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MORE ON ORTHOGONALITY IN LOCALLY PRESENTABLE CATEGORIES

by M. HEBERT, J. ADAMEK and J. ROSICKÝ**

Résumé. Nous présentons une solution nouvelle au problème des sous-catégories orthogonales dans les catégories localement présentables, essentiellement différente de la solution classique de Gabriel et Ulmer. Diverses applications sont données. En particulier nous l'utilisons pour caractériser les classes oméga-orthogonales dans les catégories localement finiment présentables, c'est-à-dire leurs sous-catégories pleines de la forme $\Sigma^{\perp}$ où les domaines et codomaines des morphismes de Σ sont finiment présentables. Nous l'utilisons aussi pour trouver une condition suffisante pour la réflexivité des sous-catégories de catégories accessibles. Finalement, nous donnons une description des catégories de fractions dans les petites catégories finiment complètes.

I. Introduction

Many "everyday" categories have the following type of presentation: a general locally finitely presentable (LFP) category $\mathcal{L}$, representing the signature in some sense, is given, together with a set Σ of morphisms having finitely presentable domains and codomains. And our category $\mathcal{K}$ is the full subcategory of $\mathcal{L}$ on all objects K orthogonal to each $s : X \to X'$ in Σ (notation: $K \perp s$), which means that every morphism $f : X \to K$ uniquely factors through s; notation: $\mathcal{K} = \Sigma^{\perp}$. Such subcategories $\mathcal{K}$ of $\mathcal{L}$ are called in [AR] the *ω-orthogonality classes.*

Example: finitary varieties. In fact, let Γ be a finitary signature and let $\mathcal{K}$ be a variety of Γ-algebras. Every equation $\alpha = \beta$ presenting

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$\mathcal{K}$ can be substituted by an orthogonality condition naturally: there are only finitely many variables contained in α and β, and we denote by X the absolutely free Γ-algebra over thoses variables, and by $\sim$ the congruence on X generated by the single pair (α, β). Then an algebra satisfies $\alpha = \beta$ iff it is orthogonal to the quotient map $X \to X/\sim$. Thus, $\mathcal{K}$ is an ω-orthogonality class in Γ-Alg, the category of all algebras of signature Γ.

A much more general example: essentially algebraic categories (see [AR], 3.34). Here again $\Gamma = \Gamma_t \cup \Gamma_p$ is a finitary signature and $\mathcal{K}$ is a category of partial Γ-algebras presented by the requirements that (a) all operations of Γ_t are total, i.e., everywhere defined, (b) for every operation $o \in \Gamma_p$ a set Def(o) of equations in Γ_t is specified, and the definition domain of o_A for any algebra $A \in \mathcal{A}$ is determined by those equations and (c) specified equations between Γ-operations are fulfilled. Then $\mathcal{K}$ is an ω-orthogonality class of the corresponding category of structures whenever each of the sets Def(o) is finite. This includes all finitary quasivarieties, the category of posets (and all other universal Horn classes), the category of small categories, etc.

Our aim is to characterize ω-orthogonality classes of a given LFP category $\mathcal{L}$. Each such class is a full subcategory which is

(1) closed under limits

and

(2) closed under filtered colimits.

The converse does not hold in general: the first example of a class of finitary structures closed in $Str\Gamma$ under limits and filtered colimits but failing to be an ω-orthogonality class has been found by H. Volger [V] (see our simple variation in II.8 below); independently, J. Jürjens found a different example in [J]. These examples are indeed quite surprising because for uncountable cardinals the corresponding result is true: suppose λ is a regular cardinal and call a full subcategory of an LFP category (or, more generally, of a locally λ-presentable category) a *λ-orthogonality class* if it is presented by orthogonality w.r.t. morphisms with λ-presentable domains and codomains. Then the following has been proved in [HR]:

Theorem. *Let $\lambda > \aleph_0$ be a regular cardinal. Then the λ-orthogonality classes in a locally λ-presentable category $\mathcal{L}$ are precisely the full subcategories of $\mathcal{L}$ closed under limits and λ-filtered colimits.*

