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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

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ALMOST ABELIAN CATEGORIES

By Wolfgang RUMP

Dedicated to K. W. Roggenkamp on the occasion of his 60th birthday

RESUME. Nous introduisons et étudions une classe de catégories additives avec des noyaux et conoyaux, catégories qui sont plus générales que les catégories abéliennes, et pour cette raison nous les appelons presque abéliennes. L'un des objectifs de ce travail est de montrer que cette notion unifie et généralise des structures associées aux catégories abéliennes: des théories de torsion (§4), des foncteurs adjoints et des bimodules (§6), la dualité de Morita et la théorie de "tilting" (§7). D'autre part, nous nous proposons de montrer qu'il y a beaucoup de catégories presque abéliennes: en algèbre topologique (§2.2), en analyse fonctionnelle (§2.3-4), dans la théorie des modules filtrés (§2.5), et dans la théorie des représentations des ordres sur les anneaux de Cohen-Macaulay de dimension inférieure ou égale à 2 (§2.1 et §2.9).

Introduction

Let $\mathcal{A}$ be an *additive category*, i. e. a category with morphism sets in the meta-category $\mathcal{A}b$ of large abelian groups and bilinear composition, and with finite biproducts [26]. $\mathcal{A}$ is said to be *preabelian* if every morphism $f: A \to B$ in $\mathcal{A}$ has a kernel and a cokernel. Then f admits a decomposition

$$f : A \xrightarrow{c} \mathrm{Coim}\, f \xrightarrow{\dot{f}} \mathrm{Im}\, f \xrightarrow{d} B \tag{1}$$

with $c = \mathrm{coim}\, f := \mathrm{cok}(\ker f)$ and $d = \mathrm{im}\, f := \ker(\mathrm{cok}\, f)$. (Arrows "$\rightarrowtail$" always refer to kernels, whereas by "$\twoheadrightarrow$" we indicate a cokernel.) For many concrete categories, $\dot{f}$ is found to be *regular*, i. e. monic and epic. If in addition, cokernels are stable under pullback, we call $\mathcal{A}$ *almost abelian.* (By Proposition 3 this concept is self-dual.) These categories will be the main object of this article.

1. Structure theory. The central part in our structure theory for almost abelian categories $\mathcal{A}$ will concern their behaviour with respect to projective and injective objects in $\mathcal{A}$. Auslander [3] defined a *variety (of annuli)* as an additive category $\mathcal{V}$ with splitting idempotents. If in addition, $\mathcal{V}$ has weak kernels (§3), a property which is needed in order to have projective resolutions, we call $\mathcal{V}$ a *projective* variety. Any abelian category $\mathcal{A}$ with enough projectives is determined by the projective variety $\mathbf{Proj}(\mathcal{A})$ of its projective objects, and each projective variety arises in this way ([12], Theorem 1.4). Dually, an abelian category $\mathcal{A}$ with enough injectives is determined by the *injective* variety $\mathbf{Inj}(\mathcal{A})$ of its injectives.

Obviously, both descriptions cannot remain valid for almost abelian categories $\mathcal{A}$. In this case, we shall describe $\mathcal{A}$ by a variety $\mathcal{V}$ consisting of projectives *and* injectives. Precisely, we shall define a *pre-PI-variety* as a variety $\mathcal{V}$ with a splitting torsion theory $(\mathcal{I}, \mathcal{P})$ such that an appropriate version of weak kernels and cokernels exists in $\mathcal{V}$. Then $\mathcal{I}$ is an injective, and $\mathcal{P}$ a projective variety, and $\mathcal{V} = \mathcal{P} \oplus \mathcal{I}$, i. e. every object of $\mathcal{V}$ is of the form $P \oplus I$ with $P \in \mathcal{P}$ and $I \in \mathcal{I}$. Furthermore, we shall prove that such a variety $\mathcal{V}$ is equivalently given by a $(\mathcal{P}, \mathcal{I})$*-bimodule*, a concept which generalizes bimodules ${}_RU_S$ over rings R, S. In fact, an (R, S)-bimodule U is just a $(\mathcal{P}, \mathcal{I})$-bimodule with $\mathcal{P} = \mathbf{Proj}(R\text{-}\mathbf{Mod})$ and $\mathcal{I}^{\mathrm{op}} = \mathbf{Proj}(\mathbf{Mod}\text{-}S)$. The property of ${}_RU_S$ to be faithful over R and S generalizes to what we call a *non-degenerate* $(\mathcal{P}, \mathcal{I})$-bimodule, and the corresponding varieties $\mathcal{V}$ will be called *PI-varieties.*

The almost abelian analogue of an abelian category with enough projectives and injectives will be called a *PI-category.* An object A of such a category $\mathcal{A}$ is determined (up to isomorphism) by a morphism $P \xrightarrow{p} A \xrightarrow{i} I$, where P, I belong to classes $\mathcal{P} \subset \mathbf{Proj}(\mathcal{A})$ and $\mathcal{I} \subset \mathbf{Inj}(\mathcal{A})$, respectively, which constitute a PI-variety $\mathcal{V}(\mathcal{A}) = \mathcal{P} \oplus \mathcal{I}$. Moreover, p is a $\mathcal{P}$*-epimorphism* (= $\mathcal{P}$*-fibration* [12]), i. e. each $P' \to A$ with $P' \in \mathcal{P}$ factors through p. Dually, i is an $\mathcal{I}$*-monomorphism* (§5).

Every PI-category $\mathcal{A}$ has the property that up to isomorphism, a morphism f has at most one decomposition $f = ip$ with a $\mathcal{P}$-epimorphism p, and an $\mathcal{I}$-monomorphism i. If such a decomposition always exists, we shall speak of an *ample* PI-category. Each PI-category $\mathcal{A}$ has a natural exact full embedding $\mathcal{A} \hookrightarrow \widetilde{\mathcal{A}}$ into an ample PI-category, and the ample PI-categories are in one-to-one correspondence with PI-varieties (Theorem 4). On the other hand, there is a unique minimal PI-category for each PI-variety, and thus any PI-category $\mathcal{A}$ is located between the maximal (i. e. ample) PI-category $\widetilde{\mathcal{A}}$, and the minimal PI-category $\underset{\sim}{\mathcal{A}}$ associated with $\mathcal{V}(\mathcal{A})$.

It will turn out that the two most interesting classes of almost abelian categories $\mathcal{A}$ have an intrinsic PI-structure. Firstly, if for each object A in $\mathcal{A}$ there is a cokernel $P \twoheadrightarrow A$ with P projective, and a kernel $A \rightarrowtail I$ with I injective, then $\mathcal{A}$ is a minimal PI-category with $\mathcal{P} = \mathbf{Proj}(\mathcal{A})$ and $\mathcal{I} = \mathbf{Inj}(\mathcal{A})$. Therefore, $\mathcal{A}$ is completely determined by a $(\mathcal{P}, \mathcal{I})$-bimodule, and we shall speak of a *strict* PI-category. Such $(\mathcal{P}, \mathcal{I})$-bimodules are in fact generalizations of classical cotilting modules. As a consequence, we get a cotilting theorem (cf. [7], Theorem 2.4) for finitely generated modules over (left resp. right) coherent rings (§7). Moreover, the cotilting modules are in one-to-one correspondence with certain PI-categories. For example, the category of Λ-lattices with Λ an order over a Dedekind domain, and the category of representations of a finite poset, are determined by a cotilting module (§7, Example 1 and 4).

The second class belongs to PI-varieties $\mathcal{V}$ whose $(\mathcal{P}, \mathcal{I})$-bimodule is a two-sided "injective cogenerator" (§6), generalizing a familiar concept from Morita duality (see [1], Theorem 24.1). Therefore, these PI-varieties $\mathcal{V}$ will be called *Morita varieties*. To any almost abelian category $\mathcal{A}$ we shall associate (§3) a pair of abelian full subcategories $\mathcal{A}_\circ$ and $\mathcal{A}^\circ$. If $\mathcal{A}$ is a PI-category, then Theorem 5 states that the underlying PI-variety $\mathcal{V}(\mathcal{A}) = \mathcal{P} \oplus \mathcal{I}$ is a Morita variety if and only if

$$\mathcal{P} = \mathbf{Proj}(\mathcal{A}_\circ); \quad \mathcal{I} = \mathbf{Inj}(\mathcal{A}^\circ).$$

Hence the PI-structure is uniquely determined by the almost abelian category $\mathcal{A}$, and we shall speak of a *Morita category*. Numerous concrete categories are of this type, for instance, the category $\mathcal{L}$ of locally compact abelian groups (§8). Here the Morita variety corresponds to the $(\mathbb{Z}, \mathbb{Z})$-bimodule T of reals modulo 1, and the minimal PI-category $\underset{\sim}{\mathcal{L}}$ consists of the objects G of $\mathcal{L}$ which do not contain the additive group R of reals as a direct summand. Note that R is the unique projective and injective indecomposable object in $\mathcal{L}$. From any given group G in $\mathcal{L}$, it can be split off by a generalization of the Krull-Schmidt theorem (Lemma 8). As a byproduct, we get a simple proof of a well-known uniqueness result in $\mathcal{L}$ (Proposition 33) proved in [5].

2. Connections with abelian categories. We already mentioned that every almost abelian category $\mathcal{A}$ has a pair of abelian full subcategories $\mathcal{A}_\circ$ and $\mathcal{A}^\circ$ which play a decisive rôle when $\mathcal{A}$ is a Morita category. In particular, each bimodule ${}_RU_S$ defining a Morita duality between R and S gives rise to a PI-category $\mathcal{A}$ with $\mathcal{A}_\circ = R$-**Mod** and $\mathcal{A}^\circ = (\textbf{Mod-}S)^{\text{op}}$. We shall see that $\mathcal{A}_\circ \cap \mathcal{A}^\circ$ coincides with the class of reflexive modules in R-**Mod** and **Mod**-S, respectively. This remains true even if we drop the assumption that ${}_RU_S$ is balanced (e.g., if $\mathcal{A} := \mathcal{L}$ and $U := {}_{\mathbf{Z}}\mathbb{T}_{\mathbf{Z}}$). Furthermore, we give a necessary and sufficient criterion for such dualities $\text{Hom}(-, U)$ which extend to a self-duality of $\widetilde{\mathcal{A}}$ (Proposition 34). This generalizes Pontrjagin's duality theorem for $\mathcal{L}$.

We also mentioned that a PI-variety is equivalent to a non-degenerate $(\mathcal{P}, \mathcal{I})$-bimodule. If $\mathcal{D}$ denotes the abelian category corresponding to the projective variety $\mathcal{P}$, and $\mathcal{C}$ the abelian category with $\textbf{Inj}(\mathcal{C}) \approx \mathcal{I}$, we shall prove that a $(\mathcal{P}, \mathcal{I})$-bimodule is tantamount to a pair of adjoint functors

$$\mathcal{D} \underset{F}{\overset{E}{\rightleftarrows}} \mathcal{C}$$

with E left adjoint to F (Proposition 23). Hence an adjoint pair $E \dashv F$ between abelian categories, being formally independent of $\mathcal{P}$ and $\mathcal{I}$, can be regarded as a "bimodule without varieties $\mathcal{P}, \mathcal{I}$". In fact, it is possible to define the corresponding ample PI-category in terms of E and F, which leads to a theory of Morita duality between arbitrary abelian categories [35].

On the other hand, to any almost abelian category $\mathcal{A}$ we shall associate a pair of exact full embeddings (§3) into abelian categories $\mathrm{Q}_l(\mathcal{A})$ and $\mathrm{Q}_r(\mathcal{A})$. Then $\mathcal{A}$ coincides with the class of reflexive objects of an adjoint functor pair

$$\mathrm{Q}_l(\mathcal{A}) \underset{F}{\overset{E}{\rightleftarrows}} \mathrm{Q}_r(\mathcal{A})$$

such that the unit $\eta : 1 \to FE$ is epic, the counit $\varepsilon : EF \to 1$ is monic, $E(\text{Ker}\,\eta) = 0$, and $F(\text{Cok}\,\varepsilon) = 0$. Such adjoint pairs $E \dashv F$ (between arbitrary abelian categories) will be called *almost equivalences.* In this way, each almost abelian category $\mathcal{A}$ gives rise to an almost equivalence, and vice versa (Theorem 3). Moreover, $\mathcal{A}$ is a strict PI-category if and only if $Q_l(\mathcal{A})$ has enough projectives which are all in $\mathcal{A}$, and $Q_r(\mathcal{A})$ has enough injectives, lying all in $\mathcal{A}$ (Proposition 11, and Corollary 1 of Proposition 30). Hence, if the abelian categories $Q_l(\mathcal{A})$ and $Q_r(\mathcal{A})$ are of the form R-**mod** and $(\mathbf{mod}\text{-}S)^{\mathrm{op}}$ with left (right) coherent rings R and S, respectively, then the "bimodule" $E \dashv F$ specializes to an (R, S)-cotilting module. In this sense, the theory of almost abelian categories can be regarded as a generalized cotilting theory.

Accordingly, our general cotilting theorem (Theorem 2) says that for any almost abelian category $\mathcal{A}$, there are natural torsion theories in $Q_l(\mathcal{A})$ and $Q_r(\mathcal{A})$ such that $\mathcal{A}$ is equivalent to the torsionfree class in $Q_l(\mathcal{A})$ and the torsion class in $Q_r(\mathcal{A})$, whereas the torsion class in $Q_l(\mathcal{A})$ is equivalent to the torsionfree class in $Q_r(\mathcal{A})$. Conversely, the torsion class, and the torsionfree class of any torsion theory in an abelian category is almost abelian.

In the study of spectral analysis, L. Waelbroeck [39] established a "calcul symbolique relatif" which was simplified by embedding the category $\mathbf{B}$ of Banach spaces into an abelian category q [40]. A different construction of q has been given by G. Noël [30]. By §3 it is easy to show that q coincides with $Q_l(\mathbf{B})$.

3. Homological algebra. A preabelian category is almost abelian if and only if short exact sequences are stable in the sense of Richman and Walker [34]. Therefore, homological algebra and K-theory naturally apply to almost abelian categories (see [16, 34, 19] and [31]).

1 Almost abelian categories

Throughout this article, all functors are assumed to be additive. Let $\mathcal{A}$ be a preabelian category. A morphism (1) will be called *strict* if $\dot{f}$ is an isomorphism. Hence $\mathcal{A}$ is abelian if and only if all its morphisms are strict. More generally, a preabelian category $\mathcal{A}$ will be called *left (right) semi-abelian* if each morphism f in $\mathcal{A}$ admits a decomposition $f = ip$ with a cokernel (epimorphism) p and a monomorphism (kernel) i. Then $p = \operatorname{coim} f$ (resp. $i = \operatorname{im} f$). Thus $\mathcal{A}$ is left and right semi-abelian if and only if for every morphism (1) in $\mathcal{A}$, the associated morphism $\dot{f}$ is *regular*, i. e. monic and epic. In this case, $\mathcal{A}$ will be called *semi-abelian.*

Now let us consider a commutative square together with the kernels and cokernels of its horizontal morphisms

$$\begin{array}{ccccccc} \operatorname{Ker} a & \overset{k}{\rightarrowtail} & A & \overset{a}{\longrightarrow} & B & \overset{e}{\twoheadrightarrow} & \operatorname{Cok} a \\ \downarrow g & & \downarrow b & & \downarrow c & & \downarrow h \\ \operatorname{Ker} d & \overset{l}{\rightarrowtail} & C & \overset{d}{\longrightarrow} & D & \overset{f}{\twoheadrightarrow} & \operatorname{Cok} d \end{array} \qquad (2)$$

If $\mathcal{A}$ is abelian, then pullbacks and pushouts (2) can be characterized in terms of the induced maps g and h. If $\mathcal{A}$ is additive, we still have (cf. [26], VIII.4, proof of Proposition 2):

Lemma 1 *If (2) is a pullback (pushout) in a preabelian category, then g (resp. h) is an isomorphism.*

Lemma 2 *Let (2) be given in a left (right) semi-abelian category $\mathcal{A}$. If a and d are kernels (cokernels), then (2) is a pullback (pushout) if and only if h is monic (resp. g is epic).*

Proof. Suppose a and d are kernels. If (2) is a pullback, then $a = \ker(he)$. In fact, if $a' : A' \to B$ is any morphism with $hea' = 0$, then $fca' = 0$ implies that ca' factors through d, whence a' factors through a. Therefore, $e = \operatorname{coim}(he)$, and h is monic since $\mathcal{A}$ is left semi-abelian. Conversely, let h be monic. Then $a = \ker(he) = \ker(fc)$, and the pullback property immediately follows. The remaining assertion is dual. □

Let us call a preabelian category $\mathcal{A}$ *left almost abelian* if for each pullback (2) in $\mathcal{A}$ where d is a cokernel, a is again a cokernel (i. e. cokernels are *stable* under pullback). Dually, $\mathcal{A}$ is called *right almost abelian* if kernels are stable under pushout. Similarly, we define a *left* (*right*) *integral* category $\mathcal{A}$ as a preabelian category such that epi- (mono-)morphisms are stable under pullback (pushout). A left and right almost abelian (integral) category will be called *almost abelian* (*integral*). By [26],VIII.4, Proposition 2, every abelian category is almost abelian and integral.

The following proposition, together with its dual, shows how semi-abelian categories are related to almost abelian and integral categories:

Proposition 1 *For $\mathcal{A}$ preabelian the following are equivalent:*

(a) *$\mathcal{A}$ is left semi-abelian.*

(b) *Any pullback (2) in $\mathcal{A}$ with a cokernel d is a pushout.*

(c) *If d is a cokernel in a pullback (2), then a is epic.*

Proof. (a) $\Rightarrow$ (b): Suppose (2) is a pullback, and d is a cokernel. Consider the pushout

$$\begin{array}{ccccccc} \operatorname{Ker} a & \xrightarrow{e} & A & \xrightarrow{a} & B & & \\ & & \downarrow{\scriptstyle b} & \text{PO} & \downarrow{\scriptstyle p} & & \\ & & C & \xrightarrow{q} & E & \xrightarrow{r} & D \end{array} \tag{3}$$

with $e = \ker a$ and the unique morphism r satisfying $rp = c$ and $rq = d$. Since $\binom{a}{-b} : A \to B \oplus C$ is the kernel of $(c\ d) : B \oplus C \to D$, we have $(p\ q) = \operatorname{coim}(c\ d)$, and r is a monomorphism since $\mathcal{A}$ is left semi-abelian. By Lemma 1, $be = \ker d$, hence $d = \operatorname{cok}(be)$. Therefore, $qbe = 0$ implies $q = sd$ for some $s : D \to E$, which yields $rsd = d$, that is, $rs = 1$. Since r is monic, we also obtain $sr = 1$, whence (2) is a pushout.

(b) $\Rightarrow$ (c): This follows immediately by Lemma 1.

(c) $\Rightarrow$ (a): For a morphism (1) in $\mathcal{A}$, consider the pullback

$$\begin{array}{ccc} A & \overset{c}{\twoheadrightarrow} & \operatorname{Coim} f \\ \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle e} \\ C & \xrightarrow{g} & \operatorname{Ker} \dot{f} \end{array}$$

with $e = \ker \dot{f}$. Then g is epic, and $fh = d\dot{f}ch = d\dot{f}eg = 0$ implies $eg = ch = 0$, whence $e = 0$. Thus $\dot{f}$ is monic, and the proof is complete. $\square$

Corollary 1. *Every left (right) almost abelian or left (right) integral category is left (right) semi-abelian.*

Let us define a sequence of morphisms

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C \tag{4}$$

in an additive category $\mathcal{A}$ to be *(short) exact* if $f = \ker g$ and $g = \operatorname{cok} f$. A functor F which preserves exact sequences will be called *exact*. If F preserves finite limits and colimits it will be called *fully exact*. In case $F : \mathcal{A} \hookrightarrow \mathcal{B}$ is a faithful embedding, we also speak of a (*fully*) *exact subcategory* $\mathcal{A}$. An object P in $\mathcal{A}$ will be called *projective* if for each cokernel $B \twoheadrightarrow C$ in $\mathcal{A}$, the induced map $\operatorname{Hom}_{\mathcal{A}}(P, B) \to \operatorname{Hom}_{\mathcal{A}}(P, C)$ is surjective. *Injective* objects are defined dually, and the corresponding full subcategories will be denoted by $\mathbf{Proj}(\mathcal{A})$ (resp. $\mathbf{Inj}(\mathcal{A})$). We shall say that $\mathcal{A}$ has *enough projectives (injectives)* if for each object A in $\mathcal{A}$, there is an epimorphism $P \to A$ with P projective (resp. a monomorphism $A \to I$ with I injective).

Corollary 2. *Let $\mathcal{A}$ be a preabelian category. If $\mathcal{A}$ has enough projectives (injectives), then $\mathcal{A}$ is left (right) semi-abelian.*

Proof. Consider a pullback (2) in $\mathcal{A}$ with a cokernel d, and let $p : P \to B$ be an epimorphism with P projective. Then $cp = dq$ for some $q : P \to C$. Hence p factors through a, and thus a is an epimorphism. By duality, this proves the corollary. □

The next result is a consequence of [24], Proposition 5.10 and 5.12:

Proposition 2 *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms in a right (left) semi-abelian category. If f and g are (co-)kernels, then gf is a (co-)kernel. If gf is a (co-)kernel, then f (resp. g) is a (co-)kernel.*

Proposition 3 *A semi-abelian category $\mathcal{A}$ is left almost abelian if and only if it is right almost abelian.*

Proof. Suppose $\mathcal{A}$ is left almost abelian, and let (2) be a pushout in $\mathcal{A}$ with a kernel a. Consider the pullback

$$\begin{array}{ccc} B \oplus C & \xrightarrow{(c\ d)} & D \\ \uparrow {\scriptstyle \binom{u}{v}} & & \uparrow {\scriptstyle i} \\ E & \xrightarrow{w} & \operatorname{Im} d \end{array}$$

with $i = \operatorname{im} d$. Then w is a cokernel, and the monomorphism $d : C \to D$ has a decomposition $d = ir$ with r regular. Application of $f = \operatorname{cok} d$

yields $0 = fiw = f(cu + dv) = fcu$. By Lemma 1, $fc = \operatorname{cok} a$. Hence, u factors through a, say, $u = au'$. Consequently, $iw = cu + dv = cau' + dv = d(bu' + v) = ir(bu' + v)$, and thus $w = r(bu' + v)$. By Proposition 2 it follows that r is a cokernel, whence r is an isomorphism. Thus d is a kernel, and we have shown that $\mathcal{A}$ is right almost abelian. The converse follows by duality. □

Remark 1. The preceding proof shows that a semi-abelian category is almost abelian if and only if for any pullback (2) the implication

$$d \text{ cokernel}, \ c \text{ kernel} \ \Rightarrow \ a \text{ cokernel}$$

holds. This property is equivalent to axiom (A6) in Raĭkov's paper [32].

The next result is due to Kelly ([24], Proposition 5.2):

Proposition 4 *Let $\mathcal{A}$ be a preabelian category. Then cokernels are stable under pushout, and kernels are stable under pullback.*

Lemma 3 *Let $\mathcal{A}$ be a left or right almost abelian category. If a commutative diagram*

$$\begin{array}{ccccc} A & \overset{a}{\rightarrowtail} & B & \overset{b}{\twoheadrightarrow} & D \\ \| & & \downarrow{\scriptstyle r} & & \| \\ A & \underset{c}{\rightarrowtail} & C & \underset{d}{\twoheadrightarrow} & D \end{array} \qquad (5)$$

has exact rows, then r is an isomorphism.

