{X_1\times Q}$, $S\big|_{X_1\times Q_1}$ are constant sheaves as well. Espaces étalés for constant sheaves on $W$ and $X_1 \times Q_1$ are disjoint unions of copies of $W$ and $X_1 \times Q_1$. For any point $(x_1, q_1) \in X_1 \times Q_1$ denote by $B$ its open ball neighbourhood in $X \times Q$. The open set $B \cap W$ is non-empty and connected. Therefore, there is only one way to glue together the above mentioned espaces étalés into espace étalé for $S$, so that $S|_{X_1 \times Q}$ is constant. □

**A.2. Lemma.** *Let $X$, $Q$ be complex algebraic varieties. Let stratification $\mathcal{S}$ of $X$ be finer than $\mathcal{X}$, and let stratification $\mathcal{T}$ of $Q$ be finer than $\mathcal{Q}$. Then*

$$\mathcal{S}h_{\mathcal{S}\times\mathcal{Q}}(X \times Q) \cap \mathcal{S}h_{\mathcal{X}\times\mathcal{T}}(X \times Q) = \mathcal{S}h_{\mathcal{X}\times\mathcal{Q}}(X \times Q).$$

*Proof.* The result follows from a particular case: $X$, $Q$ are connected complex manifolds, and $\mathcal{X} = \{X\}$, $\mathcal{Q} = \{Q\}$ are trivial stratifications. We have to prove that a sheaf $S$ on $X \times Q$, which is $\mathcal{S} \times \mathcal{Q}$-constructible and $\mathcal{X} \times \mathcal{T}$-constructible is locally constant.

Denote $X_0$, $Q_0$ the open stratum. Denote $X_1$, $Q_1$ the union of strata from $\mathcal{S}$, $\mathcal{T}$ in $X - X_0$, $Q - Q_0$ of maximal dimension; denote $X_2$, $Q_2$ the union of strata contained in $X - X_0 - X_1$, $Q - Q_0 - Q_1$ of maximal dimension, etc. Apply Lemma A.1 to submanifolds $X' = X_0 \cup X_1 \subset X$, $Q' = Q_0 \cup Q_1 \subset Q$ with stratifications $\mathcal{X}' = \{X_0, X_1\}$, $\mathcal{Q}' = \{Q_0, Q_1\}$. We deduce that $S|_{X' \times Q'}$ is locally constant. Then we apply Lemma A.1 to $(X_0, X_1; Q_0 \cup Q_1, Q_2)$ and get that $S|_{X' \times (Q_0 \cup Q_1 \cup Q_2)}$ is locally constant. Continuing we deduce that $S|_{X' \times Q}$ is locally constant. Thus the original stratification of $X$ might be replaced with $(X_0 \cup X_1, X_2, X_3, \dots)$. Continuing to simplify the stratification of $X$ we make it trivial. And the lemma follows. □

**A.3. Remark.** The above lemmas hold as well for topological pseudomanifolds $X$, $Q$ such that $X_1$, $Q_1$ has real codimension at least 2.

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