However, as mentioned above, the corresponding result is false for $\lambda = \aleph_0$ (although Theorem 1.39 in [AR] states that it holds; see [AHR] for a corrigendum to that statement). Our characterization of ω-orthogonality classes $\mathcal{K}$ of an LFP category $\mathcal{L}$ is based on the observation that the embedding $E : \mathcal{K} \to \mathcal{L}$ is a morphism of LFP categories (i.e., $\mathcal{K}$ is LFP and E is a right adjoint preserving filtered colimits). By Gabriel-Ulmer duality we obtain a theory morphism from the theory $\mathrm{Th}\,\mathcal{L}$ of $\mathcal{L}$ (= the dual to the full subcategory of all finitely presentable objects of $\mathcal{L}$) to the theory of $\mathcal{K}$: $\mathrm{Th}(E) : \mathrm{Th}\,\mathcal{L} \to \mathrm{Th}\,\mathcal{K}$. The "missing" additional condition to closedness under limits and filtered colimits to characterize ω-orthogonality classes is the following

(3) the theory morphism $\mathrm{Th}(E) : \mathrm{Th}\,\mathcal{L} \to \mathrm{Th}\,\mathcal{K}$ is a quotient.

The idea of a *quotient theory morphism* is an "up to an equivalence" adaptation (by M. Makkai [M]) of a concept due to Gabriel and Zisman [GZ] (see III.1 to III.4 below); a necessary and sufficient condition on a lex (= finite-limit preserving) functor $F : \mathcal{A} \to \mathcal{B}$ to be a quotient is that F be "essentially surjective" in the following sense: (a) every object $B \in \mathcal{B}$ is isomorphic to an object FA, $A \in \mathcal{A}$ and (b) every morphism $b : X \to FA$ in $\mathcal{B}$ allows a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\quad b \quad} & FA \\ {\scriptstyle \cong}\downarrow & \nearrow_{Fa} & \\ FA' & & \end{array}$$

for some morphism $a : A' \to A$ in $\mathcal{A}$. We are going to prove the following

Characterization Theorem. *Let $\mathcal{L}$ be a locally finitely presentable category. The ω-orthogonality classes in $\mathcal{L}$ are precisely the full subcategories $\mathcal{K}$ closed under limits and filtered colimits in $\mathcal{L}$ such that the theory of the embedding $\mathcal{K} \hookrightarrow \mathcal{L}$ is a quotient.*

Let us remark here that in [MPi] a comment to Corollary 2.4 suggests that the authors were aware of this characterization, but no proof has been published so far.

The Characterization Theorem is not quite "nice" because (3) above is not a closedness property. This contrasts with the better situation concerning *ω-injectivity classes*, see [AR] and [RAB], i.e., full subcategories of $\mathcal{L}$ (LFP) given by injectivity w.r.t. morphisms between finitely presentable objects. It is proved in [RAB] that these are precisely the full subcategories of $\mathcal{L}$ closed under

(1) products,

(2) filtered colimits

and

(3) pure subobjects (i.e., filtered colimits of split subobjects in $\mathcal{L}^{\rightarrow}$).

The main tool of our investigation is a "non-standard" orthogonal-reflection construction explained in Part II. This makes it, quite surprisingly, possible to derive that certain subcategories of finitely accessible categories are reflective. But the main goal of using that construction is to provide the above mentioned description of ω-orthogonality classes.

II. The Orthogonal-Reflection Construction

II.1 Remark. Here we consider a classical problem of category theory: given a collection Σ of morphisms in a category $\mathcal{L}$, construct a reflection of an object in the orthogonality class $\Sigma^{\perp}$. Throughout this section,

$$\mathcal{L}$$

denotes a finitely accessible (or $\aleph_0$-accessible in the terminology of [MPa]) category, and

$$\mathcal{L}_\omega$$

the full subcategory of finitely presentable objects of $\mathcal{L}$. (Recall from [MPa] that $\mathcal{L}$ is finitely accessible iff it has filtered colimits and $\mathcal{L}_\omega$ is dense in $\mathcal{L}$ and essentially small.) We mention the more general case of λ-accessible categories at the end of this section.