Proof. Let $f : E \to B$ be a morphism with $rf = 0$. Then $bf = 0$ implies that f factors through a, say, $f = ag$. Hence, $cg = rag = rf = 0$ yields $g = 0$. Thus r is a monomorphism, and by duality, r is regular. Furthermore, it is easily verified that the commutative diagram

$$\begin{array}{ccccc} A \oplus B & \xrightarrow{(a\ 1)} & B & \xrightarrow{r} & C \\ \downarrow{\scriptstyle (0\ 1)} & & & & \downarrow{\scriptstyle d} \\ B & \xrightarrow{r} & C & \overset{d}{\twoheadrightarrow} & D \end{array}$$

is a pullback. Now suppose $\mathcal{A}$ is left almost abelian. Then $r(a\ 1)$ is a cokernel since $dr = b$ is a cokernel. Hence r is an isomorphism. □

Remark 2. The statement of Lemma 3 coincides with axiom (A4) of Raĭkov [32] which is therefore redundant (cf. Remark 1). Consequently, every almost abelian category is "semiabelian" in the sense of Raĭkov, and vice versa.

In the sequel, a simultaneous pullback and pushout will be called an *exact* square.

Proposition 5 *Let $\mathcal{A}$ be a left or right almost abelian category with a commutative diagram (2). If a and d is a cokernel, and g an isomorphism, then (2) is exact.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} & & K & = & K & & \\ & & \downarrow k & \text{IV} & \downarrow rk & & \\ K & \xrightarrow{k} & A & \xrightarrow{r} & E & \xrightarrow{q} & C \\ \downarrow & \text{I} & \downarrow a & \text{II} & \downarrow p & \text{III} & \downarrow d \\ 0 & \longrightarrow & B & = & B & \xrightarrow{c} & D \end{array}$$

with $k := \ker a$, where the square (2) is divided into II and III such that III is a pullback, and r is the unique morphism with $pr = a$ and $qr = b$. Then $qrk = \ker d$ by the invertibility of g, and thus I and I+II+III are exact. Hence I+II is a pullback, i. e. $rk = \ker p$. Moreover, we infer that II+III is a pushout. Thus if $\mathcal{A}$ is right almost abelian, then $\binom{p}{q} = \operatorname{im}\binom{a}{b}$ implies that r is epic. Hence II is a pushout, and therefore p is a cokernel by Proposition 4. If $\mathcal{A}$ is left almost abelian, then p is also a cokernel since III is a pullback. Thus we have shown that the columns in II+IV are exact. By Lemma 3, we conclude that r is invertible, i. e. II+III is a pullback, hence exact. □

Let us now turn our attention to integral categories $\mathcal{A}$. Firstly, we show that the defining property can be weakened if $\mathcal{A}$ is semi-abelian:

Proposition 6 *For $\mathcal{A}$ semi-abelian the following are equivalent:*

(a) *$\mathcal{A}$ is integral.*

(b) *Regular morphisms are stable under pullbacks.*

(c) *Regular morphisms are stable under pushouts.*

Proof. If $\mathcal{A}$ is right integral, then (c) follows by Lemma 1. Hence by symmetry, it remains to prove that (c) implies that $\mathcal{A}$ is left integral. Thus assume (c), and let (2) be a pullback with an epimorphism d. Then $(c\, d) : B \oplus C \to D$ is an epimorphism with a decomposition

$$(c\, d) : B \oplus C \overset{e}{\twoheadrightarrow} E \xrightarrow{r} D,$$

where $e = \mathrm{coim}(c\, d)$, and r is regular. Let $f : B \twoheadrightarrow F$ be the cokernal of a. We shall complete the diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{\binom{a}{-b}} & B \oplus C & \overset{e}{\twoheadrightarrow} & E & \xrightarrow{r} & D \\
\| & & \downarrow{\scriptstyle (1\,0)} & & \downarrow{\scriptstyle g} & \mathrm{PO} & \downarrow{\scriptstyle h} \\
A & \xrightarrow{a} & B & \overset{f}{\twoheadrightarrow} & F & \overset{s}{\dashrightarrow} & H
\end{array}$$

by the induced morphism g and the pushout of r and g. By assumption, s is regular. Therefore, $h(c\, d) = hre = sf(1\ 0)$ implies $hd = 0$, whence $h = 0$. This implies $sf = 0$, hence $f = 0$, i. e. a is an epimorphism. □

Corollary. *A semi-abelian category is left integral if and only if it is right integral.*

By Proposition 6, a semi-abelian category $\mathcal{A}$ is integral if and only if the regular morphisms in $\mathcal{A}$ admit a calculus of left and right fractions ([15], I.2). Therefore, if $\mathrm{Q}(\mathcal{A})$ denotes the category with the same objects as $\mathcal{A}$ and formal fractions $fr^{-1} = s^{-1}g$ as morphisms, where f, g, r, s are morphisms in $\mathcal{A}$ with r, s regular and $sf = gr$, then $\mathrm{Q}(\mathcal{A})$ is an abelian category. We call it the *quotient category* of $\mathcal{A}$. Thus for an integral category $\mathcal{A}$, we have a faithful, fully exact embedding ([15], I.3):

$$\mathcal{A} \hookrightarrow \mathrm{Q}(\mathcal{A}) \tag{6}$$

This leads to another characterization of integral categories (cf. [6], V):

Proposition 7 *Let $\mathcal{A}$ be a preabelian category. Then $\mathcal{A}$ is integral if and only if there is a faithful, fully exact functor $F : \mathcal{A} \to \mathcal{B}$ into an abelian category $\mathcal{B}$. In this case, F uniquely extends to a functor $F' : \mathrm{Q}(\mathcal{A}) \to \mathcal{B}$, which is again faithful and fully exact.*

Proof. Let $F : \mathcal{A} \to \mathcal{B}$ be faithful and fully exact with $\mathcal{B}$ abelian. Then F carries monomorphisms to monomorphisms. Conversely, if Ff is monic, then $F(\operatorname{Ker} f) = 0$ implies $\operatorname{Ker} f = 0$ since F is faithful. It follows that monomorphisms are stable under pushout since this property is valid in $\mathcal{B}$. By duality we infer that $\mathcal{A}$ is integral. The remaining assertions are instantly verified. □

To check whether a subcategory is almost abelian, we use:

Lemma 4 *Let $\mathcal{A}$ be a full subcategory of an almost abelian category $\mathcal{B}$. If the object class of $\mathcal{A}$ is closed with respect to biproducts, kernels, and cokernels, then $\mathcal{A}$ is almost abelian.*

Proof. Clearly, $\mathcal{A}$ is preabelian. A morphism f in $\mathcal{A}$ is a (co-)kernel in $\mathcal{A}$ if and only if f is a (co-)kernel in $\mathcal{B}$. This follows since any kernel f is a kernel of its cokernel. Hence $\mathcal{A}$ is almost abelian. □

2 Examples

Before we start with the structure theory of almost abelian categories in the next section, let us give some instances where such categories occur.

1. Lattices over orders. Let R be a noetherian integral domain with quotient field K, and Λ an *R-order* in a finite dimensional K-algebra A, that is, an R-finite R-subalgebra with $K\Lambda = A$. Then a finitely generated Λ-module E is said to be a *Λ-lattice* if E is torsion-free over R. It is easily verified that the category Λ-**lat** of Λ-lattices is almost abelian. Furthermore, Λ-**lat** is integral with $\mathrm{Q}(\Lambda\text{-}\mathbf{lat}) = A\text{-}\mathbf{mod}$, the category of finitely generated A-modules.

For a central idempotent e of A, the category Λ-**lat**$/e\Lambda$-**lat** (investigated recently by O. Iyama [22]) is almost abelian (§7, Example 3).

2. Categories of topological abelian groups. Let **TAb** be the category of topological abelian groups with continuous group homomorphisms as morphisms. Clearly, **TAb** is preabelian: the biproduct $A \oplus B$ in **TAb** is given by the topological direct sum, and the (co-)kernel of a morphism f is just the (co-)kernel in **Ab** (the category of small abelian groups) with the induced (resp. quotient) topology. Hence, f is regular in **TAb** if and only if f is bijective. Then a straightforward verification yields that **TAb** is almost abelian and integral with $\mathrm{Q}(\mathbf{TAb}) = \mathbf{Ab}$.

Now consider the full subcategory **HAb** of Hausdorff abelian groups. Here, the cokernel of a morphism $f : A \to B$ is $B \twoheadrightarrow B \,/\, \overline{\mathrm{Im}\, f}$ with the quotient topology, and f is regular if and only if f is injective with $\mathrm{Im}\, f$ dense in B. Again, **HAb** is almost abelian since the inclusion functor $\mathbf{HAb} \hookrightarrow \mathbf{TAb}$ reflects pullbacks and those pushouts (2) for which a is a kernel. However, **HAb** is no longer integral:

$$\begin{array}{ccc} \mathsf{Z} & \overset{\vartheta}{\rightarrowtail} & \mathsf{R} \\ \uparrow & \mathrm{PB} & \uparrow \\ 0 & \rightarrowtail & \mathsf{Q} \end{array}$$

Here R denotes the additive group of real numbers, Z and Q are the discrete group of integers and rationals, respectively. If $\vartheta(1)$ is irrational, then the square PB is a pullback. Note that by means of this pullback, the proof of Proposition 6 also yields a pushout (2) for which a, but not d, is regular.

In particular, the full subcategory $\mathcal{L}$ of locally compact abelian groups is almost abelian, but not integral. For this category, a precise description will be given in §8.

3. Categories of topological linear spaces. For a topological field K, the category $\mathbf{TVS}(K)$ of (Hausdorff) topological K-vector spaces is almost abelian. In particular, let K be the field $\mathbb{R}$ or $\mathbb{C}$ with the natural absolute value. By Lemma 4, various full subcategories of $\mathbf{TVS}(K)$ are almost abelian: e. g. the categories of locally convex spaces, normed linear spaces, Fréchet spaces, Banach spaces (see Example 2 of §7), or nuclear spaces (e. g. [36]).

4. Bilinear maps and dual systems. For a bimodule ${}_RU_S$ over a pair of rings R, S, let $\mathbf{Bilin}(U)$ denote the category of bilinear maps $\beta : M \times M' \to U$, with a left R-module M, and a right S-module M'. For $x \in M$ and $y \in M'$, we simply write $\langle x, y \rangle$ instead of $\beta(x, y)$, and β itself is abbreviated by $\langle M, M' \rangle$. For a second object $\langle N, N' \rangle$ in $\mathbf{Bilin}(U)$, a morphism $\langle M, M' \rangle \to \langle N, N' \rangle$ is given by a pair of adjoint homomorphisms $f : M \to N$ and $f' : N' \to M'$, that is, a pair (f, f') with

$$\langle f(x), y \rangle = \langle x, f'(y) \rangle \qquad (7)$$

for all $x \in M$ and $y \in N'$, modulo such pairs (f, f') for which the bracket (7) is zero for all x, y. With the obvious addition and composition of morphisms, **Bilin**(U) becomes an almost abelian category. In fact, it is a PI category (§5, Proposition 19).

The full subcategory **DS**(U) of bilinear maps $M \times M' \to U$ with vanishing left and right kernel is again almost abelian. In fact, this category is equivalent to the full subcategory of objects $\langle P, P' \rangle$ in **Bilin**(U) with P, P' projective: If $K \subset P$ and $K' \subset P'$ is the left and right kernel of $\langle P, P' \rangle$, respectively, then $\langle P, P' \rangle$ corresponds to the object $\langle P/K,\ P'/K' \rangle$ in **DS**(U). Again by Proposition 19, it will follow that **DS**(U) is almost abelian. For a morphism $(f, f') : \langle M, M' \rangle \to \langle N, N' \rangle$ in **DS**(U), the maps f and f' determine each other. Therefore, a morphism (f, f') can be redefined as an R-linear map $f : M \to N$ which admits an adjoint homomorphism $f' : N' \to M'$.

In particular, if $U = {}_K K_K$ with a field K which is either $\mathbb{R}$ or $\mathbb{C}$, or discrete, then an object in **DS**(K) is just a dual system ([36], IV.1), and the morphisms $\langle M, M' \rangle \to \langle N, N' \rangle$ in **DS**(K) coincide with the K-linear maps $M \to N$ which are continuous in the weak topology.

5. Torsion theories. For any torsion theory in an abelian category $\mathcal{A}$, we shall prove in §4 that the torsion class, and the torsion-free class (as full subcategories), are both almost abelian.

For a partially ordered set Ω, consider $\hat{\Omega} := \Omega \cup \{\infty\}$ with $a < \infty$ for all $a \in \Omega$. Then the functor category $\mathcal{A}^{\hat{\Omega}}$ is abelian whenever $\mathcal{A}$ is abelian. Now let $\mathcal{A}_\Omega$ denote the full subcategory of functors $F \in \mathcal{A}^{\hat{\Omega}}$ with monomorphisms $F(i) \to F(\infty)$ for all $i \in \Omega$, and ${}_\Omega\mathcal{A}$ the full subcategory of functors F with $F(\infty) = 0$. These categories can be regarded as two types of representation categories of Ω. In case $\mathcal{A}$ is a category of finite dimensional vector spaces, $\mathcal{A}_\Omega$ has been studied by Nazarova and Roĭter [29] (see also Gabriel [14]), and ${}_\Omega\mathcal{A}$ by Loupias [25]. It is easy to verify that $({}_\Omega\mathcal{A}, \mathcal{A}_\Omega)$ is a torsion theory in $\mathcal{A}^{\hat{\Omega}}$. Hence $\mathcal{A}_\Omega$ and ${}_\Omega\mathcal{A}$ are almost abelian. Moreover, $\mathcal{A}_\Omega$ is integral with $\mathrm{Q}(\mathcal{A}_\Omega) = \mathcal{A}$.

6. Almost equivalences. See §4, Theorem 3.

7. Tilting theory. Let R be a ring, and ${}_RT$ a tilting module (see [8]) with $S := (\mathrm{End}_R T)^{\mathrm{op}}$. Then the tilting theorem ([8], Theorem 1.4; cf. [18]) implies that the functors $F = \mathrm{Hom}_R(T, -)$ and $G = T \otimes_S -$ constitute an almost equivalence $G \dashv F$ between the module categories

S-**Mod** and R-**Mod**. In particular, the categories $\mathcal{F} := \operatorname{Ker} F$ and $\mathcal{T} := \operatorname{Im} G$ are almost abelian. Furthermore, it has been observed [8, 18] that $(\mathcal{T}, \mathcal{F})$ is a torsion theory in R-**Mod**, and the torsion theories in R-**Mod** which arise from a tilting module have been characterized ([9], Theorem 2.3; cf. [2, 37]). These results easily follow from basic properties of almost abelian categories (cf. Theorem 2 and §7).

8. Morita Duality. In the next section, we shall associate with each almost abelian category $\mathcal{A}$ a pair of abelian full subcategories $\mathcal{A}_\circ$ and $\mathcal{A}^\circ$. If R, S are rings, and ${}_RU_S$ is a left and right injective cogenerator, then the above mentioned almost abelian category $\mathcal{A} = \mathbf{DS}(U)$ has the property that $\mathcal{A}_\circ$ is equivalent to R-**Mod**, and $\mathcal{A}^\circ$ is equivalent to $(\mathbf{Mod}\text{-}S)^{\mathrm{op}}$. Moreover, the intersection $\mathcal{A}_\circ \cap \mathcal{A}^\circ$ coincides with the full subcategories of reflexive modules in R-**Mod** and **Mod**-S. If $R = S = \mathbb{Z}$ and $U := \mathsf{R}/\mathsf{Z}$, then $\mathbf{DS}(U)$ contains the category $\mathcal{L}$ of locally compact abelian groups as a full subcategory.

9. Reflexive modules. Let Λ be an R-order as in **1** with R integrally closed. Then the category Λ-$\mathbf{mod}_r$ of finitely generated left Λ-modules which are reflexive over R is almost abelian and integral with $\mathrm{Q}(\Lambda\text{-}\mathbf{mod}_r) = A$-**mod**. The kernel of a morphism $f : E \to F$ in Λ-$\mathbf{mod}_r$ coincides with the kernel in Λ-**mod**, whereas $\operatorname{cok} f$ in Λ-$\mathbf{mod}_r$ is given by $F \twoheadrightarrow F/H \hookrightarrow (F/H)^{**}$ if the R-torsion part of $F/\operatorname{Im} f$ is $H/\operatorname{Im} f$. In particular, if R is a two-dimensional local Cohen-Macaulay domain, then Λ-$\mathbf{mod}_r$ coincides with the category of maximal Cohen-Macaulay modules over Λ. The fact that cokernels need not be surjective in this category justifies the convention (for R complete) to regard certain exact sequences $E \rightarrowtail F \xrightarrow{c} P$ with c non-surjective as Auslander-Reiten sequences (see, e.g., [33], 2.1). If R is a Dedekind domain, then the category Λ-$\mathbf{mod}_r$ of reflexives is just Λ-**lat**.

3 Associated abelian categories

In this section we shall associate two pairs of abelian categories with each almost abelian category $\mathcal{A}$: two full subcategories $\mathcal{A}_\circ$, $\mathcal{A}^\circ$, and two categories $\mathrm{Q}_l(\mathcal{A})$, $\mathrm{Q}_r(\mathcal{A})$ in which $\mathcal{A}$ is fully embedded. The latter pair leads to a characterization of $\mathcal{A}$ in terms of a torsion theory, whereas

$\mathcal{A}_\circ$, $\mathcal{A}^\circ$ play a rôle in the relationship between almost abelian categories and generalized Morita duality (§7).

For any almost abelian category $\mathcal{A}$, let $\mathcal{A}_\circ$ denote the full subcategory of objects D in $\mathcal{A}$ such that each monomorphism $D' \to D$ is a kernel. Dually, the full subcategory $\mathcal{A}^\circ$ consists of the objects C in $\mathcal{A}$ such that each epimorphism $C \to C'$ is a cokernel. Generalizing a well-known concept for abelian categories, let us define a *Serre subcategory* of an additive category $\mathcal{A}$ as a non-empty full subcategory $\mathcal{S}$ of $\mathcal{A}$ such that for each exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{A}$, the middle term B lies in $\mathcal{S}$ if and only if A and C are in $\mathcal{S}$. By Lemma 4, a Serre subcategory of an almost abelian category is almost abelian.

Proposition 8 *If $\mathcal{A}$ is almost abelian, then $\mathcal{A}_\circ$ and $\mathcal{A}^\circ$ are abelian Serre subcategories of $\mathcal{A}$.*

We shall call $\mathcal{A}_\circ$ the *initial* and $\mathcal{A}^\circ$ the *terminal* category of $\mathcal{A}$.

Proof. Let $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$ be an exact sequence in $\mathcal{A}$. Suppose first that B is in $\mathcal{A}_\circ$. Then A is in $\mathcal{A}_\circ$ by Proposition 2. In order to show that C is in $\mathcal{A}_\circ$, let $d : D \to C$ be a monomorphism. Consider the pullback:

$$\begin{array}{ccc} B & \overset{b}{\twoheadrightarrow} & C \\ {\scriptstyle e}\uparrow & \text{PB} & \uparrow{\scriptstyle d} \\ E & \rightarrowtail & D \end{array} \tag{8}$$

Then e is monic, hence a kernel by the assumption. Proposition 1 implies that (8) is a pushout, and thus d is a kernel since $\mathcal{A}$ is almost abelian.

Next let us show that B is in $\mathcal{A}_\circ$ whenever A and C are in $\mathcal{A}_\circ$. To this end, let $e : E \to B$ be a monomorphism. Then we have a pullback

$$\begin{array}{ccccc} A & \overset{a}{\rightarrowtail} & B & \overset{b}{\twoheadrightarrow} & C \\ {\scriptstyle f}\uparrow & \text{PB} & \uparrow{\scriptstyle e} & & \uparrow{\scriptstyle g} \\ F & \underset{i}{\rightarrowtail} & E & \underset{p}{\twoheadrightarrow} & G \end{array}$$

with $p = \operatorname{cok} i$, where i is a kernel by Proposition 4, f is monic by Lemma 1, and the induced morphism g is monic by Lemma 2. Hence f and g are kernels by the assumption. By taking the pullback of b and g, the morphism e decomposes into $e = uv$ (see diagram below). Here $j = \ker q$ according to Lemma 1, and the square PO commutes since u is monic. By Proposition 4, u is a kernel. Moreover, q is a cokernel, and the square PO is a pushout by the dual of Proposition 5. Consequently, v is a kernel, whence $e = uv$ is a kernel by Proposition 2.

$$\begin{array}{ccccc} A & \overset{a}{\rightarrowtail} & B & \overset{b}{\twoheadrightarrow} & C \\ \Vert & & \uparrow u & \text{PB} & \uparrow g \\ A & \overset{j}{\rightarrowtail} & H & \overset{q}{\twoheadrightarrow} & G \\ \uparrow f & \text{PO} & \uparrow v & & \Vert \\ F & \underset{i}{\rightarrowtail} & E & \underset{p}{\twoheadrightarrow} & G \end{array} \tag{9}$$

Thus we have shown that $\mathcal{A}_\circ$ is a Serre subcategory of $\mathcal{A}$. Finally, for every morphism $f : A \to B$ in $\mathcal{A}_\circ$, the induced monomorphism $\operatorname{Coim} f \to B$ is a kernel, whence $\mathcal{A}_\circ$ is abelian. The assertion for $\mathcal{A}^\circ$ follows by duality. □

Recall that for a morphism $f : A \to B$ in an additive category, a morphism $g : K \to A$ is said to be a *weak kernel* if $fg = 0$ and every $h : H \to A$ with $fh = 0$ factors through g:

$$\begin{array}{ccccc} K & \overset{g}{\longrightarrow} & A & \overset{f}{\longrightarrow} & B \\ \uparrow & \nearrow_{h} & & & \\ H & & & & \end{array}$$

Analogously, *weak cokernels* are defined. Following Auslander [3], we call an additive category $\mathcal{V}$ a *variety (of annuli)* if idempotents split in $\mathcal{V}$, i. e. every idempotent endomorphism $e : A \to A$ in $\mathcal{V}$ has a kernel. (In this case, there is a biproduct

$$B \underset{p}{\overset{i}{\rightleftarrows}} A \underset{q}{\overset{j}{\leftrightarrows}} C$$

with $ip = e$.) If in addition, each morphism in $\mathcal{V}$ has a weak (co-)kernel, we call $\mathcal{V}$ a *projective (injective)* variety. A complex

$$\ldots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow 0 \tag{10}$$

in a projective variety $\mathcal{P}$ will be called *acyclic* if ∂_{i+1} is a weak kernel of ∂_i for each $i \geqslant 1$. By $\mathbf{mod}(\mathcal{P})$ we denote the category of acyclic complexes in $\mathcal{P}$ modulo homotopy. Equivalently, the category $\mathbf{mod}(\mathcal{P})$ can be given by the morphisms $u = \partial_1 : P_1 \to P_0$ in $\mathcal{P}$ as objects; then the morphisms are commutative squares

$$\begin{array}{ccc} P_1 & \xrightarrow{f} & Q_1 \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ P_0 & \xrightarrow{g} & Q_0 \end{array} \tag{11}$$

modulo such squares (11) with $g = vh$ for some $h : P_0 \to Q_1$.