II.2 Remark. Gabriel and Zisman introduced in [GZ] the concept of a left calculus of fractions: a class of Σ of morphisms of a category $\mathcal{X}$ is said to *admit a left calculus of fractions* provided that

(i) Σ contains all identity morphisms,
(ii) Σ is closed under composition in $mor(\mathcal{X})$,
(iii) for every span

$$Y \xleftarrow{\ f\ } X \xrightarrow{\ s\ } X'$$

in $\mathcal{X}$ with $s \in \Sigma$ there exists a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\ s\ } & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[\ t\ ]{} & Y' \end{array}$$

with $t \in \Sigma$;

and

(iv) given parallel morphisms $h_1, h_2 \in \mathcal{X}$ such that $h_1 s = h_2 s$ for some $s \in \Sigma$, there exists $t \in \Sigma$ with $th_1 = th_2$:

$$\xrightarrow{\ s\ } \overset{h_1}{\underset{h_2}{\rightrightarrows}} \xrightarrow{\ t\ }$$

Let us remark that if $\mathcal{X}$ is a category with finite colimits, then for every class Σ of morphisms we have

$$\Sigma^{\perp} = \overline{\Sigma}^{\perp}$$

where $\overline{\Sigma}$ is the saturation of Σ (containing an arrow s iff every object of $\Sigma^{\perp}$ is orthogonal to s), which always admits a left calculus of fractions. In fact, (i) and (ii) are trivial for $\overline{\Sigma}$; for (iii), form a pushout $tf = gs$ of f and s and observe that if $s \in \overline{\Sigma}$ then $t \in \overline{\Sigma}$. (Proof: let $K \in \Sigma^{\perp}$, then given $h : Y \to K$ we have for hf a unique $h' : X' \to K$ with $hf = h's$, and the universal pushout property yields a unique h'' with

$h = h''t$.) And for (iv) form a coequalizer t of h_1, h_2 and observe that since $s \in \overline{\Sigma}$, it follows that $t \in \overline{\Sigma}$. (Similar proof.)

Thus, in finitely cocomplete categories, there is no loss of generality in assuming that Σ admits a left calculus of fractions.

II.3 The Orthogonal-Reflection Construction. Let $\mathcal{L}$ be a finitely accessible category and let Σ be a set of morphisms in $\mathcal{L}_\omega$ admitting a left calculus of fractions (in that subcategory). A reflection of an object $L \in obj(\mathcal{L})$ in $\Sigma^\perp$ can be obtained as follows:

Case (A): Reflection of a finitely presentable object L.

Denote by $L \downarrow \Sigma$ the comma category of all morphisms $f : L \to C_f$ in Σ, and let

$$D_L : L \downarrow \Sigma \to \mathcal{L}$$

be the usual forgetful functor $(L \xrightarrow{f} C_f) \longmapsto C_f$. Then a filtered colimit of D_L is a reflection of L in $\Sigma^\perp$.

That is:

(a) $L \downarrow \Sigma$ is a filtered category,

(b) a colimit

$$(C_f \xrightarrow{c_f} QL)_{f \in L \downarrow \Sigma}$$

of D_L exists in $\mathcal{L}$ and QL lies in $\Sigma^\perp$ (observe that since $\mathrm{id}_L \in \Sigma$, we have $c_{\mathrm{id}} : L \to QL$)

and

(c) $c_{\mathrm{id}} : L \to QL$ is a reflection of L in $\Sigma^\perp$.

Case (B): Reflection of an arbitrary object L.

Express $L = \operatorname{colim}_{i \in I} L_i$ as a filtered colimit of finitely presentable objects L_i, form reflections $\eta_{L_i} : L_i \to QL_i$ $(i \in I)$, and then a filtered colimit $\operatorname{colim}_{i \in I} \eta_{L_i}$ in $\mathcal{L}^\to$ is a reflection of L in $\Sigma^\perp$.

Proof. Case (B) is easy to verify: since $\Sigma^\perp$ contains $QL = \operatorname{colim} QL_i$, it is trivial to show that $\operatorname{colim} \eta_{L_i} : L \to QL$ is a reflection of L.

We prove Case (A).

Proof of (a). Firstly, $L \downarrow \Sigma \neq \emptyset$ since $\mathrm{id}_L \in \Sigma$, see II.2 (i). Next, given two objects $L \xrightarrow{f} C_f$ and $L \xrightarrow{g} C_g$ in $L \downarrow \Sigma$, we use II.2 (iii) to obtain a commutative square

$$\begin{array}{ccc} L & \xrightarrow{\quad g \quad} & C_g \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ C_f & \xrightarrow[\quad g' \quad]{} & Y \end{array}$$

with $g' \in \Sigma$, thus by II.2 (ii) we have $g'f = h \in \Sigma$. We thus found an object $L \xrightarrow{h} Y$ of $L \downarrow \Sigma$ together with morphisms $g' : (f) \to (h)$ and $f' : (g) \to (h)$.

Finally, given parallel morphisms in $L \downarrow \Sigma$:

$$\begin{array}{ccc} & L & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f'} \\ C_f & \overset{h_1}{\underset{h_2}{\rightrightarrows}} & C_{f'} \end{array}$$

then (iv) of II.2 implies that there is $t : C_{f'} \to Y$ in Σ with $th_1 = th_2$ (because $f \in \Sigma$ fulfils $h_1 f = h_2 f$). Then $f'' = tf'$ is an object of $L \downarrow \Sigma$ and $t : (f') \to (f'')$ a morphism merging h_1, h_2.

(b) We are going to prove

$$QL \perp s$$

for any $s : X \to X'$ in Σ. That is, $\hom(-, QL)$ maps s to an isomorphism in **Set**. We verify that $\hom(s, QL)$ is both an epimorphism and monomorphism.

(b1) Epimorphism: we have to show that every morphism $k : X \to QL$ factors through s. Since QL is a filtered colimit of D_L, the morphism k (whose domain is finitely presentable) factors through some of the colimit maps c_f; i.e., we have $f : L \to C_f$ in $L \downarrow \Sigma$ and $h : X \to C_f$ with

$$k = c_f h\,. \tag{1}$$

By applying II.2 (iii) to h and s we obtain a commutative square

$$(2)\qquad \begin{array}{ccc} X & \xrightarrow{\ s\ } & X' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ C_f & \xrightarrow[\ s'\ ]{} & B \end{array}$$

with $s' \in \Sigma$. By II.2 (ii) we obtain an object

$$g = s'f : L \to B \quad \text{in} \quad L \downarrow \Sigma$$

and $s' : (f) \to (g)$ is a morphism of $L \downarrow \Sigma$, thus

$$(3) \qquad c_f = c_g s' .$$

We conclude that $c_g h'$ is the desired factorization:

$$\begin{aligned} k &= c_f h && \text{by} \quad (1) \\ &= c_g s' h && \text{by} \quad (3) \\ &= c_g h' s && \text{by} \quad (2). \end{aligned}$$

(b2) Monomorphism: We have to prove that given two morphisms

$$u_1, u_2 : X' \to QL \quad \text{in} \quad \mathcal{L}$$

then

$$u_1 s = u_2 s \quad \text{implies} \quad u_1 = u_1 .$$

Since X' is finitely presentable, u_1 and u_2 factor through c_f for some $f \in L \downarrow \Sigma$:

$$(4) \qquad u_1 = c_f v_1 \quad \text{and} \quad u_2 = c_f v_2 \qquad (f \in \Sigma) .$$

From the equality

$$c_f(v_1 s) = u_1 s = u_2 s = c_f(v_2 s)$$

it follows, since X is finitely presentable, that $v_1 s$ and $v_2 s$ are merged by some morphism of the diagram $L \downarrow \Sigma$, i.e., that there exists $g \in L \downarrow \Sigma$ and $p : C_f \to C_g$ such that

(5) $$pv_1 s = pv_2 s \qquad \text{and} \quad g = pf\,.$$

Applying II.2 (iv), we obtain, since $s \in \Sigma$, the existence of $\overline{s} : C_g \to Y$ in Σ with

(6) $$\overline{s}pv_1 = \overline{s}pv_2\,.$$

Thus by II.2 (ii), $h = g\overline{s}$ is an object of $L \downarrow \Sigma$ and $\overline{s} : (g) \to (h)$ is a morphism, consequently,

(7) $$c_g = c_h \overline{s}.$$

We obtain

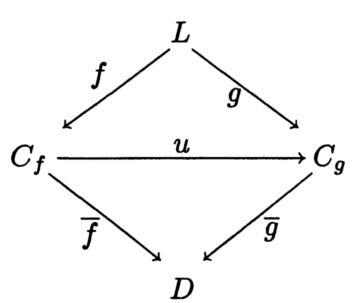
$$\begin{aligned} u_i &= c_f v_i && \text{by} \quad (4) \\ &= c_g p v_i && \text{since } p : C_f \to C_g \text{ is a morphism of } L \downarrow \Sigma \\ &= c_h \overline{s} p v_i && \text{by } (7) \end{aligned}$$

and the last line is independent of i by (6).