If $\mathcal{A}$ is an abelian category with enough projectives, then $\mathbf{Proj}(\mathcal{A})$ is a projective variety from which $\mathcal{A}$ can be recovered by the equivalence (the 0^{th} homology of the acyclic complex (10)):

$$H_0 : \mathbf{mod}(\mathbf{Proj}(\mathcal{A})) \xrightarrow{\sim} \mathcal{A}. \tag{12}$$

Conversely, P. Freyd ([12], Corollary 1.5, Theorem 3.2) has shown that for each projective variety $\mathcal{P}$, the category $\mathbf{mod}(\mathcal{P})$ is abelian, and that the embedding $P \mapsto (0 \to P)$ yields an equivalence

$$\mathcal{P} \xrightarrow{\sim} \mathbf{Proj}(\mathbf{mod}(\mathcal{P})). \tag{13}$$

By (12) and (13), the objects A of $\mathbf{mod}(\mathcal{P})$ can be interpreted as cokernels $\operatorname{Cok} u$ of morphisms $u : P_1 \to P_0$ in $\mathcal{P}$. We shall call them *modules* A over $\mathcal{P}$. By duality, we have an abelian category $\mathbf{com}(\mathcal{I})$ of *comodules* over an injective variety $\mathcal{I}$, i. e. formal kernels of morphisms $I_0 \to I_1$ in $\mathcal{I}$.

By (12) and (13), projective or injective varieties are equivalently described by means of abelian categories. In particular, this immediately gives

Proposition 9 *Let $P_2 \xrightarrow{v} P_1 \xrightarrow{u} P_0$ be a sequence of morphisms in a projective variety. Then v is a weak kernel of u if and only if the sequence is exact in* $\mathbf{mod}(\mathcal{P})$.

For a projective variety $\mathcal{P}$ and $n \in \mathbb{N}$, let $\mathbf{mod}_n(\mathcal{P})$ denote the full subcategory of objects A in $\mathbf{mod}(\mathcal{P})$ with projective dimension $\mathrm{pd}(A) \leqslant n$. Thus $\mathbf{mod}_0(\mathcal{P})$ is equivalent to $\mathcal{P}$ by (13), and the objects in $\mathbf{mod}_1(\mathcal{P})$ are given by monomorphisms $u : P_1 \to P_0$ in $\mathcal{P}$. Our next aim will be to characterize left almost abelian categories $\mathcal{A}$ in terms of $\mathbf{mod}_1(\mathcal{A})$.

A morphism $f : A \to B$ in $\mathbf{mod}_1(\mathcal{P})$ yields a commutative diagram

$$\begin{array}{ccccc} P_1 & \overset{u}{\rightarrowtail} & P_0 & \twoheadrightarrow & A \\ \downarrow{\scriptstyle h} & \mathrm{SQ} & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Q_1 & \overset{v}{\rightarrowtail} & Q_0 & \twoheadrightarrow & B \end{array} \tag{14}$$

for given projective resolutions of A and B in $\mathbf{mod}(\mathcal{P})$.

Lemma 5 *Let $\mathcal{P}$ be a projective variety and f a morphism (14) in* $\mathbf{mod}_1(\mathcal{P})$. *If the square SQ is a pushout in $\mathcal{P}$, then f is epic in* $\mathbf{mod}_1(\mathcal{P})$. *If h is an isomorphism and g a monomorphism in* $\mathbf{mod}(\mathcal{P})$, *then f is a kernel in* $\mathbf{mod}_1(\mathcal{P})$.

Proof. Suppose first that SQ is a pushout in $\mathcal{P}$, and let $e : B \to C$ be a morphism in $\mathbf{mod}_1(\mathcal{P})$ with $ef = 0$. For a projective resolution $Q_1' \overset{w}{\rightarrowtail} Q_0' \twoheadrightarrow C$ in $\mathbf{mod}(\mathcal{P})$, we get a commutative diagram

$$\begin{array}{ccccc} P_1 & \overset{u}{\rightarrowtail} & P_0 & \twoheadrightarrow & A \\ \downarrow{\scriptstyle h} & \mathrm{SQ} & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Q_1 & \overset{v}{\rightarrowtail} & Q_0 & \twoheadrightarrow & B \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\ Q_1' & \overset{w}{\rightarrowtail} & Q_0' & \twoheadrightarrow & C \end{array}$$

such that $g'g = wp$ for some $p : P_0 \to Q_1'$. Hence $wh'h = g'gu = wpu$ implies $h'h = pu$, and the pushout SQ yields a morphism $q : Q_0 \to Q_1'$

with $qv = h'$ and $qg = p$. Therefore, $wqv = g'v$ and $wqg = g'g$ and the uniqueness property of SQ gives $wq = g'$, i. e. $e = 0$.

Next suppose that h is an isomorphism, and g is a monomorphism. Then the snake lemma implies that f is monic, and that the cokernels of f and g are isomorphic. Hence we obtain a short exact sequence $A \overset{f}{\rightarrowtail} B \twoheadrightarrow C$ in $\mathbf{mod}_1(\mathcal{P})$ which completes our proof. □

Lemma 6 *If $\mathcal{A}$ is a left semi-abelian category, then for every C in $\mathbf{mod}(\mathcal{A})$ there is an epimorphism*

$$C \longrightarrow\!\!\!\!\!\rightarrow C' \tag{15}$$

with C' in $\mathbf{mod}_1(\mathcal{A})$ such that each morphism $C \to D$ with D in $\mathbf{mod}_1(\mathcal{A})$ factors through (15). (This implies that $\mathbf{mod}_1(\mathcal{A})$ is a reflective full subcategory of $\mathbf{mod}(\mathcal{A})$.)

Proof. Let C be given by a presentation $P_1 \overset{u}{\to} P_0 \overset{v}{\twoheadrightarrow} C$ with P_0, P_1 in $\mathcal{A}$. Then u has a decomposition $u : P_1 \overset{p}{\to} P \overset{q}{\to} P_0$ in $\mathcal{A}$, where $p = \operatorname{coim} u$, and q is a monomorphism. (Note that p need not be epic in $\mathbf{mod}(\mathcal{A})$!) Hence the cokernel of q in $\mathbf{mod}(\mathcal{A})$ factors through v, say, $\operatorname{cok} q : P_0 \overset{v}{\twoheadrightarrow} C \overset{c}{\twoheadrightarrow} C'$, and C' is in $\mathbf{mod}_1(\mathcal{A})$. Now let $Q_1 \rightarrowtail Q_0 \twoheadrightarrow D$ be a projective presentation of any object D in $\mathbf{mod}_1(\mathcal{A})$. Then a morphism $f : C \to D$ yields a commutative diagram

$$\begin{array}{ccccc} P_1 & \xrightarrow{u} & P_0 & \overset{v}{\twoheadrightarrow} & C \\ \downarrow{\scriptstyle h} & & \downarrow & & \downarrow{\scriptstyle f} \\ Q_1 & \rightarrowtail & Q_0 & \twoheadrightarrow & D \end{array}$$

where h annihilates the kernel of u in $\mathcal{A}$. Therefore, h factors through $p : P_1 \to P$, whence f factors through $c : C \twoheadrightarrow C'$. □

Proposition 10 *Let $\mathcal{A}$ be a left semi-abelian category. Then $\mathbf{mod}_1(\mathcal{A})$ is almost abelian, and $\mathbf{mod}_2(\mathcal{A}) = \mathbf{mod}(\mathcal{A})$. The inclusions $\mathcal{A} \hookrightarrow \mathbf{mod}_1(\mathcal{A}) \hookrightarrow \mathbf{mod}(\mathcal{A})$ preserve kernels of morphisms. Moreover, a morphism in $\mathcal{A}$ is a cokernel if and only if it is epic in $\mathbf{mod}_1(\mathcal{A})$, and a morphism in $\mathbf{mod}_1(\mathcal{A})$ is a cokernel if and only if it is epic in $\mathbf{mod}(\mathcal{A})$.*

Proof. Since $\mathcal{A}$ has kernels, it follows immediately that $\mathbf{mod}_2(\mathcal{A}) = \mathbf{mod}(\mathcal{A})$. By Proposition 9, the inclusion $\mathcal{A} \hookrightarrow \mathbf{mod}(\mathcal{A})$ preserves kernels. For a morphism $f : A \to B$ in $\mathbf{mod}_1(\mathcal{A})$, let us show that the kernel $K \rightarrowtail A$ in $\mathbf{mod}(\mathcal{A})$ lies in $\mathbf{mod}_1(\mathcal{A})$. Consider the exact sequence $K \rightarrowtail A \twoheadrightarrow \operatorname{Im} f$ in $\mathbf{mod}(\mathcal{A})$. If K would not be in $\mathbf{mod}_1(\mathcal{A})$, then $\operatorname{pd}(A) < \operatorname{pd}(K) = 2$ would imply $\operatorname{pd}(\operatorname{Im} f) = 3$ which is impossible. Hence $\mathbf{mod}_1(\mathcal{A})$ has kernels which are preserved under the embedding into $\mathbf{mod}(\mathcal{A})$. By Lemma 6, a cokernel of $f : A \to B$ in $\mathbf{mod}_1(\mathcal{A})$ is given by $B \twoheadrightarrow C \twoheadrightarrow C'$, where $B \twoheadrightarrow C$ is the cokernel in $\mathbf{mod}(\mathcal{A})$, and $C \twoheadrightarrow C'$ is given by (15). In particular, we infer that f is a cokernel in $\mathbf{mod}_1(\mathcal{A})$ if and only if f is epic in $\mathbf{mod}(\mathcal{A})$. The proof of Lemma 6 also shows that a morphism in $\mathcal{A}$ is a cokernel if and only if it is epic in $\mathbf{mod}_1(\mathcal{A})$.

Next let us show that $\mathbf{mod}_1(\mathcal{A})$ is semi-abelian. For a morphism $f : A \to B$ in $\mathbf{mod}_1(\mathcal{A})$, consider a decomposition $f : A \overset{p}{\twoheadrightarrow} C \overset{i}{\rightarrowtail} B$ in $\mathbf{mod}(\mathcal{A})$. Then i factors through the epimorphism $c : C \twoheadrightarrow C'$ of (15), whence c is an isomorphism, i. e. $p = \operatorname{coim} f$ in $\mathbf{mod}_1(\mathcal{A})$. Thus for $\mathbf{mod}_1(\mathcal{A})$ to be semi-abelian, it remains to be shown that each monomorphism $f : A \to B$ in $\mathbf{mod}_1(\mathcal{A})$ decomposes into $A \overset{r}{\to} D \overset{d}{\rightarrowtail} B$ with r regular and d a kernel in $\mathbf{mod}_1(\mathcal{A})$. For projective resolutions of A and B, the induced square (14) decomposes into a commutative diagram

$$\begin{array}{ccccc} P_1 & \xrightarrow{h} & Q_1 & \xrightarrow{1} & Q_1 \\ \downarrow u & \text{PO} & \downarrow q & & \downarrow v \\ P_0 & \xrightarrow{p} & P & \xrightarrow{s} & Q_0 \end{array} \tag{16}$$

with a pushout PO in $\mathcal{A}$ and $g = sp$. Since the square SQ in (14) is a pullback in $\mathbf{mod}(\mathcal{A})$ and hence in $\mathcal{A}$, the morphism $(p\ q) : P_0 \oplus Q_1 \to P$ is the coimage of $(g\ v) : P_0 \oplus Q_1 \to Q_0$ in $\mathcal{A}$. Therefore, s is a monomorphism. Consequently, if $D = \operatorname{Cok} q$ in $\mathbf{mod}(\mathcal{A})$, Lemma 5 yields the desired decomposition $f : A \overset{r}{\to} D \overset{d}{\rightarrowtail} B$ with a regular r and a kernel d in $\mathbf{mod}_1(\mathcal{A})$.

Finally, by Proposition 3, our proof will be complete if we show that $\mathbf{mod}_1(\mathcal{A})$ is left almost abelian. Consider a pullback (2) in $\mathbf{mod}_1(\mathcal{A})$.

By the above, this is also a pullback in $\mathbf{mod}(\mathcal{A})$. If d is a cokernel in $\mathbf{mod}_1(\mathcal{A})$, then it is epic in $\mathbf{mod}(\mathcal{A})$. Hence a is epic in $\mathbf{mod}(\mathcal{A})$ and thus a cokernel in $\mathbf{mod}_1(\mathcal{A})$. $\square$

Corollary. *If $\mathcal{A}$ is left semi-abelian, and f is a morphism (14) in $\mathbf{mod}_1(\mathcal{A})$, then f is regular if and only if the square* SQ *is exact in $\mathcal{A}$.*

Proof. If SQ is a pullback in $\mathcal{A}$, then it is a pullback in $\mathbf{mod}(\mathcal{A})$, whence f is monic. If SQ is a pushout in $\mathcal{A}$, then f is epic in $\mathbf{mod}_1(\mathcal{A})$ by Lemma 5. Conversely, let f be regular in $\mathbf{mod}_1(\mathcal{A})$. Then SQ is a pullback and decomposes as in (16), where the right-hand square induces the image of f in $\mathbf{mod}_1(\mathcal{A})$ which has to be trivial. Therefore, s is an isomorphism, and thus SQ is a pushout in $\mathcal{A}$. $\square$

Now we are ready to prove a relationship between one-sided almost abelian and two-sided integral categories:

Theorem 1 *A left semi-abelian category $\mathcal{A}$ is left almost abelian if and only if the almost abelian category $\mathbf{mod}_1(\mathcal{A})$ is integral.*

Proof. Suppose first that $\mathcal{A}$ is left almost abelian. Let $\mathcal{T}$ be the class of objects M in $\mathbf{mod}(\mathcal{A})$ such that every morphism $M \to N$ with N in $\mathbf{mod}_1(\mathcal{A})$ is zero. We show first that $\mathcal{T}$ is closed with respect to subobjects. Thus let $i : N \rightarrowtail M$ be a monomorphism in $\mathbf{mod}(\mathcal{A})$ with $M \in \mathcal{T}$. For a projective presentation $P_1 \xrightarrow{u} P_0 \overset{e}{\twoheadrightarrow} M$ and an epimorphism $f : Q_0 \twoheadrightarrow N$ with $Q_0 \in \mathcal{A}$, choose $q : Q_0 \to P_0$ with $eq = if$, and consider in $\mathcal{A}$ the pullback of q and u:

$$\begin{array}{ccccc} Q & \xrightarrow{\ v\ } & Q_0 & \overset{f}{\twoheadrightarrow} & N \\ \downarrow & \text{PB} & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle i} \\ P_1 & \xrightarrow{\ u\ } & P_0 & \overset{e}{\twoheadrightarrow} & M \end{array} \tag{17}$$

Then $Q \xrightarrow{v} Q_0 \overset{f}{\twoheadrightarrow} N$ is a projective presentation of N in $\mathbf{mod}(\mathcal{A})$. By Proposition 10, the assumption $M \in \mathcal{T}$ says that u is a cokernel in $\mathcal{A}$. Hence, v is a cokernel in $\mathcal{A}$, and thus $N \in \mathcal{T}$.

Now consider a pullback (2) in $\mathbf{mod}_1(\mathcal{A})$ with an epimorphism d. Then Proposition 10 implies that (2) is a pullback in $\mathbf{mod}(\mathcal{A})$. Therefore, we get a monomorphism $i : \operatorname{Cok} a \rightarrowtail \operatorname{Cok} d$ for the cokernels in

$\mathbf{mod}(\mathcal{A})$. Since $\operatorname{Cok} d \in \mathcal{T}$, we infer that $\operatorname{Cok} a \in \mathcal{T}$, i. e. a is epic in $\mathbf{mod}_1(\mathcal{A})$. By the corollary of Proposition 6, $\mathbf{mod}_1(\mathcal{A})$ is integral.

Conversely, let (17) be a pullback in $\mathcal{A}$ with $e = \operatorname{cok} u$ and $f = \operatorname{cok} v$ in $\mathbf{mod}(\mathcal{A})$. Then Proposition 10 implies that PB is a pullback in $\mathbf{mod}_1(\mathcal{A})$, and u or v is a cokernel in $\mathcal{A}$ if and only if it is epic in $\mathbf{mod}_1(\mathcal{A})$. This completes our proof. □

By the preceding results, every left almost abelian category $\mathcal{A}$ can be fully embedded into an integral almost abelian category $\mathbf{mod}_1(\mathcal{A})$. By the corollary of Proposition 10, a regular morphism in $\mathbf{mod}_1(\mathcal{A})$ between objects of $\mathcal{A}$ must be an isomorphism. Therefore, the embedding $\mathcal{A} \hookrightarrow \mathbf{mod}_1(\mathcal{A})$ induces a full embedding

$$\mathcal{A} \hookrightarrow \mathrm{Q}_l(\mathcal{A}) := \mathrm{Q}(\mathbf{mod}_1(\mathcal{A})). \tag{18}$$

By duality, every right almost abelian category $\mathcal{A}$ is fully embedded into the category $\mathbf{com}^1(\mathcal{A})$ of comodules of injective dimension $\leqslant 1$. This gives a full embedding

$$\mathcal{A} \hookrightarrow \mathrm{Q}_r(\mathcal{A}) := \mathrm{Q}(\mathbf{com}^1(\mathcal{A})). \tag{19}$$

The abelian categories $\mathrm{Q}_l(\mathcal{A})$ and $\mathrm{Q}_r(\mathcal{A})$ will be called the *left* (resp. *right*) *abelian cover* of $\mathcal{A}$. They have a simple description if $\mathcal{A}$ has *strictly* enough projectives (resp. injectives), i. e. if for each object A in $\mathcal{A}$, there is a cokernel $P \twoheadrightarrow A$ with P projective (resp. a kernel $A \rightarrowtail I$ with I injective):

Proposition 11 *Let $\mathcal{A}$ be a preabelian with strictly enough projectives. Then $\mathcal{A}$ is left almost abelian, and $\mathrm{Q}_l(\mathcal{A})$ is equivalent to* $\mathbf{mod}(\mathbf{Proj}(\mathcal{A}))$.

Proof. By Corollary 2 of Proposition 1, $\mathcal{A}$ is left semi-abelian. In order to show that $\mathcal{A}$ is left almost abelian, let (2) be a pullback with a cokernel d, and let $p : P \twoheadrightarrow B$ be a cokernel with P projective. Then cp factors through d, and thus p factors through a. By Proposition 2, this implies that a is a cokernel, whence $\mathcal{A}$ is left almost abelian.

Since $\mathcal{A}$ has strictly enough projectives, it follows that $\mathbf{Proj}(\mathcal{A})$ is a projective variety. Now consider the functor

$$S : \mathbf{mod}(\mathbf{Proj}(\mathcal{A})) \longrightarrow \mathbf{mod}(\mathcal{A}) \xrightarrow{R} \mathbf{mod}_1(\mathcal{A}),$$

where R is the reflector (15) of the embedding $\mathbf{mod}_1(\mathcal{A}) \hookrightarrow \mathbf{mod}(\mathcal{A})$ according to Lemma 6. Then S is fully faithful and yields an equivalence between $\mathbf{mod}(\mathbf{Proj}(\mathcal{A}))$ and the full subcategory $\mathcal{C}$ of $\mathbf{mod}_1(\mathcal{A})$

consisting of those objects M which have a presentation $A_1 \rightarrowtail A_0 \twoheadrightarrow M$ with A_1 in $\mathcal{A}$ and A_0 in $\mathbf{Proj}(\mathcal{A})$. Therefore, it remains to be shown that the faithful embedding

$$\mathbf{mod}_1(\mathcal{A}) \hookrightarrow \mathrm{Q}(\mathbf{mod}_1(\mathcal{A})) = \mathrm{Q}_l(\mathcal{A})$$

induces an equivalence between $\mathcal{C}$ and $\mathrm{Q}_l(\mathcal{A})$. Let $A_1 \rightarrowtail A_0 \twoheadrightarrow M$ be a projective resolution of any object M in $\mathbf{mod}_1(\mathcal{A})$. For a cokernel $P \twoheadrightarrow A_0$ in $\mathcal{A}$ with P in $\mathbf{Proj}(\mathcal{A})$, the pullback

$$\begin{array}{ccc} A_1 & \rightarrowtail & A_0 \\ \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\ B & \overset{v}{\rightarrowtail} & P \end{array}$$

in $\mathcal{A}$ is exact by Proposition 1. By Proposition 10, it is also a pullback in $\mathbf{mod}(\mathcal{A})$, and v is a monomorphism in $\mathbf{mod}(\mathcal{A})$. Therefore, the corollary of Proposition 10 implies that the induced morphism $\mathrm{Cok}\, v \to M$ is regular. Hence as an object in $\mathrm{Q}_l(\mathcal{A})$, M is isomorphic to some $C \in \mathcal{C}$. Moreover, for $C, D \in \mathcal{C}$, it follows that a morphism $C \to D$ in $\mathrm{Q}_l(\mathcal{A})$ is given by morphisms $C \leftarrow C' \to D$ in $\mathcal{C}$ with $C \leftarrow C'$ regular in $\mathbf{mod}_1(\mathcal{A})$. Thus to complete the proof, let us show that a regular morphism $r : M \to N$ in $\mathcal{C}$ is an isomorphism. Consider the induced morphism between projective presentations of M and N:

$$\begin{array}{ccccc} A & \overset{i}{\rightarrowtail} & P & \overset{p}{\twoheadrightarrow} & M \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle r} \\ B & \overset{j}{\rightarrowtail} & Q & \overset{q}{\twoheadrightarrow} & N \end{array}$$

Here A, B are in $\mathcal{A}$, and P, Q in $\mathbf{Proj}(\mathcal{A})$. Since $(f\ j) : P \oplus B \twoheadrightarrow Q$ is a cokernel in $\mathcal{A}$, we have a section $\binom{s}{t} : Q \to P \oplus B$, i. e. $fs + jt = 1$. Then $rpsj = qfsj = q(1 - jt)j = 0$ implies $psj = 0$, and thus $ps = uq$ for some $u : N \to M$. Now $ruq = rps = qfs = q$ yields $ru = 1$, whence r is an isomorphism. □

Corollary. *If $\mathcal{A}$ is a left semi-abelian category, then* $\mathbf{Proj}(\mathbf{mod}_1(\mathcal{A})) \approx \mathcal{A}$. *Moreover,* $\mathrm{Q}_l(\mathbf{mod}_1(\mathcal{A}))$ *is equivalent to* $\mathbf{mod}(\mathcal{A})$.