(c) $c_{\mathrm{id}} : L \to QL$ is a reflection of L in $\Sigma^{\perp}$. In fact, let $d : L \to D$ be a morphism of $\mathcal{L}$ with $D \in \Sigma^{\perp}$. For each $f : L \to C_f$ in $L \downarrow \Sigma$ we have, since $f \in \Sigma$, a unique morphism $\overline{f} : C_f \to D$ with

$$\overline{f} f = d \qquad (f \in L \downarrow \Sigma)$$

and those morphisms form a cocone of D_L



In fact, every morphism $u : C_f \to C_g$ of $L \downarrow \Sigma$ fulfils $g = uf$, and from $\overline{f}f = d = \overline{g}g = (\overline{g}u)f$ we conclude, since $D \perp f$, that $\overline{f} = \overline{g}u$. The unique morphism $d^* : QL \to D$ with $d^*c_f = \overline{f}$ (for each f) fulfils

$$d^*c_{\text{id}} = d^*c_f f = \overline{f}f = d.$$

Conversely, from $d^*c_{\text{id}} = d$ we have d^* uniquely determined (since $d^*c_f = \overline{f}$ for all f). □

II.4 Corollary. *Every ω-orthogonality class in a finitely accessible category $\mathcal{L}$ presented by a set of morphisms of $\mathcal{L}_\omega$ admitting a left calculus of fractions (in $\mathcal{L}_\omega$) is reflective in $\mathcal{L}$.*

II.5 Remark. The above result is somewhat surprising because no sufficient conditions for reflectivity in (non-complete) accessible categories have been known.

But also for locally finitely presentable categories the above construction brings new information. We will see more of this in the proof of the main characterization theorem in Part III. Here we derive some direct consequences.

II.6 Notation. For a full subcategory $\mathcal{K}$ of $\mathcal{L}$ we denote by

$$Orth_\omega \mathcal{K}$$

the class of all morphisms in $\mathcal{L}_\omega$ to which all objects of $\mathcal{K}$ are orthogonal, and by

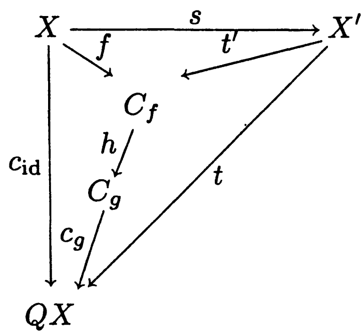
$$Inj_\omega \mathcal{K}$$

the class of all morphisms in $\mathcal{L}_\omega$ to which all objects of $\mathcal{K}$ are injective.

II.7 Corollary. *A full subcategory $\mathcal{K}$ of an* LFP *category $\mathcal{L}$ is an ω-orthogonality class iff $\mathcal{K}$ is closed under limits and filtered colimits, and every morphism $s : X \to X'$ in $Inj_\omega \mathcal{K}$ can be prolongated to a morphism $ts : X \to X''$ in $Orth_\omega \mathcal{K}$.*

Proof. I. Necessity. Let $\mathcal{K} = \Sigma^\perp$ for a set Σ of morphism in $\mathcal{L}_\omega$; as remarked in II.2, since $\mathcal{L}_\omega$ has finite colimits, we can assume that Σ admits a left calculus of fractions (by taking the saturation in $\mathcal{L}_\omega$).

Given $s : X \to X'$ in $Inj_\omega \mathcal{K}$ apply II.3 to obtain a



reflection $c_{\mathrm{id}} : X \to QX = \operatorname{colim} D_X$. Since $QX \in \mathcal{K}$ is injective to s, we have $t : X' \to QX$ with $ts = c_{\mathrm{id}}$. Now X' is finitely presentable and QX is a filtered colimit, thus, t factors through one of the colimit maps $c_f : C_f \to QX$:

$$t = c_f t' \quad \text{with} \quad t' : X' \to C_f \, .$$

The equality

$$c_f f = c_{\mathrm{id}} = ts = c_f(t's) : X \to QX$$

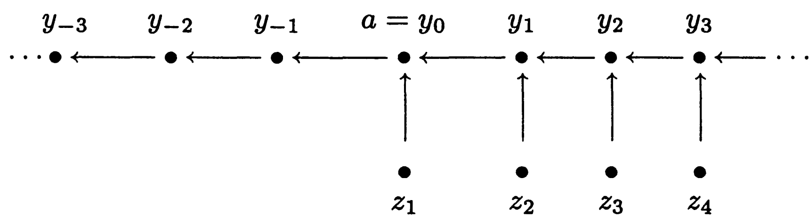
implies, since X is finitely presentable, that some morphism of the diagram D_X merges f and $t's$; say, $h : (X \xrightarrow{f} C_f) \to (X \xrightarrow{g} C_g)$, where $g = hf$, fulfils $hf = ht's$. Then g, being in $X \downarrow \Sigma$, and hence in $Orth_\omega \mathcal{K}$, is the required prolongation of s.

II. Sufficiency. Let $\mathcal{K}$ be closed under limits and filtered colimits. Then it is also closed under pure subobjects because every pure subobject $A \rightarrowtail B$ is an equalizer of morphisms $B \rightrightarrows B^*$ where B^* is a filtered colimit of powers of B, see [AR], 2.31. Thus, by [RAB], $\mathcal{K}$ is an ω-injectivity class, presented e.g. by $Inj_\omega \mathcal{K}$. For each $s : X \to X'$ in $Inj_\omega \mathcal{K}$ choose a prolongation $t_s s : X \to X''$ in $Orth_\omega \mathcal{K}$, then obviously every object orthogonal to $t_s s$ is injective to s, therefore, $\mathcal{K}$ is equal to the ω-orthogonality class $\{t_s s; s \in Inj_\omega \mathcal{K}\}^\perp$. □

II.8 Example of a class of algebras which is not an ω-orthogonality class, although it is closed in Alg$(1,0)$ (the category of all algebras on one unary operation α and one constant a) under limits and filtered colimits. The class $\mathcal{K}$ consists of all algebras (A, α, a) which have a sequence $a = y_0, y_1, y_2 \dots$ of elements with $\alpha y_{n+1} = y_n$ such that whenever $\alpha^k z = a$ for $z \in A$ and $k > 0$, then $\alpha z = y_n$ for some $n \in \omega$. More succintly, $\mathcal{K}$ can be axiomatized by the following sentences (indexed by $n \in \omega$):

$$\exists! y_0, y_1, \dots, y_{n-1} (\exists y_n, y_{n+1} (a = y_0 \wedge \bigwedge_{k=0}^{n} \alpha y_{k+1} = y_k)) \, .$$

$\mathcal{K}$ is clearly closed under limits and filtered colimits. To prove that $\mathcal{K}$ is not an ω-orthogonality class, we use II.7. Consider the following algebra K_0 which obviously lies in $\mathcal{K}$:



Denote by $f : A \to A'$ the unique morphism from the initial algebra A of Alg$(1, 0)$ to the subalgebra A' of K_0 on $\{y_n\}_{n \leq 1}$. Every algebra $K \in \mathcal{K}$ is injective w.r.t. f : consider the homomorphism $A' \to K$ given by $y_1 \to y_1$ (thus $y_0 \mapsto \alpha y_1 = a, y_{-1} \mapsto \alpha a, \dots$). If $\mathcal{K}$ were an ω-orthogonality class, there would exist $g : A' \to A''$ in Alg$(1,0)_\omega$ such that $K_0 \perp gf$; in particular we would conclude that

there exists exactly one homomorphism h from A'' to K_0.

We derive a contradiction. Since A'' is finitely presentable in Alg$(1, 0)$, there is a largest $m \in \omega$ with $\alpha^m z = a$ for some $z \in A''$. Observe that $m \geq 1$ because for $z = g(y_1)$ we conclude from $\alpha y_1 = a$ in A' that $\alpha g(y_1) = a$ in A'', and $g(y_1) \neq a$ (else $g(y_1)$ would be a fixed point of α, but K_0 has no fixed points of α).

Every z in A'' with $\alpha^m z = a$ fulfils $\alpha^m h(z) = a$ in K_0, thus, $h(z) = y_m$ or z_m; observe that then $h(\alpha z) = y_{m-1}$. Consequently, if we swap y_m and z_m, we obviously obtain another homomorphism $\overline{h} : A'' \to K_0$:

$$\overline{h}(z) = \begin{cases} h(z) & \text{if} \quad \alpha^m z \neq a \\ z_m & \text{if} \quad h(z) = y_m \\ y_m & \text{if} \quad h(z) = z_m \, . \end{cases}$$

This contradicts the uniqueness of h.