Proof. By Proposition 10, the cokernels in $\mathbf{mod}_1(\mathcal{A})$ are epic in $\mathbf{mod}(\mathcal{A})$. Hence $\mathcal{A} \subset \mathbf{Proj}(\mathbf{mod}_1(\mathcal{A}))$. Conversely, let P be projective in $\mathbf{mod}_1(\mathcal{A})$, and let $A_1 \rightarrowtail A_0 \overset{p}{\twoheadrightarrow} P$ be a presentation with A_0, A_1

in $\mathcal{A}$. Then p is a cokernel in $\mathbf{mod}_1(\mathcal{A})$. Hence $A_0 = A_1 \oplus P$, and thus P is isomorphic to an object in $\mathcal{A}$. Furthermore, this shows that $\mathbf{mod}_1(\mathcal{A})$ has strictly enough projectives. Therefore, Proposition 11 proves the remaining assertion. □

We conclude this section with a universal characterization of $\mathrm{Q}_l(\mathcal{A})$:

Proposition 12 *Let $\mathcal{A}$ be a left almost abelian category. Then the embedding $\mathcal{A} \hookrightarrow \mathrm{Q}_l(\mathcal{A})$ is exact and preserves monomorphisms. Every exact functor $F : \mathcal{A} \to \mathcal{B}$ into an abelian category $\mathcal{B}$ which preserves monomorphisms has a unique extension to an exact functor $\mathrm{Q}_l(\mathcal{A}) \to \mathcal{B}$.*

Proof. By Proposition 10, the inclusion $\mathcal{A} \hookrightarrow \mathrm{Q}_l(\mathcal{A})$ preserves kernels, hence monomorphisms. A short exact sequence $A \overset{a}{\rightarrowtail} B \twoheadrightarrow C$ in $\mathcal{A}$ is tantamount to an exact square

$$\begin{array}{ccc} A & \overset{a}{\longrightarrow} & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C \end{array}$$

in $\mathcal{A}$. By the corollary of Proposition 10, the induced morphism $\mathrm{Cok}\, a \to C$ in $\mathbf{mod}_1(\mathcal{A})$ is regular, hence an isomorphism in $\mathrm{Q}_l(\mathcal{A})$. Therefore, $A \rightarrowtail B \twoheadrightarrow C$ is exact in $\mathrm{Q}_l(\mathcal{A})$. Now let $F : \mathcal{A} \to \mathcal{B}$ be a functor with the properties given in the proposition. Every object M in $\mathbf{mod}_1(\mathcal{A})$ has a presentation $A_1 \overset{u}{\rightarrowtail} A_0 \twoheadrightarrow M$ with A_0, A_1 in $\mathcal{A}$. Hence if we define $F'(M)$ as the cokernel of $F(u)$ in $\mathcal{B}$, we obtain an extended functor $F' : \mathbf{mod}_1(\mathcal{A}) \to \mathcal{B}$ which makes regular morphisms invertible. In order to show that F' is exact, let $M' \rightarrowtail M \twoheadrightarrow M''$ be a short exact sequence in $\mathbf{mod}_1(\mathcal{A})$, and $A_1 \rightarrowtail A_0 \twoheadrightarrow M$ a projective presentation of M. Since the pullback

$$\begin{array}{ccc} A & \overset{a}{\rightarrowtail} & A_0 \\ \downarrow & \mathrm{PB} & \downarrow \\ M' & \rightarrowtail & M \end{array}$$

in $\mathbf{mod}(\mathcal{A})$ is exact, it follows that $\mathrm{Cok}\, a \cong M''$. Hence A is isomorphic to an object in $\mathcal{A}$, and we obtain a projective presentation $A_1 \rightarrowtail A \twoheadrightarrow M'$. Applying F' gives a short exact sequence $F'(M') \rightarrowtail F'(M) \twoheadrightarrow$

$F'(M'')$ in $\mathcal{B}$. Therefore, F' yields a unique exact extension $Q_l(\mathcal{A}) \to \mathcal{B}$ of F. □

Corollary 1 *Let $\mathcal{A}$ be left almost abelian, and $A_1 \overset{u}{\rightarrowtail} A_0 \overset{v}{\twoheadrightarrow} M$ a presentation of an object M in $Q_l(\mathcal{A})$ with A_0, A_1 in $\mathcal{A}$. Then M is isomorphic to an object in $\mathcal{A}$ if and only if u is a kernel in $\mathcal{A}$.*

Proof. If M is in $\mathcal{A}$, then u splits. Conversely, if u is a kernel in $\mathcal{A}$, then the cokernel of u in $\mathcal{A}$ is isomorphic to M by the exactness of the embedding $\mathcal{A} \hookrightarrow Q_l(\mathcal{A})$. □

Corollary 2 *If $\mathcal{A}$ is left almost abelian, and $u : A \rightarrowtail B$ a monomorphism in $Q_l(\mathcal{A})$ with B in $\mathcal{A}$, then A is isomorphic to an object in $\mathcal{A}$.*

Proof. Let $A_1 \overset{i}{\rightarrowtail} A_0 \overset{p}{\twoheadrightarrow} A$ be a presentation of A in $Q_l(\mathcal{A})$ with A_0, A_1 in $\mathcal{A}$. Then $i = \ker(up)$, and the assertion follows by Corollary 1. □

For a class $\mathcal{C}$ of objects in an additive category $\mathcal{A}$, let add $\mathcal{C}$ denote the class of direct summands of finite biproducts $C_1 \oplus \ldots \oplus C_n$ with $C_1, \ldots, C_n \in \mathcal{C}$. If $\mathcal{A}$ is almost abelian, we define a *subquotient* of an object C as an object A for which there exists a kernel $B \rightarrowtail C$ and a cokernel $B \twoheadrightarrow A$. By Proposition 4, a subquotient A of C can also be represented in the form $A \rightarrowtail D \twoheadleftarrow C$. For an object class $\mathcal{C}$ in $\mathcal{A}$, the full subcategory of subquotients of objects in add $\mathcal{C}$ will be denoted by ab $\mathcal{C}$. Thus ab $\mathcal{C}$ is closed with respect to biproducts and subquotients. In particular, if $\mathcal{A}$ is abelian, then ab $\mathcal{C}$ is an exact abelian full subcategory of $\mathcal{A}$.

Corollary 3 *Let $\mathcal{A}$ be a left almost abelian category with an exact full embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ into an abelian category $\mathcal{B}$ such that $\mathcal{A}$ is closed with respect to subobjects in $\mathcal{B}$, and $\mathcal{B} = \text{ab}\,\mathcal{A}$. Then $\mathcal{A} \hookrightarrow \mathcal{B}$ extends to an equivalence $Q_l(\mathcal{A}) \xrightarrow{\sim} \mathcal{B}$.*

Proof. If $f : A \to B$ is a monomorphism in $\mathcal{A}$, then its kernel in $\mathcal{B}$ belongs to $\mathcal{A}$. Hence f is monic in $\mathcal{B}$. By the proposition, $\mathcal{A} \hookrightarrow \mathcal{B}$ extends to an exact functor $E : Q_l(\mathcal{A}) \to \mathcal{B}$. Firstly, let M be an object in $Q_l(\mathcal{A})$ with $E(M) = 0$. For a presentation $A_1 \overset{u}{\rightarrowtail} A_0 \twoheadrightarrow M$ in $\mathbf{mod}_1(\mathcal{A})$ with A_0, A_1 in $\mathcal{A}$, this implies that $E(u)$ is an isomorphism. Since E is full and exact on $\mathcal{A}$, we infer that u is an isomorphism, and thus $M = 0$. Now let $f : M \to N$ be a morphism in $\mathsf{Q}_l(\mathcal{A})$ with $E(f) = 0$. Then $E(\operatorname{Im} f) = \operatorname{Im} E(f) = 0$ implies $\operatorname{Im} f = 0$. Thus we have shown that E is faithful.

Since each object M' of $\mathcal{B}$ admits a presentation $A_1 \overset{u}{\rightarrowtail} A_0 \twoheadrightarrow M'$ with A_0, A_1 in $\mathcal{A}$, the cokernel $A_0 \twoheadrightarrow M$ of u in $\mathsf{Q}_l(\mathcal{A})$ yields an object M with $E(M) \cong M'$. Thus E is dense.

In order to show that E is full, let M, N be objects in $\mathsf{Q}_l(\mathcal{A})$, and $g : E(M) \to E(N)$ a morphism in $\mathcal{B}$. Consider presentations $A_1 \rightarrowtail A_0 \overset{p}{\twoheadrightarrow} M$ and $B_1 \overset{v}{\rightarrowtail} B_0 \overset{q}{\twoheadrightarrow} N$ in $\mathsf{Q}_l(\mathcal{A})$ with A_0, A_1, B_0, B_1 in $\mathcal{A}$. The pullback

$$\begin{array}{ccc} A_0' & \xrightarrow{\;r\;} & A_0 \\ \Big\downarrow{\scriptstyle s} & & \Big\downarrow{\scriptstyle gE(p)} \\ B_0 & \xrightarrow{E(q)} & E(N) \end{array}$$

yields a presentation $A_1' \overset{w}{\rightarrowtail} A_0' \overset{pr}{\twoheadrightarrow} M$ with A_0', A_1' in $\mathcal{A}$, and a commutative diagram

$$\begin{array}{ccccc} A_1' & \overset{w}{\rightarrowtail} & A_0' & \overset{pr}{\twoheadrightarrow} & M \\ \Big\downarrow & & \Big\downarrow{\scriptstyle s} & & \vdots{\scriptstyle f} \\ B_1 & \overset{v}{\rightarrowtail} & B_0 & \overset{q}{\twoheadrightarrow} & N \end{array}$$

with exact rows in $\mathsf{Q}_l(\mathcal{A})$. In fact, $E(qsw) = E(q)sw = gE(prw) = 0$ implies $qsw = 0$, whence sw factors through v, which yields an induced morphism f. Now $E(f)E(pr) = E(q)s = gE(pr)$ implies $E(f) = g$, whence E is an equivalence. □

4 Torsion theories and almost equivalences

Let $\mathcal{A}$ be an additive category. For a class $\mathcal{C}$ of objects in $\mathcal{A}$, we define a $\mathcal{C}$-*epimorphism* as a morphism $f : A \to B$ in $\mathcal{A}$ such that each $g : C \to B$ with $C \in \mathcal{C}$ factors through f. Dually, f will be called a $\mathcal{C}$-*monomorphism* if each $g : A \to C$ with $C \in \mathcal{C}$ factors through f. A $\mathcal{C}$-epimorphism $C \to A$ with $C \in \mathcal{C}$ will be called a $\mathcal{C}$-*cover* [12], and a $\mathcal{C}$-monomorphism $A \to C$ with $C \in \mathcal{C}$ a $\mathcal{C}$-*hull*. Let us call $\mathcal{C}$ a *pre-torsion*

(-free) class in $\mathcal{A}$ if $\mathcal{C}$ is closed with respect to isomorphisms (i. e. every object isomorphic to some $C \in \mathcal{C}$ belongs to $\mathcal{C}$), and each object in $\mathcal{A}$ has a monic $\mathcal{C}$-cover (resp. an epic $\mathcal{C}$-hull). In what follows, we frequently identify an object class $\mathcal{C}$ with its corresponding full subcategory.

Remarks. 1. Every pre-torsion(-free) class $\mathcal{C}$ is closed with respect to biproducts. In fact, if $A, B \in \mathcal{C}$, and $c : C \to A \oplus B$ a monic $\mathcal{C}$-cover, then $\binom{1}{0} : A \to A \oplus B$ and $\binom{0}{1} : B \to A \oplus B$ factor through c, i. e. c admits a section, whence c is an isomorphism.

2. If $\mathcal{C}$ is a pre-torsion(-free) class in a semi-abelian category $\mathcal{A}$, and if $C \twoheadrightarrow A$ is a cokernel (resp. $A \rightarrowtail C$ a kernel) in $\mathcal{A}$ with $C \in \mathcal{C}$, then $A \in \mathcal{C}$. This follows since a cokernel $C \twoheadrightarrow A$ factors through a (monic) $\mathcal{C}$-cover $D \to A$, whence $D \to A$ is a cokernel by Proposition 2, and thus $A \cong D \in \mathcal{C}$. In particular, if $\mathcal{A}$ is abelian, every pre-torsion(-free) class is closed with respect to quotient (resp. sub-) objects.

3. A monic $\mathcal{C}$-cover (as well as an epic $\mathcal{C}$-hull) is always unique up to isomorphism: If $c : C \to A$ and $d : D \to A$ are monic $\mathcal{C}$-covers, then by definition, there are morphisms $f : C \to D$ and $g : D \to C$ with $c = df$ and $d = cg$. Then $cgf = c$ implies $gf = 1$, and dually, $fg = 1$.

A pair $(\mathcal{T}, \mathcal{F})$ of object classes in $\mathcal{A}$ is said to be a *torsion theory* if $\mathcal{T}$ and $\mathcal{F}$ are closed with respect to isomorphisms, and for each object A in $\mathcal{A}$ there exists a short exact sequence $T \overset{u}{\rightarrowtail} A \overset{v}{\twoheadrightarrow} F$ with a $\mathcal{T}$-cover u, and an $\mathcal{F}$-hull v. Then the pre-torsion class $\mathcal{T}$ and the pre-torsionfree class $\mathcal{F}$ determine each other, and we shall speak of a *torsion class* $\mathcal{T}$, and a *torsionfree class* $\mathcal{F}$ in this case.

If $\mathcal{A}$ is abelian, this definition is equivalent with the original one given by Dickson [10] (cf. also [27]). If in addition, $\mathcal{T}$ is closed with respect to subobjects, then the torsion theory $(\mathcal{T}, \mathcal{F})$ is said to be *hereditary*.

Lemma 7 *For a pre-torsionfree class $\mathcal{F}$ in an abelian category $\mathcal{A}$, the following are equivalent:*

(a) *$\mathcal{F}$ is a torsionfree class.*

(b) *$\mathcal{F}$ is closed with respect to extensions in $\mathcal{A}$.*

(c) *For each $\mathcal{F}$-hull $f : A \twoheadrightarrow F$ in $\mathcal{A}$, the kernel* $\operatorname{Ker} f$ *has an $\mathcal{F}$-hull* $\operatorname{Ker} f \to 0$.

Proof. (a) $\Rightarrow$ (b): If $\mathcal{F}$ belongs to a torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, then $\mathcal{T}$ consists of the objects T in $\mathcal{A}$ with $\mathcal{F}$-hull $T \to 0$. Therefore, if $F \overset{i}{\rightarrowtail} A \twoheadrightarrow F'$ is a short exact sequence in $\mathcal{A}$ with $F, F' \in \mathcal{F}$, then every morphism $t : T \to A$ with $T \in \mathcal{T}$ factors through i, whence $t = 0$. This implies $A \in \mathcal{F}$.

(b) $\Rightarrow$ (c): Let $K \rightarrowtail A \overset{f}{\twoheadrightarrow} F$ be a short exact sequence in $\mathcal{A}$ with an $\mathcal{F}$-hull f, and let $g : K \twoheadrightarrow F_0$ be an $\mathcal{F}$-hull of K. Consider the pushout

$$\begin{array}{ccccc} K & \rightarrowtail & A & \overset{f}{\twoheadrightarrow} & F \\ \downarrow{\scriptstyle g} & \text{PO} & \downarrow{\scriptstyle h} & & \Vert \\ F_0 & \rightarrowtail & B & \twoheadrightarrow & F \end{array}$$

in $\mathcal{A}$. Then $B \in \mathcal{F}$ implies that h factors through f, and thus $g = 0$.

(c) $\Rightarrow$ (a): Let $\mathcal{T}$ be the class of objects T in $\mathcal{A}$ with $\mathcal{F}$-hull $T \to 0$. Then every object A in $\mathcal{A}$ with $\mathcal{F}$-hull $f : A \twoheadrightarrow F$ yields an exact sequence $\operatorname{Ker} f \rightarrowtail A \twoheadrightarrow F$ with a $\mathcal{T}$-cover $\operatorname{Ker} f \rightarrowtail A$. □

Theorem 2 *Let $\mathcal{A}$ be an abelian category with a pre-torsionfree class $\mathcal{F}$. Then the full subcategory $\mathcal{F}$ is left almost abelian with* ab $\mathcal{F}$ *equivalent to* $\mathrm{Q}_l(\mathcal{F})$. *If $\mathcal{F}$ defines a (hereditary) torsion theory in $\mathcal{A}$, then $\mathcal{F}$ is (integral and) almost abelian. Conversely, every left almost abelian category $\mathcal{C}$ is equivalent to a pre-torsionfree class in* $\mathrm{Q}_l(\mathcal{C})$, *and $\mathcal{C}$ is (integral and) almost abelian if and only if $\mathcal{C}$ defines a (hereditary) torsion theory in* $\mathrm{Q}_l(\mathcal{C})$.

Proof. By the above remarks **1** and **2**, every pre-torsionfree class $\mathcal{F}$ in an abelian category $\mathcal{A}$ is closed with respect to biproducts and subobjects. For any morphism $f : F_1 \to F_2$ in $\mathcal{F}$, the kernel in $\mathcal{A}$ is a kernel in $\mathcal{F}$, and if $c : F_2 \twoheadrightarrow C$ is a cokernel of f in $\mathcal{A}$, then the $\mathcal{F}$-hull $C \twoheadrightarrow F(C)$ of C yields a cokernel $F_2 \overset{c}{\twoheadrightarrow} C \twoheadrightarrow F(C)$ of f in $\mathcal{F}$. In particular, f is a cokernel in $\mathcal{F}$ if and only if f is epic in $\mathcal{A}$. As an immediate consequence, we infer that $\mathcal{F}$ is left almost abelian. Since $\mathcal{F} \hookrightarrow \mathcal{A}$ is exact, Corollary 3 of Proposition 12 implies that ab $\mathcal{F} \approx \mathrm{Q}_l(\mathcal{F})$.

Now let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in $\mathcal{A}$. In order to show that $\mathcal{F}$ is right almost abelian, consider a pushout (2) in $\mathcal{F}$ with a kernel a.

Then the short exact sequence $A \overset{a}{\rightarrowtail} B \twoheadrightarrow \operatorname{Cok} a$ in $\mathcal{F}$ is also exact in $\mathcal{A}$. By Lemma 7, $\mathcal{F}$ is closed with respect to extensions in $\mathcal{A}$. Hence (2) is a pushout in $\mathcal{A}$, and thus d is a kernel in $\mathcal{F}$. This implies that $\mathcal{F}$ is almost abelian. Next we assume that $(\mathcal{T}, \mathcal{F})$ is hereditary. Any pullback (2) in $\mathcal{F}$ is also a pullback in $\mathcal{A}$, and a morphism in $\mathcal{F}$ is epic if and only if its cokernel in $\mathcal{A}$ belongs to $\mathcal{T}$. Since the induced morphism $\operatorname{Cok} a \to \operatorname{Cok} d$ is monic in $\mathcal{A}$, it follows that $\mathcal{F}$ is left integral. By the corollary of Proposition 6, $\mathcal{F}$ is integral.

Conversely, let $\mathcal{C}$ be a left almost abelian category. For any object M in $\mathbf{mod}_1(\mathcal{C})$, consider a presentation $C_1 \overset{u}{\rightarrowtail} C_0 \twoheadrightarrow M$ with $C_0, C_1 \in \mathcal{C}$. Then u has a decomposition $u : C_1 \overset{r}{\to} C \overset{v}{\rightarrowtail} C_0$ in $\mathcal{C}$ with $v = \operatorname{im} u$. This yields a commutative diagram

$$\begin{array}{ccccc} C_1 & \overset{r}{\rightarrowtail} & C & \twoheadrightarrow & \operatorname{Cok} r \\ \Vert & & \downarrow{\scriptstyle v} & & \downarrow \\ C_1 & \overset{u}{\rightarrowtail} & C_0 & \twoheadrightarrow & M \end{array}$$

in $\mathbf{mod}_1(\mathcal{C})$ with exact rows, and an exact right-hand square. Hence we obtain an exact sequence

$$\operatorname{Cok} r \rightarrowtail M \twoheadrightarrow \operatorname{Cok} v \tag{20}$$

in $\mathbf{mod}_1(\mathcal{C})$, hence in $\mathrm{Q}_l(\mathcal{C})$. Now let $\mathcal{F}$ be the class of objects M in $\mathrm{Q}_l(\mathcal{C})$ which admit a presentation $C_1 \overset{u}{\rightarrowtail} C_0 \twoheadrightarrow M$ in $\mathbf{mod}_1(\mathcal{C})$ where u is a kernel in $\mathcal{C}$. Then for every M in $\mathbf{mod}_1(\mathcal{C})$, (20) provides an $\mathcal{F}$-hull $M \twoheadrightarrow \operatorname{Cok} v$ in $\mathrm{Q}_l(\mathcal{C})$, and by Corollary 1 of Proposition 12, $\mathcal{C}$ is equivalent to $\mathcal{F}$. Furthermore, Lemma 7 shows that the pre-torsionfree class $\mathcal{F}$ determines a torsion theory in $\mathrm{Q}_l(\mathcal{C})$ if $\mathcal{C}$ is right semi-abelian.

Finally, let the almost abelian category $\mathcal{C}$ be integral, and let $i : M \rightarrowtail N$ be a monomorphism in $\mathbf{mod}_1(\mathcal{C})$. There is a commutative diagram

$$\begin{array}{ccccc} C_1 & \overset{u}{\rightarrowtail} & C_0 & \twoheadrightarrow & M \\ \downarrow & \text{PB} & \downarrow & & \downarrow{\scriptstyle i} \\ D_1 & \overset{v}{\rightarrowtail} & D_0 & \twoheadrightarrow & N \end{array}$$

in $\mathbf{mod}(\mathcal{C})$ with exact rows and $C_0, C_1, D_0, D_1 \in \mathcal{C}$. By Proposition 10, the square PB is a pullback in $\mathcal{C}$, and u, v are monic in $\mathcal{C}$. Moreover, u (resp. v) is regular in $\mathcal{C}$ if and only if M (resp. N) is a torsion object in the torsion theory of $\mathsf{Q}_l(\mathcal{C})$ given by $\mathcal{C}$. Hence $\mathcal{C}$ defines a hereditary torsion theory in $\mathsf{Q}_l(\mathcal{C})$. □

As an immediate consequence we note:

Corollary. *For a category $\mathcal{A}$ the following are equivalent:*

(a) *$\mathcal{A}$ is almost abelian.*

(b) *$\mathcal{A}$ is equivalent to the torsion class of a torsion theory for an abelian category.*

(c) *$\mathcal{A}$ is equivalent to the torsionfree class of a torsion theory for an abelian category.*

For an almost abelian category $\mathcal{A}$, the torsion class corresponding to $\mathcal{A}$ in $\mathsf{Q}_l(\mathcal{A})$ will be denoted by $\mathcal{R}(\mathcal{A})$. The objects of this full subcategory of $\mathsf{Q}_l(\mathcal{A})$ can be given by regular morphisms $r : A_1 \to A_0$ in $\mathcal{A}$. These objects constitute a full subcategory $\mathcal{C}$ of $\mathbf{mod}_1(\mathcal{A})$, with commutative squares modulo homotopy as morphisms. Then $\mathcal{R}(\mathcal{A})$ is a category of fractions [15] of $\mathcal{C}$ such that exact squares (i. e. morphisms in $\mathcal{C}$ which are regular in $\mathbf{mod}_1(\mathcal{A})$) are made invertible. Since this description of $\mathcal{R}(\mathcal{A})$ is entirely self-dual, we obtain a pair of full embeddings:

$$\mathsf{Q}_l(\mathcal{A}) \hookleftarrow \mathcal{R}(\mathcal{A}) \hookrightarrow \mathsf{Q}_r(\mathcal{A}).$$

Thus we infer that there is a torsion theory $(\mathcal{R}(\mathcal{A}), \mathcal{F})$ of $\mathsf{Q}_l(\mathcal{A})$, and a torsion theory $(\mathcal{T}, \mathcal{R}(\mathcal{A}))$ of $\mathsf{Q}_r(\mathcal{A})$, such that the categories $\mathcal{T}$ and $\mathcal{F}$ are both equivalent to $\mathcal{A}$. Let us call $\mathcal{R}(\mathcal{A})$ the *regular category* of $\mathcal{A}$.

Proposition 13 *The regular category $\mathcal{R}(\mathcal{A})$ of an almost abelian category $\mathcal{A}$ is almost abelian. $\mathcal{A}$ is abelian if and only if $\mathcal{R}(\mathcal{A}) = 0$, and $\mathcal{A}$ is integral if and only if $\mathcal{R}(\mathcal{R}(\mathcal{A})) = 0$.*

Proof. Since $\mathcal{R}(\mathcal{A})$ is the torsionfree class of a torsion theory for $\mathsf{Q}_r(\mathcal{A})$, Theorem 2 implies that $\mathcal{R}(\mathcal{A})$ is almost abelian.