II.9 Remark. The above example is an adaptation of an example presented by H. Volger [V] to demonstrate that a class of (relational) structures of a given signature Σ can be closed under limits and filtered colimits in $Str\,\Sigma$ (the category of all structures and homomorphisms) without being axiomatizable in a "uniform" way – thus, Volger's argument shows more than the fact that this example is not an ω-orthogonality class. The signature H. Volger used had infinitely many relations, and it has been later simplified in [MPi] to an example with a binary relation and a constant. Our example above is in the same spirit, but our argument is simpler that in the previous work.

Let us remark here that H. Volger characterized classes of Σ-structures closed under limits and filtered colimits in a spirit closely related to Corollary II.7. Recall from Coste's paper [C] the concept of a *limit sentence* in first-order logic: it is a sentence of the form

$$\forall \vec{x} (\varphi(\vec{x}) \to \exists! \vec{y} (\psi(\vec{x}, \vec{y}))) \tag{1}$$

where φ and ψ are (finite) conjunctions of atomic formulas and $\vec{x}$ and $\vec{y}$ are (finite) strings of variables. A class of Σ-structures is an ω-orthogonality class in $Str\,\Sigma$ iff it can be axiomatized by limit sentences, see 5.6 in [AR]. Now drop the uniqueness requirement in the above formula and obtain a formula that we might call a *weak limit sentence*: it has the following form

$$\forall \vec{x} (\varphi(\vec{x}) \to \exists \vec{y} (\psi(\vec{x}, \vec{y}))) \, . \tag{2}$$

The ω-injectivity classes in $Str\,\Sigma$ are precisely those axiomatizable by weak limit sentences. This is shown in detail in 5.33 of [AR], and the reader familiar with that passage between syntax and semantics will have no difficulty in translating II.7 above as follows:

II.10 Corollary. *A full subcategory $\mathcal{K}$ of $Str\Sigma$ is an ω-orthogonality class iff it can be axiomatized by weak limit sentences and for each of these sentences (2) there exists a (finite) conjunction $\psi'(\vec{x},\vec{y},\vec{z})$ of atomic formulas such that $\mathcal{K}$ satisfies the following limit sentence:*

$$(3) \qquad \forall\vec{x}(\varphi(\vec{x}) \to \exists!(\vec{y},\vec{z})(\psi(\vec{x},\vec{y}) \wedge \psi'(\vec{x},\vec{y},\vec{z}))) .$$

The above Corollary is closely related to the following result [V].

Theorem (H. Volger). *A full subcategory $\mathcal{K}$ of $Str\Sigma$ is closed under limits and filtered colimits iff it can be axiomatized by weak limit sentences, and for each of these sentences* (2) *there exists a (finite) conjunction $\psi'(\vec{x},\vec{y},\vec{z})$ of atomic formulas such that $\mathcal{K}$ satisfies the following sentence*

$$(4) \qquad \forall\vec{x}(\varphi(\vec{x}) \to \exists!\vec{y}(\psi(\vec{x},\vec{y}) \wedge \exists\vec{z}\psi'(\vec{x},\vec{y},\vec{z}))) .$$

II.11 Remark. The results of this section immediately generalize to λ-orthogonality classes for all regular cardinals λ. Let us say that a class Σ of morphisms of $\mathcal{X}$ admits a *λ-strong left calculus of fractions* if it satisfies, besides (i)– (iv) of II.2,

(v) for each $f_i : X \to Y_i$, $i \in I$, $|I| < \lambda$ from Σ there are $g_i : Y_i \to Z$, $i \in I$ in Σ such that $g_i f_i = g_j f_j$ for each $i, j \in I$.

Given a λ-accessible category $\mathcal{L}$ and a set Σ of morphisms in $\mathcal{L}_\lambda$ admitting a λ-strong left calculus of fractions, we can construct, for every λ-presentable object L of $\mathcal{L}$, a reflection of L in $\Sigma^\perp$ as follows: form the comma category $L \downarrow \Sigma$ as above, and prove that it is λ-filtered. Then the forgetful functor $D_L : L \downarrow \Sigma \to \mathcal{L}$ has a colimit $(C_f \xrightarrow{c_f} QL)_{f \in L\downarrow\Sigma}$, and we have $QL \in \Sigma^\perp$, and $c_{\mathrm{id}} : L \to QL$ is a reflection of L in $\Sigma^\perp$.