If $\mathcal{A}$ is abelian, then every monomorphism in $\mathcal{A}$ is a kernel. Hence $\mathcal{A}$ is equivalent to $\mathsf{Q}_l(\mathcal{A})$, and $\mathcal{R}(\mathcal{A}) = 0$. Conversely, $\mathcal{R}(\mathcal{A}) = 0$ implies that $\mathcal{A}$ is equivalent to the abelian category $\mathsf{Q}_l(\mathcal{A})$. If $\mathcal{A}$ is integral, then Theorem 2 implies that $\mathcal{R}(\mathcal{A})$ is closed with respect to subobjects in $Q_l(\mathcal{A})$, whence $\mathcal{R}(\mathcal{A})$ is abelian. If $\mathcal{A}$ is not integral, then there is a monomorphism $a : A \rightarrowtail B$ in $Q_l(\mathcal{A})$ with $B \in \mathcal{R}(\mathcal{A})$ but $A \notin \mathcal{R}(\mathcal{A})$. Hence if $c : B \twoheadrightarrow C$ is the cokernel of a in $Q_l(\mathcal{A})$, then $C \in \mathcal{R}(\mathcal{A})$, and there exists an exact sequence $T \rightarrowtail A \twoheadrightarrow F$ in $Q_l(\mathcal{A})$ with $T \in \mathcal{R}(\mathcal{A})$ and $F \in \mathcal{A}$, such that $k : T \rightarrowtail A \overset{a}{\rightarrowtail} B$ is the kernel of c in $\mathcal{R}(\mathcal{A})$. Now if $\mathcal{R}(\mathcal{A})$ would be abelian, then c would coincide with the cokernel of k in $Q_l(\mathcal{A})$, whence $A \cong T \in \mathcal{R}(\mathcal{A})$, a contradiction. □

For a functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories, define the *kernel* $\operatorname{Ker} F$ as the full subcategory of objects A in $\mathcal{A}$ with $F(A) = 0$, and the *image* $\operatorname{Im} F$ as the full subcategory of objects B in $\mathcal{B}$ which are isomorphic to some $F(A)$ with $A \in \mathcal{A}$.

Next we consider a pair $E \dashv F$ of adjoint functors

$$\mathcal{D} \underset{F}{\overset{E}{\rightleftarrows}} \mathcal{C} \tag{21}$$

between abelian categories $\mathcal{C}$ and $\mathcal{D}$. Let $\mathcal{D}_{\text{ref}}$ denote the full subcategory of *reflexive* objects in $\mathcal{D}$, i. e. those objects D for which the unit morphism $\eta_D : D \to FE(D)$ is an isomorphism. Dually, $\mathcal{C}_{\text{ref}}$ denotes the full subcategory of reflexive objects in $\mathcal{C}$. Then the restrictions of E and F yield an equivalence between $\mathcal{D}_{\text{ref}}$ and $\mathcal{C}_{\text{ref}}$. We shall say that $E \dashv F$ is a *left (right) almost equivalence* if for each object D in $\mathcal{D}$, the unit morphism η_D is epic with $E(\operatorname{Ker} \eta_D) = 0$ (resp. for each object C in $\mathcal{C}$, the counit morphism $\varepsilon_C : EF(C) \to C$ is monic with $F(\operatorname{Cok} \varepsilon_C) = 0$). A left and right almost equivalence (21) will be called an *almost equivalence* $E \dashv F$. Now we have the following representation theorem for almost abelian categories:

Theorem 3 *Let (21) be a left almost equivalence between abelian categories, with E left adjoint to F. Then the category $\mathcal{D}_{\text{ref}}$ of reflexive objects in $\mathcal{D}$ is almost abelian and coincides with* $\operatorname{Im} F$. *Conversely, for each almost abelian category $\mathcal{A}$, there exists an almost equivalence (21) with $\mathcal{D} = Q_l(\mathcal{A})$ and $\mathcal{C} = Q_r(\mathcal{A})$, such that $\mathcal{A} \approx \operatorname{Im} F \approx \operatorname{Im} E$ and $\mathcal{R}(\mathcal{A}) \approx \operatorname{Ker} E \approx \operatorname{Ker} F$.*

Proof. Suppose first that (21) is a left almost equivalence. Then the composition $F \overset{\eta F}{\twoheadrightarrow} FEF \overset{F\varepsilon}{\to} F$ is the identity, whence ηF is an isomorphism. Therefore, $\mathcal{D}_{\text{ref}} = \operatorname{Im} F$. Since η_D is epic for every D, it follows that $\operatorname{Im} F$ is a pre-torsionfree class in $\mathcal{D}$. By the adjunction $E \dashv F$, each morphism $\operatorname{Ker} \eta_D \to F(C)$ corresponds to a morphism $E(\operatorname{Ker} \eta_D) \to C$ which is zero by assumption. Therefore, Lemma 7 implies that $(\operatorname{Ker} E, \operatorname{Im} F)$ is a torsion theory in $\mathcal{D}$, whence $\mathcal{D}_{\text{ref}}$ is almost abelian by Theorem 2.

To prove the converse, let $\mathcal{A}$ be an almost abelian category. Then the inclusion $J : \mathcal{A} \hookrightarrow \mathrm{Q}_r(\mathcal{A})$ admits a right adjoint $T : \mathrm{Q}_r(\mathcal{A}) \to \mathcal{A}$, and the inclusion $J' : \mathcal{A} \hookrightarrow \mathrm{Q}_l(\mathcal{A})$ has a left adjoint T'. By Theorem 2 it follows immediately that the functors $E := JT'$ and $F := J'T$ yield an almost equivalence

$$\mathrm{Q}_l(\mathcal{A}) \underset{F}{\overset{E}{\rightleftarrows}} \mathrm{Q}_r(\mathcal{A})$$

with the desired properties. □

5 PI-categories

If an abelian category $\mathcal{A}$ has enough projectives or injectives, we know that the whole category is determined by $\mathbf{Proj}(\mathcal{A})$ or $\mathbf{Inj}(\mathcal{A})$, and an object in $\mathcal{A}$ is given by a morphism in $\mathbf{Proj}(\mathcal{A})$ or $\mathbf{Inj}(\mathcal{A})$, respectively. For an almost abelian category $\mathcal{A}$, a similar representation is possible if the variety $\mathbf{Proj}(\mathcal{A})$ or $\mathbf{Inj}(\mathcal{A})$ is replaced by a variety $\mathcal{V}$ consisting of projectives *and* injectives. Then an object in $\mathcal{A}$ will be given by a morphism $P \to I$ in $\mathcal{V}$ with P projective and I injective.

Let us define a *PI-system* in an additive category $\mathcal{A}$ as a pair $(\mathcal{P}, \mathcal{I})$ with $\mathcal{P} \subset \mathbf{Proj}(\mathcal{A})$, $\mathcal{I} \subset \mathbf{Inj}(\mathcal{A})$ such that the following axioms are satisfied:

(PI_0) $\mathcal{P}$ and $\mathcal{I}$ are closed w.r.t. biproducts and direct summands.

(PI_1) Every A in $\mathcal{A}$ has a $\mathcal{P}$-cover $P \to A$ and an $\mathcal{I}$-hull $A \to I$.

(PI_2) Every $\mathcal{P}$-epic and $\mathcal{I}$-monic f in $\mathcal{A}$ is an isomorphism.

Clearly, if $(\mathcal{P}, \mathcal{I})$ with $\mathcal{P} \subset \mathbf{Proj}(\mathcal{A})$ and $\mathcal{I} \subset \mathbf{Inj}(\mathcal{A})$ satisfies (PI_1) and (PI_2), then $(\operatorname{add} \mathcal{P}, \operatorname{add} \mathcal{I})$ is a PI-system. Therefore, the relevant axioms are (PI_1) and (PI_2). A preabelian category with a PI-system $(\mathcal{P}, \mathcal{I})$ will be called a *PI-category*. We define $\mathcal{P}(\mathcal{A}) := \mathcal{P}$ and $\mathcal{I}(\mathcal{A}) := \mathcal{I}$.

Proposition 14 *Let $\mathcal{A}$ be a PI-category with PI-system $(\mathcal{P}, \mathcal{I})$. Then $\mathcal{P}$-epimorphisms and $\mathcal{I}$-monomorphisms are both stable under pullback and pushout.*

Proof. We prove the assumption for a pullback (2). Suppose first that d is a $\mathcal{P}$-epimorphism, and let $p : P \to B$ be a morphism with $P \in \mathcal{P}$. then cp factors through d, whence p factors through a. Hence, a is $\mathcal{P}$-epic.

Next let d be $\mathcal{I}$-monic, and let $i : A \to I$ be a morphism with $I \in \mathcal{I}$. Then i factors through the kernel $\binom{a}{b} : A \to B \oplus C$ of $(-c \ \ d) : B \oplus C \to D$, i. e. there exists a morphism $(r \ \ s) : B \oplus C \to I$ with $i = ra + sb$. Since d is $\mathcal{I}$-monic, $s = td$ for some $t : D \to I$, and thus $i = ra + tdb = (r + tc)a$. Hence a is $\mathcal{I}$-monic. □

As a consequence we have

Proposition 15 *Let $\mathcal{A}$ be a PI-category with PI-system $(\mathcal{P}, \mathcal{I})$. Then $\mathcal{P}$ is a projective, $\mathcal{I}$ an injective variety, and $\mathcal{A}$ is almost abelian. Moreover, every $\mathcal{P}$-epimorphism ($\mathcal{I}$-monomorphism) is epic (monic). If $p : P \to A$ is a $\mathcal{P}$-cover, and $i : A \to I$ an $\mathcal{I}$-hull, then up to isomorphism, A is uniquely determined by the morphism $ip : P \to I$.*

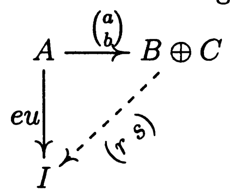
Proof. By (PI_0), $\mathcal{P}$ and $\mathcal{I}$ are varieties. Since $\mathcal{A}$ has kernels and cokernels, and by (PI_1), $\mathcal{P}$ is a projective, and $\mathcal{I}$ an injective variety. In order to show that $\mathcal{A}$ is almost abelian, consider a pullback

$$\begin{array}{ccccc} K & \overset{i}{\rightarrowtail} & A & \overset{a}{\longrightarrow} & B \\ \| & & \downarrow b & \text{PB} & \downarrow c \\ K & \underset{j}{\rightarrowtail} & C & \underset{d}{\twoheadrightarrow} & D \end{array}$$

with $i = \ker a$, and a cokernel d. Then a has a decomposition

$$a : A \overset{u}{\twoheadrightarrow} E \overset{v}{\rightarrow} B$$

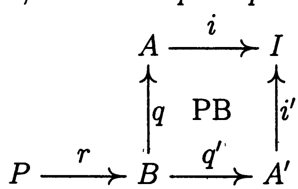
with $u = \operatorname{cok} i$. By Proposition 14, $v : E \to B$ is a $\mathcal{P}$-epimorphism. On the other hand, consider a morphism $e : E \to I$ with $I \in \mathcal{I}$. As in the proof of Proposition 14, there exists a lifting



with $eu = ra + sb$. Hence $sj = sbi = (eu - ra)i = 0$ implies $s = td$ for some $t : D \to I$. Therefore, $eu = ra + tdb = (r + tc)vu$ yields $e = (r + tc)v$, and thus v is $\mathcal{I}$-monic. By (PI_2), we infer that a is a cokernel. Thus we have shown that $\mathcal{A}$ is left almost abelian, and by symmetry, it follows that $\mathcal{A}$ is almost abelian.

Next lat $f : A \to B$ be a $\mathcal{P}$-epimorphism with cokernel $c : B \twoheadrightarrow C$. By ($PI_1$) there is a $\mathcal{P}$-cover $p : P \to C$. Then p factors through c and also through f. Hence $p = 0$, and thus $0 \to C$ is $\mathcal{P}$-epic and $\mathcal{I}$-monic. Consequently, (PI_2) implies $C = 0$, whence f is epic. Dually, every $\mathcal{I}$-monomorphism is monic.

Finally, let $p : P \to A$ and $p' : P \to A'$ be $\mathcal{P}$-covers, and let $i : A \to I$ and $i' : A' \to I$ be $\mathcal{I}$-hulls, such that $ip = i'p'$. Consider the pullback



with $p = qr$ and $p' = q'r$. Then q, q' are $\mathcal{P}$-epic, and by Proposition 14 also $\mathcal{I}$-monic. Hence q and q' are isomorphisms. □

We shall say that a PI-category $\mathcal{A}$ is *ample* if each morphism $A \to B$ in $\mathcal{A}$ has a decomposition $A \xrightarrow{p} E \xrightarrow{i} B$ with a $\mathcal{P}(\mathcal{A})$-epimorphism p, and an $\mathcal{I}(\mathcal{A})$-monomorphism i. By Proposition 15, such a decomposition is unique up to isomorphism.

Remark. Alternatively, an ample PI-category can be characterized as a preabelian category $\mathcal{A}$ with a *proper factorization system* $(\mathcal{E}, \mathcal{M})$ in the sense of [13] (cf. [23]) such that $\mathcal{E}$ is a *projective*, and $\mathcal{M}$ an *injective* class [11]. Namely, $\mathcal{E}$ (resp. $\mathcal{M}$) consists of the $\mathcal{P}(\mathcal{A})$-epimorphisms (resp. $\mathcal{I}(\mathcal{A})$-monomorphisms). Consequently, $\mathcal{P}(\mathcal{A})$ and $\mathcal{I}(\mathcal{A})$ determine each other.

Our next aim is to give a structure theorem for PI-categories. To this end, we define a *bivariety* as a variety $\mathcal{V}$ together with a pair $\mathcal{P} = \mathcal{P}(\mathcal{V})$ and $\mathcal{I} = \mathcal{I}(\mathcal{V})$ of object classes, closed with respect to isomorphisms, such that $(\mathcal{I}, \mathcal{P})$ is a *splitting* torsion theory, i. e. for each object A in $\mathcal{V}$, there is a split exact sequence

$$I \overset{u}{\rightarrowtail} A \overset{v}{\twoheadrightarrow} P$$

with an $\mathcal{I}$-cover u, and a $\mathcal{P}$-hull v. In other words, the following axioms are satisfied:

(BV_1) $\mathrm{Hom}_{\mathcal{V}}(I, P) = 0$ for all $P \in \mathcal{P}$ and $I \in \mathcal{I}$.

(BV_2) Each A in $\mathcal{V}$ is of the form $P \oplus I$ with $P \in \mathcal{P}$ and $I \in \mathcal{I}$.

For a morphism $f : A \to B$ in a bivariety $\mathcal{V}$, we define a *P-kernel* of f as a morphism $p : P \to A$ with $P \in \mathcal{P}$ and $fp = 0$ such that each $q : Q \to A$ with $Q \in \mathcal{P}$ and $fq = 0$ factors through p. Dually, an *I-cokernel* of f is a morphism $i : B \to I$ with $I \in \mathcal{I}$ and $if = 0$ such that each $j : B \to J$ with $J \in \mathcal{I}$ and $jf = 0$ factors through i. For an object $I \in \mathcal{I}$, a sequence $P_1 \overset{p}{\to} P_0 \overset{u}{\to} I$ will be called a *P-presentation* if u is a $\mathcal{P}$-cover and p a P-kernel of u. Dually, we define an *I-presentation* of an object $P \in \mathcal{P}$ as a sequence $P \overset{u}{\to} I_0 \overset{i}{\to} I_1$ with an $\mathcal{I}$-hull u and an I-cokernel i of u. We shall call $\mathcal{V}$ a *pre-PI-variety* if $\mathcal{V}$ is a bivariety with the additional property:

(BV_3) Each morphism in $\mathcal{V}$ has a P-kernel and an I-cokernel.

There is a functorial description of bivarieties. For varieties $\mathcal{P}$ and $\mathcal{I}$, consider a bifunctor

$$H : \mathcal{P}^{\mathrm{op}} \times \mathcal{I} \longrightarrow \mathcal{A}b \tag{22}$$

which is additive in each variable. For $P \in \mathcal{P}$ and $I \in \mathcal{I}$, an element $u \in H(P, I)$ can be interpreted as a morphism $u : P \to I$, and for morphisms $p : Q \to P$ and $i : I \to J$ in $\mathcal{P}$ and $\mathcal{I}$, respectively, the maps $H(p, I) : H(P, I) \to H(Q, I)$ and $H(P, i) : H(P, I) \to H(P, J)$ define a composition $u \circ p : Q \to I$ and $i \circ u : P \to J$. If we extend H by

$$H(P, Q) = \mathrm{Hom}_{\mathcal{P}}(P, Q); \quad H(I, J) = \mathrm{Hom}_{\mathcal{I}}(I, J); \quad H(I, P) = 0 \tag{23}$$

for $P, Q \in \mathcal{P}$ and $I, J \in \mathcal{I}$, then it is easily verified that H defines the Hom-functor of a bivariety $\mathcal{V}$ with $\mathcal{P}(\mathcal{V}) = \mathcal{P}$ and $\mathcal{I}(\mathcal{V}) = \mathcal{I}$. We shall denote this bivariety $\mathcal{V}$ by $\mathcal{P} \oplus_H \mathcal{I}$. Clearly, each bivariety is of this form.

Proposition 16 *A bivariety $\mathcal{V}$ is a pre-PI-variety if and only if $\mathcal{P}(\mathcal{V})$ is a projective, $\mathcal{I}(\mathcal{V})$ an injective variety, each $I \in \mathcal{I}(\mathcal{V})$ has a P-presentation, and each $P \in \mathcal{P}(\mathcal{V})$ has an I-presentation.*

Proof. If $\mathcal{V}$ is a bivariety, then $\mathcal{P}(\mathcal{V})$ and $\mathcal{I}(\mathcal{V})$ are *subvarieties*, i. e. closed with respect to biproducts and direct summands. Hence the condition for $\mathcal{V}$ to be pre-PI is necessary. To prove sufficiency, let $u : P \to I$ be a morphism with $P \in \mathcal{P}(\mathcal{V})$ and $I \in \mathcal{I}(\mathcal{V})$, and $P_1 \xrightarrow{p} P_0 \xrightarrow{v} I$ a P-presentation of I. Then $u = vr$ for some $r : P \to P_0$. Now let $\binom{s}{t} : P' \to P \oplus P_1$ be a weak kernel of $(r\ p) : P \oplus P_1 \to P_0$ in $\mathcal{P}(\mathcal{V})$. Then $us = vrs = -vpt = 0$, and it is readily seen that $s : P' \to P$ is a P-kernel of u.

Now consider a general morphism $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) : P_1 \oplus I_1 \to P_2 \oplus I_2$ in $\mathcal{V}$. Let $p : P \to P_1$ be a weak kernel of $a : P_1 \to P_2$ in $\mathcal{P}(\mathcal{V})$, and $q : Q \to I_1$ a $\mathcal{P}$-cover of I_1. By the above, there exists a P-kernel $\binom{p'}{q'} : P' \to P \oplus Q$ of $(bp\ cq) : P \oplus Q \to I_2$. Then it is easily verified that $\binom{pp'}{qq'} : P' \to P_1 \oplus I_1$ is a P-kernel of $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right)$. Thus we have shown that every morphism in $\mathcal{V}$ has a P-kernel. By duality, our proof is complete. □

For a projective resp. injective variety $\mathcal{P}$ and $\mathcal{I}$, a biadditive functor (22) will be called a $(\mathcal{P}, \mathcal{I})$-*bimodule* if $\mathcal{P} \oplus_H \mathcal{I}$ is a pre-PI-variety. We call (22) *small* if its values are in **Ab**. In fact, this definition generalizes the familiar concept of bimodule over rings:

Proposition 17 *Let R, S be rings, $\mathcal{P} =$ **Proj**$(R$-**Mod**$)$, and $\mathcal{I}^{\mathrm{op}} =$ **Proj**$($**Mod**-$S)$. Then each small $(\mathcal{P}, \mathcal{I})$-bimodule is of the form*

$$H(P, Q) = \mathrm{Hom}_R(P, \mathrm{Hom}_S(Q, U)) \qquad (24)$$

with $U = H({}_RR, S_S)$, an (R, S)-bimodule. Conversely, every (R, S)-bimodule ${}_RU_S$ yields a small $(\mathcal{P}, \mathcal{I})$-bimodule by (24).

Proof. Suppose H is a small $(\mathcal{P}, \mathcal{I})$-bimodule. Then $U := H({}_RR, S_S)$ is an (R, S)-bimodule since $R^{\mathrm{op}} \otimes S = \mathrm{End}({}_RR) \otimes \mathrm{End}(S_S)$ operates from the right on U. An I-presentation ${}_RR \to Q_0 \xrightarrow{i} Q_1$ in $\mathcal{P} \oplus_H \mathcal{I}$ induces an exact sequence of functors $\mathcal{I} \to$ **Ab**:

$$\mathrm{Hom}_{\mathcal{I}}(Q_1, -) \to \mathrm{Hom}_{\mathcal{I}}(Q_0, -) \twoheadrightarrow H({}_RR, -). \qquad (25)$$

Note that i corresponds to a homomorphism $q : Q_1 \to Q_0$ of right S-modules, and $\mathrm{Hom}_{\mathcal{J}}(Q, -) = \mathrm{Hom}_S(-, Q)$. Inserting S_S into (25) gives an exact sequence $Q_1 \xrightarrow{q} Q_0 \twoheadrightarrow U$ in **Mod**-S, and then (25) yields a natural isomorphism of functors on $\mathcal{J}$:

$$H({}_RR, -) = \mathrm{Hom}_S(-, U). \tag{26}$$

Similarly, each object Q in $\mathcal{J}$ has a P-presentation $P_1 \to P_0 \to Q$ in $\mathcal{P} \oplus_H \mathcal{J}$ which gives an exact sequence of functors $\mathcal{P}^{\mathrm{op}} \to \mathbf{Ab}$:

$$\mathrm{Hom}_R(-, P_1) \to \mathrm{Hom}_R(-, P_0) \twoheadrightarrow H(-, Q). \tag{27}$$

By virtue of (26), inserting of ${}_RR$ into (27) yields an exact sequence

$$P_1 \to P_0 \twoheadrightarrow \mathrm{Hom}_S(Q, U), \tag{28}$$

and the equivalence (24) follows by (27).

Conversely, let (24) be given. Then any projective presentation (28) of the R-module $\mathrm{Hom}_S(Q, U)$ gives an exact sequence (27), and by Yoneda's lemma, this amounts to a P-presentation $P_1 \to P_0 \to Q$ of Q in $\mathcal{P} \oplus_H \mathcal{J}$. In view of the natural isomorphism

$$\mathrm{Hom}_R(P, \mathrm{Hom}_S(Q, U)) = \mathrm{Hom}_S(Q, \mathrm{Hom}_R(P, U)), \tag{29}$$

a symmetry argument completes the proof. □

A bifunctor (22) will be called *non-degenerate* if for each morphism p in $\mathcal{P}$ and i in $\mathcal{J}$, the implications $H(p, -) = 0 \Rightarrow p = 0$ and $H(-, i) = 0 \Rightarrow i = 0$ are true. If H is a non-degenerate $(\mathcal{P}, \mathcal{J})$-bimodule, the associated pre-PI-variety $\mathcal{P} \oplus_H \mathcal{J}$ will be called a *PI-variety*.

Proposition 18 *For a $(\mathcal{P}, \mathcal{J})$-bimodule H, the following are equivalent:*

(a) *H is non-degenerate.*

(b) *Any $u \in H(P, I)$ is monic if u is $\mathcal{J}$-monic, and epic if u is $\mathcal{P}$-epic in $\mathcal{P} \oplus_H \mathcal{J}$.*

If $\mathcal{P} = \mathbf{Proj}(R\text{-}\mathbf{Mod})$ and $\mathcal{J}^{\mathrm{op}} = \mathbf{Proj}(\mathbf{Mod}\text{-}S)$ with rings R, S, then H is non-degenerate if and only if $U := H({}_RR, S_S)$ is faithful as a left R-module, and as a right S-module.

Proof. By Yoneda's lemma, an element $u \in H(P, I)$ corresponds to a natural transformation $\nu_u : \mathrm{Hom}_{\mathcal{P}}(-, P) \to H(-, I)$, and u is $\mathcal{P}$-epic in $\mathcal{P} \oplus_H \mathcal{I}$ if and only if ν_u is componentwise surjective. Hence in this case, a morphism $i : I \to J$ in $\mathcal{I}$ satisfies $iu = 0$ if and only if $H(-, i) = 0$. By duality, this proves (a) $\Leftrightarrow$ (b).

Now let H be of the form (24), and $q : Q' \to Q$ a morphism in **Mod**-S with corresponding $i : Q \to Q'$ in $\mathcal{I}$. Then $H(-, i) = 0$ is tantamount to $\mathrm{Hom}_S(q, U) = 0$, and the latter states that every S-linear map $Q \to U$ annihilates q. Clearly, if the implication $\mathrm{Hom}_S(q, U) = 0 \Rightarrow q = 0$ holds for $Q = S_S$, then it is also valid for all $q : Q' \to Q$. Hence, the implication $H(-, i) = 0 \Rightarrow i = 0$ just says that U_S is faithful. By duality, the proof is complete. □

Let us define the category $\mathbf{DS}(H)$ of *H-dual systems* (cf. §2.4) for any $(\mathcal{P}, \mathcal{I})$-bimodule H. Objects are the elements $u \in H(P, I)$ with $P \in \mathcal{P}$ and $I \in \mathcal{I}$. If $v \in H(Q, J)$, then the morphisms $u \to v$ are given by commutative squares

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow u & & \downarrow v \\ I & \xrightarrow{g} & J \end{array} \tag{30}$$

in $\mathcal{P} \oplus_H \mathcal{I}$, modulo such squares with $vf = gu = 0$. In the special case of Proposition 17, we also write $\mathbf{DS}(U)$ instead of $\mathbf{DS}(H)$.

By $\mathcal{P}(\mathbf{DS}(H))$ we denote the full subcategory of objects $u : P \to I$ in $\mathbf{DS}(H)$ for which u is an $\mathcal{I}$-monomorphism in $\mathcal{P} \oplus_H \mathcal{I}$. Dually, we define $\mathcal{I}(\mathbf{DS}(H))$. For given $P \in \mathcal{P}$ (resp. $I \in \mathcal{I}$), a particular choice of $u : P \to I$ in $\mathcal{P}(\mathbf{DS}(H))$ (resp. $\mathcal{I}(\mathbf{DS}(H))$) yields a pair of full and dense functors

$$\mathcal{P} \longrightarrow \mathcal{P}(\mathbf{DS}(H)); \quad \mathcal{I} \longrightarrow \mathcal{I}(\mathbf{DS}(H)). \tag{31}$$

Proposition 19 *Let $\mathcal{P}$ be a projective, $\mathcal{I}$ an injective variety, and H a $(\mathcal{P}, \mathcal{I})$-bimodule. Then* $\mathbf{DS}(H)$ *is an ample PI-category. Moreover, H is non-degenerate if and only if the functors (31) are equivalences.*

Proof. Clearly, $\mathbf{DS}(H)$ is an additive category. For a morphism (30), choose a weak kernel $p : P' \to P$ of vf. Then it is easily verified that

$$\begin{array}{ccc} P' & \xrightarrow{p} & P \\ \downarrow{\scriptstyle up} & & \downarrow{\scriptstyle u} \\ I & \xrightarrow{1} & I \end{array} \tag{32}$$

is a kernel of (30). By duality, we infer that $\mathbf{DS}(H)$ is a category with kernels and cokernels. Moreover, the representation of a kernel in the form (32) immediately implies:

$$\mathcal{P}(\mathbf{DS}(H)) \subset \mathbf{Proj}(\mathbf{DS}(H)); \quad \mathcal{J}(\mathbf{DS}(H)) \subset \mathbf{Inj}(\mathbf{DS}(H)).$$

Obviously, (PI_0) is satisfied for these subcategories. For an object $u : P \to I$, choose a $\mathcal{P}$-cover $P' \to I$ and an $\mathcal{J}$-hull $P \to I'$ in $\mathcal{P} \oplus_H \mathcal{J}$. Then (PI_1) follows by the commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{1} & P & \longrightarrow & P' \\ \downarrow & & \downarrow{\scriptstyle u} & & \downarrow \\ I' & \longrightarrow & I & \xrightarrow{1} & I. \end{array}$$

Next let (30) be a $\mathcal{P}(\mathbf{DS}(H))$-epimorphism. If $v' : Q \to J'$ is in $\mathcal{P}(\mathbf{DS}(H))$, then v factors through v', and the morphism

$$\begin{array}{ccc} Q & \xrightarrow{1} & Q \\ \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} \\ J' & \longrightarrow & J \end{array}$$

in $\mathbf{DS}(H)$ factors through (30). Hence there is a morphism $r : Q \to P$ with $v(1 - fr) = 0$. Similarly, if (30) is $\mathcal{J}(\mathbf{DS}(H))$-monic, then there exists a morphism $s : J \to I$ with $(1 - sg)u = 0$. Consequently, $ur = sgur = svfr = sv$, and the pair (r, s) yields a morphism $v \to u$. Furthermore, $v = vfr = gur$ implies $(1 - gs)v = (1 - gs)gur = g(1 -$

$sg)ur = 0$, and $u = sgu = svf$ yields $u(1-rf) = svf(1-rf) = sv(1-fr)f = 0$. Hence (30) is an isomorphism, and (PI_2) is verified.

Since each morphism (30) admits a decomposition

$$\begin{array}{ccccc} P & \xrightarrow{1} & P & \xrightarrow{f} & Q \\ \downarrow{\scriptstyle u} & & \downarrow & & \downarrow{\scriptstyle v} \\ I & \xrightarrow{g} & J & \xrightarrow{1} & J \end{array} \tag{33}$$

into a $\mathcal{P}(\mathbf{DS}(H))$-epimorphism and an $\mathcal{I}(\mathbf{DS}(H))$-monomorphism, we infer that $\mathbf{DS}(H)$ is ample. Now Proposition 18 completes the proof. □

Proposition 20 *For a PI-category $\mathcal{A}$, the* Hom*-functor*

$$\mathrm{Hom}_{\mathcal{A}} : \mathcal{P}(\mathcal{A})^{\mathrm{op}} \times \mathcal{I}(\mathcal{A}) \longrightarrow \mathcal{A}b \tag{34}$$

is a non-degenerate bimodule.

Proof. Trivially, (34) is a bimodule. The non-degeneracy follows by Proposition 18 since every $\mathcal{P}$-epimorphism ($\mathcal{I}$-monomorphism) is epic (monic). □

Let us define a *PI-subcategory* $\mathcal{A}$ of a PI-category $\mathcal{B}$ as a full subcategory which contains $\mathcal{P}(\mathcal{B}) \cup \mathcal{I}(\mathcal{B})$ and is closed with respect to biproducts, kernels, and cokernels. Thus $\mathcal{A}$ is again a PI-category with $\mathcal{P}(\mathcal{A}) = \mathcal{P}(\mathcal{B})$ and $\mathcal{I}(\mathcal{A}) = \mathcal{I}(\mathcal{B})$. A fully faithful, fully exact functor $F : \mathcal{A} \to \mathcal{B}$ between PI-categories will be called a *PI-embedding* if F induces equivalences $\mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ and $\mathcal{I}(\mathcal{A}) \to \mathcal{I}(\mathcal{B})$. If F is dense, we speak of a *PI-equivalence.* Similarly, we define a *PI-equivalence* $\mathcal{P} \oplus_H \mathcal{I} \xrightarrow{\sim} \mathcal{P}' \oplus_{H'} \mathcal{I}'$ between pre-PI-varieties as an equivalence which restricts to equivalences $\mathcal{P} \xrightarrow{\sim} \mathcal{P}'$ and $\mathcal{I} \xrightarrow{\sim} \mathcal{I}'$. Now we are ready to prove:

Theorem 4 *Each PI-category $\mathcal{A}$ admits a PI-embedding $\mathcal{A} \hookrightarrow \widetilde{\mathcal{A}}$ into an ample PI-category $\widetilde{\mathcal{A}}$, and every such embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ extends to a PI-equivalence $\widetilde{\mathcal{A}} \xrightarrow{\sim} \mathcal{B}$. Up to PI-equivalence, ample PI-categories are in one-to-one correspondence with PI-varieties.*

Proof. Suppose $\mathcal{A}$ is a PI-category. Then (34) is a non-degenerate $(\mathcal{P}(\mathcal{A}), \mathcal{I}(\mathcal{A}))$-bimodule. Since for each object A in $\mathcal{A}$, we can choose a $\mathcal{P}(\mathcal{A})$-cover $p : P \to A$ and an $\mathcal{I}(\mathcal{A})$-hull $i : A \to I$, there is a functor

$$F : \mathcal{A} \longrightarrow \widetilde{\mathcal{A}} := \mathbf{DS}(\mathrm{Hom}_{\mathcal{A}}) \tag{35}$$

with $F(A) := ip$, and F is faithful by Proposition 15. In order to show that F is full, consider a commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{p} & A & \xrightarrow{i} & I \\ \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle g} \\ Q & \xrightarrow{q} & B & \xrightarrow{j} & J \end{array} \tag{36}$$

with $\mathcal{P}(\mathcal{A})$-covers p, q, and $\mathcal{I}(\mathcal{A})$-hulls i, j. We take the pushout

$$\begin{array}{ccccc} P & \xrightarrow{p} & A & \xrightarrow{i} & I \\ \downarrow{\scriptstyle f} & \text{PO} & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle g} \\ Q & \xrightarrow{r} & C & \xrightarrow{t} & J \end{array}$$

with $gi = ts$ and $jq = tr$, and then the pullback

$$\begin{array}{ccccc} Q & \xrightarrow{r} & C & \xrightarrow{t} & J \\ & \searrow^{w} & \uparrow{\scriptstyle u} & \text{PB} & \uparrow{\scriptstyle j} \\ & & D & \xrightarrow{v} & B \end{array}$$

with $q = vw$ and $r = uw$. By Proposition 14 we infer that r is $\mathcal{P}(\mathcal{A})$-epic, and u is $\mathcal{I}(\mathcal{A})$-monic. Hence u is an isomorphism. Consequently, $vu^{-1}s : A \to B$ splits (36) into two commutative squares, and thus F is full. By the description of kernels (32) and cokernels in $\mathbf{DS}(\mathrm{Hom}_{\mathcal{A}})$, it follows easily that F is fully exact. Hence, Proposition 19 and 20 imply that (35) is a PI-embedding. Furthermore, F is a PI-equivalence if and only if $\mathcal{A}$ is ample. Therefore, any other PI-embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ into an ample PI-category $\mathcal{B}$ leads to PI-equivalent categories $\widetilde{\mathcal{A}} \approx \widetilde{\mathcal{B}} \approx \mathcal{B}$.

Finally, if we start with a PI-variety $\mathcal{P} \oplus_H \mathcal{I}$, then H is a non-degenerate $(\mathcal{P}, \mathcal{I})$-bimodule. By Proposition 19, $\mathbf{DS}(H)$ is an ample PI-category, and (31) are equivalences. Under these equivalences, the bimodule (34) corresponds to H, i. e. $\mathcal{P} \oplus_H \mathcal{I}$ is recovered by $\mathbf{DS}(H)$. □

Note: In a PI-category $\mathcal{A}$, the varieties $\mathcal{P}(\mathcal{A})$ and $\mathcal{I}(\mathcal{A})$ may have a non-trivial intersection, in fact, they may coincide. In the corresponding PI-variety $\mathcal{P}(\mathcal{A}) \oplus_H \mathcal{I}(\mathcal{A})$, however, these varieties are disjoint. Nevertheless, the intersection $\mathcal{P}(\mathcal{A}) \cap \mathcal{I}(\mathcal{A})$ in $\mathcal{A}$ is determined by the PI-variety.

By the preceding theorem, the ample PI-categories $\mathcal{A}$ are just the maximal ones, i. e. those for which each PI-embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ into a PI-category $\mathcal{B}$ is an equivalence. In contrast to this, let us define a PI-category to be *minimal* if it has no proper PI-subcategories. Like for the ample PI-categories, the theorem establishes a one-to-one correspondence between minimal PI-categories and PI-varieties, up to PI-equivalence. Moreover, any PI-category $\mathcal{A}$ has a smallest (hence minimal) PI-subcategory $\underset{\sim}{\mathcal{A}}$. For a $(\mathcal{P}, \mathcal{I})$-bimodule H and $\mathcal{A} = \mathbf{DS}(H)$, this category $\underset{\sim}{\mathcal{A}}$ will be denoted by $\mathbf{Lat}(H)$. Its objects can be viewed as generalized lattices (cf. Example 4 of §7).

Proposition 21 *A PI-subcategory $\mathcal{C}$ of a PI-category $\mathcal{A}$ is closed with respect to subquotients. In particular, the smallest PI-subcategory of $\mathcal{A}$ is $\underset{\sim}{\mathcal{A}} = \mathrm{ab}(\mathcal{P}(\mathcal{A}) \cup \mathcal{I}(\mathcal{A}))$.*

Proof. Since every kernel $A \rightarrowtail B$ in $\mathcal{A}$ is a kernel of a morphism $B \to I$ with $I \in \mathcal{I}(\mathcal{A})$, and every cokernel $B \twoheadrightarrow C$ is a cokernel of a morphism $P \to B$ with $P \in \mathcal{P}(\mathcal{A})$, a PI-subcategory $\mathcal{C}$ of $\mathcal{A}$ is closed with respect to subquotients. Now the proposition follows. □

A particular class of minimal PI-categories will be discussed in §7.

6 Bimodules and adjoint functors

Continuing the results of the last section, we shall now establish a correspondence between ample PI-categories and pairs of adjoint functors between $\mathbf{mod}(\mathcal{P})$ and $\mathbf{com}(\mathcal{I})$.

Proposition 22 *Let $\mathcal{C}, \mathcal{D}$ be abelian categories, $\mathcal{C}$ with enough injectives, and $\mathcal{D}$ with enough projectives. Let $E : \mathcal{D} \to \mathcal{C}$ be a functor which is left adjoint to $F : \mathcal{C} \to \mathcal{D}$. Then*

$$H(P, I) := \mathrm{Hom}_{\mathcal{C}}(E(P), I) = \mathrm{Hom}_{\mathcal{D}}(P, F(I)) \qquad (37)$$

is a $(\mathbf{Proj}(\mathcal{D}), \mathbf{Inj}(\mathcal{C}))$-bimodule.

Proof. For any object I in $\mathbf{Inj}(\mathcal{C})$, a projective presentation $P_1 \to P_0 \twoheadrightarrow F(I)$ in $\mathcal{D}$ yields a P-presentation of I in $\mathbf{Proj}(\mathcal{D}) \oplus_H \mathbf{Inj}(\mathcal{C})$. By duality and Proposition 16, the assertion follows. □

In particular, every adjoint pair $E \dashv F$ defines an ample PI-category $\mathbf{DS}(H)$, and a minimal PI-category $\mathbf{Lat}(H)$. Conversely, let H be a $(\mathcal{P}, \mathcal{I})$-bimodule, and $\mathcal{A}$ an arbitrary PI-category corresponding to H, i. e. up to PI-equivalence, $\mathcal{A}$ can be realized as a PI-subcategory of $\mathbf{DS}(H)$. We define a pair of functors

$$\mathbf{mod}(\mathcal{P}) \xrightarrow{H_\circ} \mathcal{A} \xleftarrow{H^\circ} \mathbf{com}(\mathcal{I}) \tag{38}$$

as follows. The functors (31) induce full and dense functors

$$H_\circ : \mathcal{P} \longrightarrow \mathcal{P}(\mathcal{A}); \quad H^\circ : \mathcal{I} \longrightarrow \mathcal{I}(\mathcal{A}) \tag{39}$$

which define (38) on $\mathcal{P}$ and $\mathcal{I}$, respectively. For an object M in $\mathbf{mod}(\mathcal{P})$, choose a projective presentation $P_1 \to P_0 \twoheadrightarrow M$. Then $H_\circ(M)$ is given by the cokernel $H_\circ(P_1) \to H_\circ(P_0) \twoheadrightarrow H_\circ(M)$ in $\mathcal{A}$, and H° is defined dually. If $\mathcal{A}$ is abelian and H non-degenerate, $H_\circ$ coincides with (12).

In addition, we introduce a pair of functors into the reverse direction:

$$\mathbf{mod}(\mathcal{P}) \xleftarrow{S_\circ} \mathcal{A} \xrightarrow{S^\circ} \mathbf{com}(\mathcal{I}). \tag{40}$$

Since $\mathcal{A}$ is fully embedded into $\mathbf{DS}(H)$, each object A in $\mathcal{A}$ is given by a morphism $u \in H(P, I)$ in $\mathcal{P} \oplus_H \mathcal{I}$. Let $p : P' \to P$ be a P-kernel of u. Then we define $S_\circ(A)$ as the cokernel $\operatorname{Cok} p$ in $\mathbf{mod}(\mathcal{P})$. Dually, $S^\circ(A)$ is defined as the kernel of an I-cokernel of u.

Note: The definition of the functors (38) and (40) takes its simplest form in terms of the pre-PI-variety $\mathcal{P} \oplus_H \mathcal{I}$. Namely, if the objects of $\mathbf{mod}(\mathcal{P})$ are given by projective presentations $p : P_1 \to P_0$ in $\mathcal{P}$, and the objects of $\mathcal{A}$ are given by morphisms $u \in H(P, I)$ in $\mathcal{P} \oplus_H \mathcal{I}$, then $H_\circ(p)$ is obtained as an I-cokernel of p, and $S_\circ(u)$ is a P-kernel of u.

By this remark it follows immediately that $(H_\circ, S_\circ)$ and (S°, H°) are adjoint pairs of functors:

$$H_\circ \dashv S_\circ; \quad S^\circ \dashv H^\circ. \tag{41}$$

Hence the composition $E := S^\circ H_\circ$ and $F := S_\circ H^\circ$ yields a pair of adjoint functors $E \dashv F$:

$$\mathbf{mod}(\mathcal{P}) \underset{F}{\overset{E}{\rightleftarrows}} \mathbf{com}(\mathcal{I}). \tag{42}$$

Proposition 23 *Let $\mathcal{P}$ be a projective, $\mathcal{I}$ an injective variety. Then (37) and (42) give a one-to-one correspondence between $(\mathcal{P},\mathcal{I})$-bimodules H and adjoint pairs $E \dashv F$, up to natural isomorphism. Moreover, H is non-degenerate if and only if E, F are faithful on $\mathcal{P}$ and $\mathcal{I}$, respectively.*

Proof. If H is given, then $E = S^\circ H_\circ$ and $F = S_\circ H^\circ$ yield the $(\mathcal{P},\mathcal{I})$-bimodule $\mathrm{Hom}_{\mathbf{mod}(\mathcal{P})}(P, S_\circ H^\circ I) = \mathrm{Hom}_{\mathcal{A}}(H_\circ P, H^\circ I) = H(P,I)$. Conversely, an arbitrary adjoint functor pair (42) defines a $(\mathcal{P},\mathcal{I})$-bimodule (37). For $P \in \mathcal{P}$, a projective resolution in $\mathbf{mod}(\mathcal{P})$ is $0 \to P$, and an I-cokernel of $0 \to P$ is given by the unit morphism $u_P : P \to F(I)$ of the adjunction $E \dashv F$, with $I = E(P)$. Thus $H_\circ(P)$ is given by the object u_P in $\mathbf{Lat}(H)$. By (37), u_P can also be represented by the identical morphism $1 : E(P) \to I$, which has an I-cokernel $I \to 0$ in $\mathcal{P} \oplus_H \mathcal{I}$. Hence $S^\circ H_\circ(P)$ coincides with $E(P)$.

Finally, H is non-degenerate if and only if for each $P \in \mathcal{P}$, the unit morphism $\eta_P : P \to FE(P)$ of $E \dashv F$ is monic, and for each $I \in \mathcal{I}$, the counit morphism $EF(I) \to I$ is epic. Clearly, if η_P is monic for all $P \in \mathcal{P}$, then E is faithful on $\mathcal{P}$. Conversely, suppose E is faithful on $\mathcal{P}$. If η_P would not be monic, there would exist a non-zero morphism $q : Q \to P$ in $\mathcal{P}$ with $\eta_P q = 0$. This would imply $E(q) = 0$, whence $q = 0$, a contradiction. By duality, we are done. □

Remark. If $\mathcal{P} = \mathbf{Proj}(R\text{-}\mathbf{Mod})$ and $\mathcal{I}^{\mathrm{op}} = \mathbf{Proj}(\mathbf{Mod}\text{-}S)$ with rings R, S, then the explicit description (24) of H yields

$$E = \mathrm{Hom}_R(-,U); \quad F = \mathrm{Hom}_S(-,U) \tag{43}$$

with $U = H({}_RR, S_S)$. Then $\mathbf{com}(\mathcal{I}) = (\mathbf{Mod}\text{-}S)^{\mathrm{op}}$.

Recall that a class $\mathcal{C}$ of objects in an additive category is said to *(co-)generating* if for every morphism $f : A \to B$ which annihilates each morphism $C \to A$ (resp. $B \to C$) with $C \in \mathcal{C}$, it follows that $f = 0$.

Now let H be a $(\mathcal{P},\mathcal{I})$-bimodule. We shall introduce three properties of H which generalize the relevant concepts for Morita duality. Firstly, an object $P \in \mathcal{P}$ will be called *strict* if there is an $\mathcal{I}$-hull $P \to I$ in $\mathcal{P} \oplus_H \mathcal{I}$ which is a P-kernel. (In this case, each $\mathcal{I}$-hull $P \to I$ is a P-kernel!) Then we define H to be *left balanced* if $\mathcal{P}$ is generated by the strict objects in $\mathcal{P}$. Secondly, we call H a *left cogenerator* if every morphism $P' \to P$ in $\mathcal{P}$ is a P-kernel of some $u \in H(P,I)$ in $\mathcal{P} \oplus_H \mathcal{I}$. Thirdly, we shall say that H is *left injective* if every morphism $p : P \to P'$ in $\mathcal{P}$ has the property that each morphism $u \in H(P, I)$ in $\mathcal{P} \oplus_H \mathcal{I}$ which annihilates a P-kernel of p factors through p. The corresponding right notions are defined dually. If the respective left and right property is satisfied, we simply speak of a *balanced*, *cogenerating*, or *injective* bimodule H.

Proposition 24 *Let R, S be rings, $\mathcal{P} =$ $\mathbf{Proj}(R\text{-}\mathbf{Mod})$), and $\mathcal{I}^{\text{op}} =$ $\mathbf{Proj}(\mathbf{Mod}\text{-}S)$. For a small $(\mathcal{P}, \mathcal{I})$-bimodule H and its corresponding (R, S)-bimodule $U = H({}_RR, S_S)$, the following equivalences hold:*

(a) *H is left (right) balanced* $\Leftrightarrow$ *${}_RU_S$ is balanced.*
(b) *H is a left (right) cogenerator* $\Leftrightarrow$ *${}_RU$ (resp. U_S) is a cogenerator.*
(c) *H is left (right) injective* $\Leftrightarrow$ *${}_RU$ (resp. U_S) is injective.*

Proof. By duality, it suffices to prove the "left" statements.

(a) Since an I-presentation of $P \in \mathcal{P}$ coincides with a projective presentation of the S-module $\text{Hom}_R(P, U)$, it follows that P is strict if and only if for each $Q \in \mathcal{P}$, the S-linear maps $\text{Hom}_R(P, U) \to \text{Hom}_R(Q, U)$ are induced by R-linear maps $Q \to P$. But this assertion holds for all $Q \in \mathcal{P}$ iff it is true for $Q = R$. Hence P is strict if and only if the natural map

$$P \longrightarrow \text{Hom}_S(\text{Hom}_R(P, U), U)$$

is epic. For $P = R$ this says that $R \to \text{End}(U_S)$ is surjective. On the other hand, ${}_RR$ is strict if and only if $\mathcal{P}$ is generated by strict objects.

(b) By definition, H is a left cogenerator if and only if for each morphism $p : P' \to P$ in $\mathcal{P}$ there exists an exact sequence of R-modules $P' \xrightarrow{p} P \to \text{Hom}_S(Q, U)$ with $Q \in \mathcal{I}^{\text{op}}$. Here, Q can be replaced by a free S-module, and then the assertion states that the cokernel of p is cogenerated by ${}_RU$.

(c) Finally, H is left injective iff each R-module $\text{Hom}_S(Q, U)$ with $Q \in \mathcal{I}^{\text{op}}$ is injective, that is, if ${}_RU$ is injective. $\square$

By virtue of (38) and (40), the above concepts have the following functorial description:

Proposition 25 *Let H be a $(\mathcal{P}, \mathcal{I})$-bimodule, and $\mathcal{A}$ a PI-subcategory of $\mathbf{DS}(H)$. With the notations of (38), (40), and (42), the following are equivalent:*

(a) *H is a left cogenerator* (c) $H_\circ$ *is faithful*
(b) *E is faithful* (d) $S_\circ H_\circ \cong 1$

Proof. (a) ⇒ (b): Recall that for an object M in $\mathbf{mod}(\mathcal{P})$, given by a projective presentation $p : P_1 \to P_0$, an I-cokernel $u : P_0 \to I_0$ of p represents $H_\circ(M)$ in $\mathbf{DS}(H)$, and an I-cokernel $i : I_0 \to I_1$ of u yields an injective presentation of $E(M)$. Therefore, the application of E to a morphism (f, g) in $\mathbf{mod}(\mathcal{P})$ is given by a commutative diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{p} & P_0 & \xrightarrow{u} & I_0 & \xrightarrow{i} & I_1 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow \\ Q_1 & \xrightarrow{q} & Q_0 & \xrightarrow{v} & J_0 & \xrightarrow{j} & J_1 \end{array}$$

with an I-cokernel v of q, and an I-cokernel j of v. If E carries (f, g) to zero, then h factors through i. Hence $vf = hu = 0$, and (a) implies that f factors through q, i. e. (f, g) represents a zero morphism in $\mathbf{mod}(\mathcal{P})$.

(c) ⇒ (a): Let $p : P' \to P$ be a morphism in $\mathcal{P}$ and $u \in H(P, I)$ an I-cokernel of p. In order to show that p is a P-kernel of u, let $q : Q \to P$ be a morphism in $\mathcal{P}$ with $uq = 0$. For an $\mathcal{J}$-monomorphism $u_0 : Q \to I_0$ we obtain a commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & Q & \xrightarrow{u_0} & I_0 \\ \downarrow & & \downarrow{\scriptstyle q} & & \downarrow \\ P' & \xrightarrow{p} & P & \xrightarrow{u} & I \end{array}$$

Since $H_\circ$ is faithful, it follows that q factors through p.

The implications (a) ⇒ (d) ⇒ (c) ⇐ (b) are trivial. □

By Proposition 23, we infer

Corollary. *If a $(\mathcal{P}, \mathcal{J})$-bimodule H is a cogenerator, then H is non-degenerate.*

Proposition 26 *Let H be a $(\mathcal{P}, \mathcal{J})$-bimodule, and $\mathcal{A}$ a PI-subcategory of* $\mathbf{DS}(H)$. *With the associated functors (42), the left- and right-hand statements are equivalent:*

(a) *H is left injective*
(b) *E is exact*
(c) $F(\mathcal{I}) \subset \mathbf{Inj}(\mathbf{mod}(\mathcal{P}))$

(a′) *H is right injective*
(b′) *F is exact*
(c′) $E(\mathcal{P}) \subset \mathbf{Proj}(\mathbf{com}(\mathcal{I}))$.

Proof. We prove the equivalence of the left-hand statements.

(a) $\Rightarrow$ (c): Let $i : M \rightarrowtail P$ be a monomorphism in $\mathbf{mod}(\mathcal{P})$ with $P \in \mathcal{P}$, and $P_1 \xrightarrow{p'} P_0 \overset{p}{\twoheadrightarrow} M$ a projective presentation of M. Then (a) states that every morphism $P_0 \to F(I)$ with $I \in \mathcal{I}$ which annihilates p' factors through ip. Hence every $M \to F(I)$ factors through i. This implies that $F(I)$ is injective in $\mathbf{mod}(\mathcal{P})$.

(c) $\Rightarrow$ (b): Since E is left adjoint to F, we already know that E is right exact. Therefore, it suffices to prove that E respects monomorphisms. Thus let $i : M \rightarrowtail N$ be a monomorphism in $\mathbf{mod}(\mathcal{P})$, and $k : K \to E(M)$ the kernel of $E(i) : E(M) \to E(N)$ in $\mathbf{com}(\mathcal{I})$. Choose a monomorphism $j : K \rightarrowtail I$ in $\mathbf{com}(\mathcal{I})$ with $I \in \mathcal{I}$. Then $j = lk$ for some $l : E(M) \to I$. By the adjunction $E \dashv F$, this morphism l corresponds to a morphism $l' : M \to F(I)$, and l' factors through i since $F(I)$ is injective. Hence l factors through $E(i)$, and thus $j = 0$. We infer that $K = 0$, i. e. $E(i)$ is monic.

(b) $\Rightarrow$ (a): Let $P_2 \xrightarrow{p'} P_1 \xrightarrow{p} P_0$ be an exact sequence of projectives in $\mathbf{mod}(\mathcal{P})$, and $u : P_1 \to F(I)$ a morphism with $I \in \mathcal{I}$ and $up' = 0$. Then $E(P_2) \to E(P_1) \to E(P_0)$ is exact, and u induces a morphism $u' : E(P_1) \to I$ in $\mathbf{com}(\mathcal{I})$ which annihilates $E(p')$. Hence u' factors through $E(p)$, and thus u factors through p. □

7 Morita varieties and strict almost abelian categories

In this section we shall encounter two classes of almost abelian categories which possess a natural structure of a PI-category. In view of Proposition 24, let us define a *Morita variety* as a PI-variety $\mathcal{P} \oplus_H \mathcal{I}$ corresponding to an injective cogenerator H. However, we do not assume that H is balanced. If H is merely a cogenerator, we shall speak of a *faithful* PI-variety. Any PI-category with an underlying faithful (resp. Morita) PI-variety will be called a *faithful* PI-category (resp. a *Morita* *category*). By Proposition 22, 25, and 26, an ample faithful PI-category is tantamount to an adjoint pair (42) of faithful functors E and F which are exact in case of a Morita category.

Now let us focus our attention upon the initial category $\mathcal{A}_\circ$, and the terminal category $\mathcal{A}^\circ$ (cf. §3) of a PI-category $\mathcal{A}$. If $\mathcal{A}$ is faithful, then $\mathbf{mod}(\mathcal{P}(\mathcal{A}))$ and $\mathbf{com}(\mathcal{I}(\mathcal{A}))$ can be regarded as full subcategories of $\mathcal{A}$ (Proposition 25), and we have

$$\mathcal{A}_\circ \subset \mathbf{mod}(\mathcal{P}(\mathcal{A})); \quad \mathcal{A}^\circ \subset \mathbf{com}(\mathcal{I}(\mathcal{A})). \tag{44}$$

Our next theorem shows that for a Morita category $\mathcal{A}$, the PI-system is determined by $\mathcal{A}_\circ$ and $\mathcal{A}^\circ$:

Theorem 5 *Let $\mathcal{A}$ be a PI-category with PI-variety $\mathcal{P}(\mathcal{A})\oplus_H\mathcal{I}(\mathcal{A})$. The following are equivalent:*

(a) *$\mathcal{A}$ is a Morita category.*

(b) *(38) gives equivalences: $\mathbf{mod}(\mathcal{P}(\mathcal{A})) \approx \mathcal{A}_\circ$ and $\mathbf{com}(\mathcal{I}(\mathcal{A})) \approx \mathcal{A}^\circ$.*

Proof. By Proposition 25, the functors (38) are fully faithful if and only if H is a cogenerator. Under this assumption, H is injective if and only if the morphisms in $\mathcal{P}(\mathcal{A})$ and $\mathcal{I}(\mathcal{A})$ are strict. This means that $\mathcal{P}(\mathcal{A}) \subset \mathcal{A}_\circ$ and $\mathcal{I}(\mathcal{A}) \subset \mathcal{A}^\circ$. By Proposition 8, the latter implies that the inclusions (44) are equations. □

Note: By the preceding theorem, Morita categories can be regarded as almost abelian categories with an intrinsic PI-structure.

Via the equivalences of the theorem, the functors (42) induce an adjoint pair of functors:

$$\mathcal{A}_\circ \overset{M^\circ}{\underset{M_\circ}{\rightleftarrows}} \mathcal{A}^\circ. \tag{45}$$

Corollary. *If $\mathcal{A}$ is a Morita category, then both of the abelian categories $\mathcal{A}_\circ$ and $\mathcal{A}^\circ$ have enough projectives and injectives.*

Proof. Since $\mathcal{A}_\circ$ is equivalent to $\mathbf{mod}(\mathcal{P}(\mathcal{A}))$, we only have to show that $\mathcal{A}_\circ$ has enough injectives. Let A be an object in $\mathcal{A}_\circ$. Since M° is faithful, the unit morphism $A \to M_\circ M^\circ(A)$ of the adjunction $M^\circ \dashv M_\circ$ is monic ([26], IV.3). Since $\mathcal{A}^\circ$ has enough injectives, there is an $I \in$

$\mathcal{I}(\mathcal{A})$ and a monomorphism $M^\circ(A) \rightarrowtail I$. By the left exactness of $M_\circ$, this yields a monomorphism $A \rightarrowtail M_\circ M^\circ(A) \rightarrowtail M_\circ(I)$, where $M_\circ(I)$ is injective in $\mathcal{A}_\circ$ by Proposition 26. □

Now let $\mathcal{P}$ be a projective, $\mathcal{I}$ an injective variety, and

$$\mathbf{mod}(\mathcal{P}) \underset{F}{\overset{E}{\rightleftarrows}} \mathbf{com}(\mathcal{I}) \tag{46}$$

an adjoint pair of functors $E \dashv F$ with corresponding $(\mathcal{P}, \mathcal{I})$-bimodule

$$H(P, I) = \mathrm{Hom}_{\mathbf{mod}(\mathcal{P})}(P,\ F(I)) \tag{47}$$

according to Proposition 22. We retain the notations of §6. In particular, let $\mathcal{A}$ be a PI-category belonging to H. For any object A in $\mathcal{A}$, the counit of $H_\circ \dashv S_\circ$ and the unit of $S^\circ \dashv H^\circ$ (cf. (38) and (40)) yield regular morphisms

$$H_\circ S_\circ(A) \xrightarrow{\varepsilon_\circ} A \xrightarrow{\eta^\circ} H^\circ S^\circ(A), \tag{48}$$

where $\varepsilon_\circ$ is $\mathcal{P}(\mathcal{A})$-epic, and η° is $\mathcal{I}(\mathcal{A})$-monic. (By abuse of notation, we write $\varepsilon_\circ$ for the natural transformation $H_\circ S_\circ \to 1$ as well as for its A-component, etc.)

Note: If $\mathcal{A}$ is a Morita category, then $H_\circ S_\circ(A) \in \mathcal{A}_\circ$ and $H^\circ S^\circ(A) \in \mathcal{A}^\circ$ for all objects A in $\mathcal{A}$. The restriction of $H^\circ S^\circ$ to $\mathcal{A}_\circ$ coincides with M°, and the restriction of $H_\circ S_\circ$ to $\mathcal{A}^\circ$ coincides with $M_\circ$ of (45).

We shall say that the object A of $\mathcal{A}$ is *strict* if the morphisms (48) are isomorphisms. Equivalently, this means that there is a cokernel $P \twoheadrightarrow A$ and a kernel $A \rightarrowtail I$ with $P \in \mathcal{P}(\mathcal{A})$ and $I \in \mathcal{I}(\mathcal{A})$. Clearly, this concept is compatible with the notion of strict object in $\mathcal{P}$ or $\mathcal{I}$ introduced in §6, namely, $P \in \mathcal{P}$ is strict if and only if $H_\circ(P)$ is strict in $\mathcal{A}$, and similarly for $I \in \mathcal{I}$. By $\mathcal{A}_s$ we denote the full subcategory of strict objects in $\mathcal{A}$.

Proposition 27 *Let $E \dashv F$ be a pair of adjoint functors (46) with corresponding $(\mathcal{P}, \mathcal{I})$-bimodule (47), and let $\mathcal{A}$ be any PI-category with a PI-embedding $\mathcal{A} \hookrightarrow \mathbf{DS}(H)$. Then the functors (38) and (40) yield equivalences:*

$$\mathbf{mod}(\mathcal{P})_{\mathrm{ref}} \underset{S_\circ}{\overset{H_\circ}{\rightleftarrows}} \mathcal{A}_s \underset{S^\circ}{\overset{H^\circ}{\leftrightarrows}} \mathbf{com}(\mathcal{I})_{\mathrm{ref}}. \tag{49}$$

Proof. Let A be an object in $\mathcal{A}_s$. Then $S_\circ(A) = S_\circ H^\circ S^\circ(A) = S_\circ H^\circ S^\circ H_\circ S_\circ(A) = FES_\circ(A)$. Hence $S_\circ(A)$ is in $\mathbf{mod}(\mathcal{P})_{\mathrm{ref}}$, and $A = H_\circ S_\circ(A)$. Conversely, let A be in $\mathbf{mod}(\mathcal{P})_{\mathrm{ref}}$. If $\eta_\circ : A \to S_\circ H_\circ(A)$ denotes the unit morphism with respect to $H_\circ \dashv S_\circ$, then we have an isomorphism

$$A \xrightarrow{\eta_\circ} S_\circ H_\circ(A) \xrightarrow{S_\circ \eta^\circ H_\circ} S_\circ H^\circ S^\circ H_\circ(A). \tag{50}$$

Since $\varepsilon_\circ$ of (48) is a natural transformation, the diagram

$$\begin{array}{ccc}
H_\circ(A) & \xrightarrow{\quad \eta^\circ H_\circ \quad} & H^\circ S^\circ H_\circ(A) \\
\uparrow{\scriptstyle \varepsilon_\circ H_\circ} & & \uparrow{\scriptstyle \varepsilon_\circ H^\circ S^\circ H_\circ} \\
H_\circ S_\circ H_\circ(A) & \xrightarrow{H_\circ S_\circ \eta^\circ H_\circ} & H_\circ S_\circ H^\circ S^\circ H_\circ(A)
\end{array}$$

is commutative. By (48), $\varepsilon_\circ H^\circ S^\circ H_\circ$ is $\mathcal{P}(\mathcal{A})$-epic, and $\eta^\circ H_\circ$ is $\mathcal{I}(\mathcal{A})$-monic. Now consider the morphism $H_\circ \eta_\circ : H_\circ(A) \to H_\circ S_\circ H_\circ(A)$. By virtue of (50), its composition with $H_\circ S_\circ \eta^\circ H_\circ$ is an isomorphism, whereas its composition with $\varepsilon_\circ H_\circ$ is the identity. Consequently, $\eta^\circ H_\circ$ and $\varepsilon_\circ H^\circ S^\circ H_\circ$ are isomorphisms by (PI_2). Since $\varepsilon_\circ H_\circ$ is also an isomorphism, $H_\circ(A) \in \mathcal{A}_s$. Moreover, we infer that $S_\circ \eta^\circ H_\circ$ is an isomorphism, whence (50) implies that $\eta_\circ$ is an isomorphism. By duality, this completes our proof. □

For a Morita category $\mathcal{A}$, we have a rather nice description of $\mathcal{A}_s$, hence of the reflexive objects in $\mathcal{A}_\circ$ and $\mathcal{A}^\circ$:

Proposition 28 *Let H be a $(\mathcal{P}, \mathcal{I})$-bimodule which is an injective cogenerator, and $\mathcal{A}$ a corresponding Morita category. Then $\mathcal{A}_s = \mathcal{A}_\circ \cap \mathcal{A}^\circ$. In particular, $\mathcal{A}_s$ is an abelian Serre subcategory of $\mathcal{A}$.*

Proof. Clearly, $\mathcal{A}_\circ \cap \mathcal{A}^\circ \subset \mathcal{A}_s$. The reverse inclusion follows by Theorem 5 and Proposition 8 since for each A in $\mathcal{A}_s$, there is a cokernel $P \twoheadrightarrow A$ and a kernel $A \rightarrowtail I$ with $P \in \mathbf{Proj}(\mathcal{A}_\circ)$ and $I \in \mathbf{Inj}(\mathcal{A}^\circ)$. □

We conclude this section with the discussion of *strict* PI-categories $\mathcal{A}$ which we define by $\mathcal{A}_s = \mathcal{A}$. Equivalently, this says that every $\mathcal{P}(\mathcal{A})$-epimorphism is a cokernel, and every $\mathcal{I}(\mathcal{A})$-monomorphism is a kernel.

In this case, we obviously have:

$$\mathcal{P}(\mathcal{A}) = \mathbf{Proj}(\mathcal{A}); \quad \mathcal{I}(\mathcal{A}) = \mathbf{Inj}(\mathcal{A}). \tag{51}$$

Therefore, the strict PI-categories coincide with the almost abelian categories with strictly enough projectives and injectives, and we have reason to speak of *strict almost abelian categories.*

Example 1. For an abelian category $\mathcal{A}$ with enough projectives and injectives, and a finite poset Ω, the almost abelian category $\mathcal{A}_\Omega$ of Example 2.5 is strict. In fact, the objects P in $\mathcal{A}_\Omega$ with $P(i)$ projective for all $i \in \Omega$ and $P(i) \rightarrowtail P(j)$ split for $i \leqslant j$ are projective, and for each $F \in \mathcal{A}_\Omega$, there is a cokernel $P \twoheadrightarrow F$ with such a P. Namely, if $\sum_{j<i} F(j) \rightarrowtail F(i) \twoheadrightarrow C_i$ is exact in $\mathcal{A}$, then the $P(i)$ can be chosen as $\bigoplus_{j\leqslant i} P_j$ with $P_j \in \mathbf{Proj}(\mathcal{A})$ such that $P(i) \to F(i)$ is induced by morphisms $P_i \to F(i)$ which arise from epimorphisms $P_i \twoheadrightarrow C_i$. Thus $\mathcal{A}_\Omega$ has strictly enough projectives. The dual assertion follows by the obvious equivalence $(\mathcal{A}^{\mathrm{op}})_\Omega \approx \mathcal{A}_{\Omega^{\mathrm{op}}}$.

Example 2. Let $\mathbf{B}$ denote the almost abelian category of (real or complex) Banach spaces. For any set X, consider the spaces $l^1(X)$ and $l^\infty(X)$ of summable resp. bounded functions on X. For a Banach space E, the continuous linear maps $l^1(X) \to E$ correspond to maps $X \to E$ with bounded image, and the continuous linear maps $E \to l^\infty(X)$ correspond to maps $X \to E'$ with bounded image. Hence $l^1(X)$ is projective and $l^\infty(X)$ injective in $\mathbf{B}$. Moreover, the unit ball $E_1 \hookrightarrow E$ gives rise to a cokernel $l^1(E_1) \twoheadrightarrow E$, and similarly, $E \hookrightarrow l^\infty(E_1')$ is a kernel in $\mathbf{B}$. Therefore, $\mathbf{B}$ is a strict almost abelian category.

Example 3. Let R be a noetherian integral domain with quotient field K, and Λ an R-order in a finite dimensional K-algebra A. Let e be a central idempotent of A. Recently, O. Iyama [22] considered the quotient category $\mathcal{C} := \Lambda\text{-}\mathbf{lat}/e\Lambda\text{-}\mathbf{lat}$ of Λ-lattices modulo homomorphisms factoring through some $e\Lambda$-lattice. His results [22] imply that $\mathcal{C}$ is a strict almost abelian category. If $\mathcal{C}$ has a projective object Q which generates $\mathbf{Proj}(\mathcal{C})$, he proves that $\mathcal{C}$ is of the form Γ-$\mathbf{lat}$ with an R-order Γ.

Proposition 29 *Every strict PI-category $\mathcal{A}$ is minimal. If $\mathcal{A}$ is ample or faithful, then $\mathcal{A}$ is abelian.*

Proof. The minimality follows by Proposition 21. If $\mathcal{A}$ is ample, then every morphism in $\mathcal{A}$ is strict; if $\mathcal{A}$ is faithful, then $H_\circ : \mathbf{mod}(\mathbf{Proj}(\mathcal{A})) \hookrightarrow \mathcal{A}$ is a full embedding. Hence $\mathcal{A}$ is abelian. □

We shall give a characterization of the PI-variety of a strict PI-category. Let us call a P-kernel $p : P \to A$ in a PI-variety $\mathcal{V}$ *strict* if each morphism $P \to I$ with $I \in \mathfrak{I}(\mathcal{V})$, which annihilates a P-kernel of p, factors through p. Note that if $A \in \mathfrak{I}(\mathcal{V})$, this just means that p is an I-cokernel. Dually, a *strict* I-cokernel is defined.

Proposition 30 *For a PI-variety $\mathcal{V} = \mathcal{P} \oplus_H \mathfrak{I}$, the following are equivalent:*

(a) *The PI-category* $\mathbf{Lat}(H)$ *is strict.*
(b) *The P-kernels and I-cokernels in $\mathcal{V}$ are strict.*
(c) *Every P-kernel, and every I-cokernel of a morphism $u \in H(P, I)$ in $\mathcal{V}$ is strict, and the objects in $\mathcal{P}$ and $\mathfrak{I}$ are strict.*
(d) *For each kernel $A \rightarrowtail P$ in* $\mathbf{Lat}(H)$ *with $P \in \mathcal{P}$, the object A is strict, and for each cokernel $I \twoheadrightarrow B$ in* $\mathbf{Lat}(H)$ *with $I \in \mathfrak{I}$, the object B is strict.*

Proof. (a) ⇒ (b): Let $f : A \to B$ be a morphism in $\mathcal{V}$. By Proposition 19, there is a faithful embedding $\mathcal{V} \hookrightarrow \mathbf{Lat}(H)$. Let $g : K \rightarrowtail A$ be the kernel of f, and $p : P \twoheadrightarrow K$ a cokernel in $\mathbf{Lat}(H)$ with $P \in \mathcal{P}$. Then $gp : P \to A$ lies in $\mathcal{V}$ and is a P-kernel of f. Now let $q : Q \to A$ be an arbitrary P-kernel of f. Then $q = gr$ for some $r : Q \to K$, and $gp = gr \circ s$ for some $s : P \to Q$. Hence $p = rs$, and by Proposition 2, r is a cokernel. Therefore, if $u : Q \to I$ is any morphism in $\mathcal{V}$ with $I \in \mathfrak{I}$ such that u annihilates a P-kernel of q, then u factors through r. Since I is injective in $\mathbf{Lat}(H)$, u factors through $q = gr$. Thus we have shown that q is strict in $\mathcal{V}$, and by duality, this also follows for I-cokernels.

(b) ⇒ (c): This holds since an $\mathfrak{I}$-monomorphism ($\mathcal{P}$-epimorphism) $u \in H(P, I)$ is just the same as an I-cokernel of $0 \to P$ (resp. a P-kernel of $I \to 0$).

(c) ⇒ (d): The conditions in (c) imply that for $u \in H(P, I)$, the P-kernels and I-cokernels of u are strict in $\mathbf{Lat}(H)$, and that the objects of $\mathcal{P}$ and $\mathcal{I}$ are strict in $\mathbf{Lat}(H)$. Since each kernel $A \rightarrowtail P$ in $\mathbf{Lat}(H)$ with $P \in \mathcal{P}$ is a kernel of some $u \in H(P, I)$, it follows that A is strict in $\mathbf{Lat}(H)$. By duality, this implies (d).

(d) $\Rightarrow$ (a): By Proposition 21, and by duality, it suffices to show that for each kernel $A \rightarrowtail B$ in $\mathbf{Lat}(H)$ the object A is strict whenever B is. If $B \rightarrowtail I$ is a kernel with $I \in \mathcal{I}$, then $A \rightarrowtail B \rightarrowtail I$ is also a kernel. If $P \twoheadrightarrow B$ is a cokernel with $P \in \mathcal{P}$, then A is a subquotient of P, say, $A \twoheadleftarrow C \rightarrowtail P$. Then by assumption, C is strict. Therefore, a cokernel $Q \twoheadrightarrow C$ with $Q \in \mathcal{P}$ yields a cokernel $Q \twoheadrightarrow C \twoheadrightarrow A$, whence A is strict. □

The following shows how Proposition 30 is related to Theorem 3:

Corollary 1 *Let $E \dashv F$ be a pair of adjoint functors (46) with corresponding $(\mathcal{P}, \mathcal{I})$-bimodule (47) non-degenerate. Then* $\mathbf{Lat}(H)$ *is strict if and only if E, F constitute an almost equivalence.*

Proof. If $\mathbf{Lat}(H)$ is strict, then Proposition 27 implies that $\mathbf{Lat}(H)$ is equivalent to $\mathbf{mod}(\mathcal{P})_{\text{ref}}$. By Proposition 11, $Q_l(\mathbf{Lat}(H))$ is equivalent to $\mathbf{mod}(\mathcal{P})$. Hence Theorem 3 implies that E, F constitute an almost equivalence. Conversely, suppose the latter holds, and $\mathcal{A} := \mathbf{Lat}(H)$. Then Proposition 27 shows that $\mathbf{mod}(\mathcal{P})_{\text{ref}}$ and $\mathcal{A}_s$ are equivalent almost abelian categories. Since H is non-degenerate, there are full embeddings $\mathcal{P} \hookrightarrow \mathcal{A}_s \hookleftarrow \mathcal{I}$. Thus by Proposition 30, it remains to prove that every P-kernel of a given $u \in H(P, I)$ is strict. If $k : K \rightarrowtail P$ is the kernel of $u : P \to F(I)$ in $\mathbf{mod}(\mathcal{P})$, this means that each morphism $K \to F(J)$ in $\mathbf{mod}(\mathcal{P})$ with $J \in \mathcal{I}$ factors through k. If $P \twoheadrightarrow C$ is the cokernel of k in $\mathbf{mod}(\mathcal{P})$, then we have an exact sequence $K \rightarrowtail P \twoheadrightarrow C$ in $\mathbf{mod}(\mathcal{P})$ which entirely lies in $\mathbf{mod}(\mathcal{P})_{\text{ref}}$. Hence $E(K) \rightarrowtail E(P) \twoheadrightarrow E(C)$ is exact in $\mathbf{com}(\mathcal{I})$. Then a morphism $f : K \to F(J)$ corresponds to a morphism $E(K) \to J$ which lifts along $E(K) \rightarrowtail E(P)$. Hence f factors through k. □

Recall that a ring R is said to be *left (right) coherent* if every finitely generated left (right) ideal of R is finitely presented. Equivalently, this says that the category R-**proj** (**proj**-R) of finitely generated projective left (right) R-modules is a projective variety. Now let R be a left, and S a right coherent ring. Consider the varieties $\mathcal{P} = R$-**proj** and $\mathcal{I} =$ (**proj**-S)$^{\text{op}}$. Then $\mathbf{mod}(\mathcal{P}) = R$-**mod**, the category of finitely presented left R-modules, and $\mathbf{com}(\mathcal{I}) = (\mathbf{mod}\text{-}S)^{\mathrm{op}}$. A slight modification of Proposition 17 shows that a small $(\mathcal{P}, \mathcal{I})$-bimodule H is tantamount to a bimodule ${}_RU_S$ with ${}_RU$ and U_S finitely presented, and the corresponding adjoint functor pair is given by (43). Therefore, we write $\mathbf{lat}(U)$ instead of $\mathbf{Lat}(H)$. Clearly, H is non-degenerate if and only if ${}_RU$ and U_S are faithful (cf. Proposition 18). Let us call ${}_RU_S$ a *cotilting bimodule* (cf. [18, 37, 7]) if U is finitely presented and faithfully balanced (i. e. ${}_RU$ and U_S are faithful with $R = \mathrm{End}(U_S)$ and $S = (\mathrm{End}_RU)^{\mathrm{op}}$) with

$$\mathrm{Ext}^1_R(U,U) = \mathrm{Ext}^2_R(-,U) = 0; \quad \mathrm{Ext}^1_S(U,U) = \mathrm{Ext}^2_S(-,U) = 0. \quad (52)$$

Corollary 2 *Let R be a left, S a right coherent ring, and ${}_RU_S$ a bimodule, finitely presented and faithful over R and S. Then* $\mathbf{lat}(U)$ *is strict if and only if U is a cotilting bimodule.*

Proof. By assumption, the varieties $\mathcal{P} = R\text{-}\mathbf{proj}$ and $\mathcal{I} = (\mathbf{proj}\text{-}S)^{\mathrm{op}}$ constitute a PI-variety $\mathcal{V} := \mathcal{P} \oplus_H \mathcal{I}$, and Proposition 30 applies. For $P \in \mathcal{P}$, $Q \in \mathcal{I}$, a morphism $u : P \to Q$ in $\mathcal{V}$ is given by an R-linear map $u : P \to \mathrm{Hom}_S(Q, U)$. Thus a P-kernel of u amounts to an exact sequence $P' \xrightarrow{p} P \xrightarrow{u} \mathrm{Hom}_S(Q, U)$ with $P' \in \mathcal{P}$. Hence p is strict if and only if each R-linear map $P' \to \mathrm{Hom}_S(Q', U)$ with $Q' \in \mathcal{I}^{\mathrm{op}}$ which annihilates the kernel of p factors through p. Clearly, this statement is not altered if Q' is replaced by S, and Q by some finitely generated free S-module. Hence, the assertion states that every short exact sequence $K \hookrightarrow P \twoheadrightarrow M$ of R-modules, with $P \in \mathcal{P}$ and M finitely cogenerated by U, the map $\mathrm{Hom}_R(P, U) \to \mathrm{Hom}_R(K, U)$ is surjective. But this amounts to $\mathrm{Ext}_R(M, U) = 0$. Thus we have shown that the first assertion in Proposition 30(c) states that $\mathrm{Ext}_R(M, U) = \mathrm{Ext}_S(N, U) = 0$ whenever M is finitely cogenerated, and N finitely generated by U. In fact, this statement is equivalent to (52). By symmetry, it suffices to prove:

$$\mathrm{Ext}_R(M, U) = 0 \text{ for } M \hookrightarrow U^n \iff \mathrm{Ext}_R(U, U) = \mathrm{Ext}^2_R(-, U) = 0.$$

To show "$\Rightarrow$", let $M \in R\text{-}\mathbf{mod}$ be given. Consider a free presentation $\Omega M \hookrightarrow A^n \twoheadrightarrow M$. Since ${}_RU$ is faithful, we have an embedding $A \hookrightarrow U^m$, whence $\mathrm{Ext}^2_R(M, U) = \mathrm{Ext}^1_R(\Omega M, U) = 0$. Conversely, let us assume $\mathrm{Ext}_R(U, U) = \mathrm{Ext}^2_R(-, U) = 0$, and consider an injective resolution

${}_RU \hookrightarrow I_0 \overset{\partial}{\twoheadrightarrow} I_1$. For $M \hookrightarrow U^n$ in R-**mod**, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_R(M, I_0) & \xrightarrow{\partial_*} & \mathrm{Hom}_R(M, I_1) \\ \uparrow & & \uparrow \\ \mathrm{Hom}_R(U^n, I_0) & \twoheadrightarrow & \mathrm{Hom}_R(U^n, I_1). \end{array}$$

Hence ∂_* is epic, i. e. $\mathrm{Ext}_R(M, U) = 0$.

Finally, the argument in the proof of Proposition 24(a) shows that the objects of $\mathcal{P}$ are strict if and only if $R = \mathrm{End}(U_S)$. By duality, we are done. □

Example 4. For an order Λ over a Dedekind domain R, the (Λ, Λ)-bimodule $\Lambda^* = \mathrm{Hom}_R(\Lambda, R)$ is cotilting, and $\mathbf{lat}(\Lambda^*)$ is equivalent to Λ-**lat**.

Remark. In conjunction with Theorem 3, the above corollaries yield a cotilting theorem (cf. [7], Theorem 2.4) for modules over coherent rings. Moreover, we get the following symmetric characterization of (co-)tilting torsion theories (cf. [2, 37, 9]):

Theorem 6 *Let R be a left coherent ring. An almost abelian category $\mathcal{A}$ is equivalent to a torsionfree class in R-**mod** induced by a cotilting bimodule ${}_RU_S$ if and only if $Q_l(\mathcal{A}) \approx R$-**mod** and $Q_r(\mathcal{A}) \approx (\mathbf{mod}\text{-}S')^{\mathrm{op}}$ for some right coherent ring S'. (Then $S' = S$).*

Remarks. 1. Equivalently, the condition of the theorem says that $\mathcal{A}$ is strict with $\mathbf{Proj}(\mathcal{A}) \approx R$-**proj** and $\mathbf{Inj}(\mathcal{A}) = (\mathbf{proj}\text{-}S)^{\mathrm{op}}$ for some right coherent ring S.

2. In the case of an artinian algebra S, Assem [2] considers a torsion class $\mathcal{T}$ in S-**mod** $= (\mathbf{mod}\text{-}S)^{\mathrm{op}}$. Then his first tilting condition states that $\mathcal{T}$ contains the injective left S-modules, that is, $Q_r(\mathcal{T}) = (\mathbf{mod}\text{-}S)^{\mathrm{op}}$. His second condition, saying that $\mathcal{T}$ is functorially finite in S-**mod**, is equivalent to $Q_l(\mathcal{T}) \approx R$-**mod** for some artinian algebra R.

8 Locally compact abelian groups

As an example to illustrate some of the preceding results, let us consider the category $\mathcal{L}$ of locally compact abelian groups, with continuous homomorphisms as morphisms. As already noticed in §2.2, this category is almost abelian, with the topological direct sum as biproduct, and for a morphism $f : A \to B$ in $\mathfrak{L}$, the kernel and cokernel are given by

$$\operatorname{Ker} f \hookrightarrow A \xrightarrow{f} B \twoheadrightarrow B \big/ \overline{\operatorname{Im} f}\,, \tag{53}$$

where $\operatorname{Ker} f$ and $\operatorname{Im} f$ are to be understood in the group theoretical sense. Then $\operatorname{Ker} f$ is endowed with the induced topology, and $B \big/ \overline{\operatorname{Im} f}$ with the quotient topology. Hence the coimage of f is given by the quotient $A \twoheadrightarrow \operatorname{Im} f$, whereas the (categorial) image of f is $\overline{\operatorname{Im} f} \hookrightarrow B$. The three most fundamental objects in $\mathfrak{L}$, the group Z of integers, the reals R, and the circle group T, are connected by a short exact sequence

$$\mathsf{Z} \rightarrowtail \mathsf{R} \twoheadrightarrow \mathsf{T}. \tag{54}$$

The projective and injective objects in $\mathfrak{L}$ are well-known ([28], Theorem 3.2 and 3.3):

Proposition 31 *An object P in $\mathfrak{L}$ is projective if and only if $P \cong \mathsf{R}^n \oplus \mathsf{Z}^{(\mathfrak{m})}$ for some $n \in \mathbb{N}$ and a cardinal $\mathfrak{m}$; an object I in $\mathfrak{L}$ is injective if and only if $I \cong \mathsf{R}^n \oplus \mathsf{T}^{\mathfrak{m}}$ with $n \in \mathbb{N}$ and a cardinal $\mathfrak{m}$. In both cases, n and $\mathfrak{m}$ are invariants for P and I, respectively.*

In particular, the bijectives in $\mathfrak{L}$ are just the vector groups R^n. There is a natural way to split off bijectives from an arbitrary group A in $\mathfrak{L}$. Define $\mathrm{C}(A)$ as the connected component of zero in A, and $\mathrm{B}(A)$ as the union of compact subgroups of A. Then $\mathrm{C}(A)$ is the smallest closed subgroup of A which contains all the images of morphisms $\mathsf{R} \to A$ ([21], Theorem 25.20); and $\mathrm{B}(A)$ is the largest closed subgroup of A which is annihilated by every morphism $A \to \mathsf{R}$ ([21], Theorem 24.34).

Proposition 32 *For A in $\mathfrak{L}$, the following are equivalent:*

(a) *A does not have R as a direct summand.*

(b) $\mathrm{C}(A) \subset \mathrm{B}(A)$.

Proof. (a) $\Rightarrow$ (b): Suppose $\mathrm{C}(A) \not\subset \mathrm{B}(A)$. Then there exists a morphism $i : \mathsf{R} \to A$ with image not in $\mathrm{B}(A)$, i. e. there exists a morphism $p : A \to \mathsf{R}$ with $pi \neq 0$. Hence $pi : \mathsf{R} \to \mathsf{R}$ is a topological automorphism of R, and thus R is a direct summand of A.

(b) $\Rightarrow$ (a): If R would be a direct summand of A, there would be morphisms $\mathsf{R} \xrightarrow{i} A \xrightarrow{p} \mathsf{R}$ with $pi = 1$. Thus $\mathrm{C}(A) \not\subset \mathrm{B}(A)$. □

The following cancellation lemma generalizes the Krull-Schmidt Theorem:

Lemma 8 *Let C be an object in an additive category with* $\mathrm{End}(C)$ *local. Then $A \oplus C \cong B \oplus C$ implies $A \cong B$.*

Proof. The isomorphism $A \oplus C \cong B \oplus C$ is given by mutually inverse morphisms $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : A \oplus C \to B \oplus C$ and $\begin{pmatrix} p & q \\ r & s \end{pmatrix} : B \oplus C \to A \oplus C$. Hence $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ yields $rb + sd = 1$. Since $\mathrm{End}(C)$ is local, there are two cases:

<u>Case 1:</u> sd is invertible. Then $q = -pbd^{-1}$, $c = -s^{-1}ra$, and $pa + qc = 1$ implies $p(1 + bd^{-1}s^{-1}r)a = 1$, and $b = -aqs^{-1}$ yields $a \cdot p(1 + bd^{-1}s^{-1}r) = ap - aqs^{-1}r = ap + br = 1$. Hence $a : A \to B$ is an isomorphism.

<u>Case 2:</u> rb is invertible. Then $B = B' \oplus B''$ with induced isomorphisms $b : C \xrightarrow{\sim} B''$ and $r : B'' \xrightarrow{\sim} C$. Hence by Case 1, we obtain $A \cong B' \oplus C \cong B$. □

As an immediate consequence we get the uniqueness result proved in [5] (cf. [21], Theorem 24.30):

Proposition 33 *Any locally compact abelian group G is of the form $\mathsf{R}^n \oplus A$ with $n \in \mathbb{N}$, where A has no direct summand isomorphic to R. The isomorphism class of A, and $n \in \mathbb{N}$ are uniquely determined by G.*

Proof. Let U be a compact symmetric neighbourhood of zero in G. Then $U + U$ is compact. Hence $U + U$ is contained in $U + F$ for a finite set $F \subset G$. Therefore, the subgroup $\langle U \rangle$ generated by U is contained in $U + \langle F \rangle$. Since $\langle U \rangle$ is open, we have $\mathrm{C}(G) \subset \langle U \rangle \subset U + \langle F \rangle$. Now if $G = \mathsf{R}^n \oplus A$, then the projection $p : \mathsf{R}^n \oplus A \twoheadrightarrow \mathsf{R}^n$ maps $U + \langle F \rangle$ onto $\mathsf{R}^n = p(U) + \langle p(F) \rangle$. Since $p(U)$ is compact, we infer that the $\mathbb{R}$-subspace generated by $p(F)$ cannot be proper in R^n. Hence $n \leqslant |p(F)|$, and the assertion follows by Lemma 8. □

Note: Nevertheless, the $\mathbb{R}$-dimension of $\mathrm{Hom}_{\mathcal{L}}(\mathsf{R}, G)$ or $\mathrm{Hom}_{\mathcal{L}}(G, \mathsf{R})$ can be infinite! Notice also that we have not made use of Proposition 32.

By Proposition 33, the structure of $\mathcal{L}$ reduces to that of the full subcategory of objects A with $\mathrm{C}(A) \subset \mathrm{B}(A)$. In fact, this category is just a minimal PI-category:

Theorem 7 *The category $\mathcal{L}$ of locally compact abelian groups is a Morita category with $\mathcal{L}_\circ = \mathbf{Ab}$ and $\mathcal{L}^\circ = \mathbf{Ab}^{\mathrm{op}}$, the category of compact abelian groups. The minimal PI-subcategory $\underset{\sim}{\mathcal{L}}$ of $\mathcal{L}$ coincides with the full subcategory of objects A with $\mathrm{C}(A) \subset \mathrm{B}(A)$, i. e. without direct summands R. The defining $(\mathbb{Z}, \mathbb{Z})$-bimodule for $\underset{\sim}{\mathcal{L}}$ is T.*

Proof. For any A in $\mathcal{L}$, there are regular morphisms $A_d \to A \to A_b$, where A_d denotes the group A with the discrete topology, and A_b is the Bohr compactification of A (see [21], 26.11). Hence $\mathcal{L}_\circ = \mathbf{Ab}$, and $\mathcal{L}^\circ$ consists of the compact groups in $\mathcal{L}$. Therefore, let $\mathcal{P}(\mathcal{L}) = \mathbf{Proj}(\mathbf{Ab})$ be the full subcategory of free abelian groups in $\mathcal{L}$, and $\mathcal{J}(\mathcal{L})$ the full subcategory of products $\mathsf{T}^{\mathfrak{m}}$ for some cardinal $\mathfrak{m}$. For sufficiently large $\mathfrak{m}$, we have an epimorphism $\mathsf{Z}^{(\mathfrak{m})} \twoheadrightarrow A_d$, i. e. a cokernel in $\mathcal{L}$, and a kernel $A_b \hookrightarrow \mathsf{T}^{\mathfrak{m}}$. Hence we obtain a $\mathcal{P}(\mathcal{L})$-cover $p : \mathsf{Z}^{(\mathfrak{m})} \to A$ and an $\mathcal{J}(\mathcal{L})$-hull $A \to \mathsf{T}^{\mathfrak{m}}$. In order to verify (PI$_2$), let $r : A \to B$ be a $\mathcal{P}(\mathcal{L})$-epimorphism and $\mathcal{J}(\mathcal{L})$-monomorphism in $\mathcal{L}$. Then r induces an isomorphism $A_d \xrightarrow{\sim} B_d$, and an isomorphism $\hat{B} \xrightarrow{\sim} \hat{A}$ of the character groups. By a result of Kaplansky and Glicksberg [17] (cf. [4], 10.6), this implies that r is an isomophism in $\mathcal{L}$, and thus $\mathcal{L}$ is a PI-category. By Theorem 5, $\mathcal{L}$ is a Morita category.

Next let us determine $\underset{\sim}{\mathcal{L}}$. Since R is bijective, the objects without R as direct summand form a PI-subcategory of $\mathcal{L}$. For each such object A, the argument in the proof of Proposition 33 shows that $\mathrm{C}(A)$ is compact, and since $A/\mathrm{C}(A)$ is 0-dimensional, this implies that there is a compact open neighbourhood U of 0 in A. Hence there is a symmetric neighbourhood V of 0 with $U + V = U$, and thus V generates a compact open subgroup C of A. This yields a short exact sequence $C \rightarrowtail A \twoheadrightarrow D$ with a discrete group D. If $C \rightarrowtail I$ is a kernel with $I \in \mathcal{J}(\mathcal{L})$, the pushout

$$\begin{array}{ccccc} C & \rightarrowtail & A & \twoheadrightarrow & D \\ \downarrow & \text{PO} & \downarrow & & \| \\ I & \rightarrowtail & I \oplus D & \twoheadrightarrow & D \end{array}$$

shows that A is a subquotient of some $I \oplus P$ with $P \in \mathcal{P}(\mathcal{L})$, whence $A \in \underset{\sim}{\mathcal{L}}$ by Proposition 21. Finally, Proposition 17 yields the $(\mathbb{Z}, \mathbb{Z})$-bimodule $H(\mathbb{Z}, \mathbb{Z}) = \mathrm{Hom}_{\mathcal{L}}(\mathsf{Z}, \mathsf{T}) = \mathsf{T}$. □

By a *self-duality* of a category $\mathcal{A}$ we understand a functor $D : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ with a natural isomorphism $D^2 \xrightarrow{\sim} 1$. In view of the preceding theorem, Pontrjagin's duality theorem generalizes as follows:

Proposition 34 *An ample Morita category $\mathcal{A}$ with $\mathcal{P} = \mathbf{Proj}(\mathcal{A}_\circ)$ and $\mathcal{I} = \mathbf{Inj}(\mathcal{A}^\circ)$ is self-dual if and only if there exists an equivalence $D : \mathcal{P}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{I}$ and a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(P, D(Q)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(Q, D(P)) \tag{55}$$

for $P, Q \in \mathcal{P}$.

Proof. If $\mathcal{A}$ admits a self-duality $D : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$, then D induces an equivalence $(\mathcal{A}_\circ)^{\mathrm{op}} \xrightarrow{\sim} \mathcal{A}^\circ$, hence an equivalence $D : \mathcal{P}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{I}$, and thus an isomorphism (55) which is natural in P and Q. Conversely, if these conditions are satisfied, then (55) defines a self-duality of the ample category $\widetilde{\mathcal{A}}$. □

Corollary. *Let S be a commutative ring, $\mathcal{P} = \mathbf{Proj}(S\text{-}\mathbf{Mod}) = \mathcal{I}^{\mathrm{op}}$, and $_SU$ an injective cogenerator. Then the Morita category $\mathbf{DS}(_SU_S)$ is self-dual.*

Proof. In view of Proposition 17, $H(P, Q) = \mathrm{Hom}_S(P, \mathrm{Hom}_S(Q, U)) = \mathrm{Hom}_S(P \otimes_S Q, U) = H(Q, P)$, whence the assertion. □

For the Morita category $\mathcal{L}$, this implies that $\underset{\sim}{\mathcal{L}}$ is self-dual (Proposition 21). Since the unique indecomposable bijective object R is carried into itself under the duality of $\widetilde{\mathcal{L}} = \mathbf{DS}(\mathsf{T})$, the corollary reduces to Pontrjagin's theorem if U is specialized to the $\mathbb{Z}$-module T.

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