For all other objects of $\mathcal{L}$ a reflection in $\Sigma^\perp$ is constructed by λ-filtered colimits (analogously to Case (B) of II.3 above). Thus, one obtains the following

Theorem. *For every λ-accessible category $\mathcal{L}$, the orthogonality class $\Sigma^\perp$, where Σ is a set of morphisms in $\mathcal{L}_\lambda$ admitting a λ-strong left calculus of fractions, is reflective in $\mathcal{L}$.*

Concluding Remark. The above procedure is a general solution of the orthogonal subcategory problem in locally presentable categories $\mathcal{L}$:

Let Σ be a set of morphisms in $\mathcal{L}$. There exists λ such that $\mathcal{L}$ is locally λ-presentable, and domains and codomains of members of Σ are λ-presentable objects. Then every object L of $\mathcal{L}$ has a reflection in $\Sigma^{\perp}$: if L is λ-presentable, this reflection is a λ-filtered colimit of the diagram of all morphisms $f : L \to C_f$ in $\overline{\Sigma} \cap \mathcal{L}_{\lambda}$. For a general object L, a reflection is obtained as a λ-filtered colimit of reflections of λ-presentable objects (forming L as a λ-filtered colimit).

Instead of this two-stage approach, we can say directly how a reflection of any object L of $\mathcal{L}$ is formed: consider the full subcategory $\mathcal{D}_L$ of $L \downarrow \overline{\Sigma}$ ($\overline{\Sigma}$ being the saturation of Σ in $\mathcal{L}$) formed by all arrows which are λ-presentable in $L \downarrow \mathcal{L}$. Then $\mathcal{D}_L$ is λ-filtered, and a colimit of the obvious forgetful functor $\mathcal{D}_L \to \mathcal{L}$ gives a reflection of L in $\Sigma^{\perp}$. The proof is analogous to that in II.3 above.

III. Characterization Theorem

III.1 Quotient Functors. In [M] an interesting factorization system for the 2-category

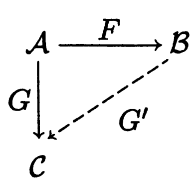
$$\text{Lex}$$

of small lex (= finitely complete) categories, lex functors, and natural transformations has been introduced: every morphism (1-cell) of Lex has an essentially unique factorization into a quotient functor followed by a conservative one. Here a *conservative* functor is one that reflects isomorphisms. And a *quotient functor* is a lex functor $F : \mathcal{A} \to \mathcal{B}$ for which a set Σ of morphisms in $\mathcal{A}$ exists such that

(a) F turns Σ-morphisms into isomorphisms

and

(b) F is lax universal w.r.t. (a), i.e.:



for every small lex category $\mathcal{C}$ the precomposition with F is an equivalence of categories

$$(-)F : Lex(\mathcal{B},\mathcal{C}) \to Lex_{\Sigma^{-1}}(\mathcal{A},\mathcal{C})$$

Here $Lex_{\Sigma^{-1}}(\mathcal{A},\mathcal{C})$ is the full subcategory of $Lex(\mathcal{A},\mathcal{C})$ formed by lex functors turning Σ-morphisms into isomorphisms. Thus, (b) implies that every such functor $G : \mathcal{A} \to \mathcal{C}$ determines an essentially unique lex functor G' with $G \cong G'F$.

III.2 Proposition (A. Pitts, see [M]). *A lex functor $F : \mathcal{A} \to \mathcal{B}$ is a quotient iff it is essentially onto in the following sense: for every morphism $b : X \to FA$ in $\mathcal{B}$, with $A \in \mathcal{A}$, there exists a morphism $a : A' \to A$ in $\mathcal{A}$ and a commutative triangle*

$$\begin{array}{ccc} X & \xrightarrow{\;b\;} & FA \\ {\scriptstyle\cong}\downarrow & \nearrow{\scriptstyle Fa} & \\ FA' & & \end{array}$$

in $\mathcal{B}$.

III.3 Remark. (a) and (b) above are very close to the concept of category of fractions of Gabriel and Zisman [GZ]: Recall that given a category $\mathcal{A}$ and morphisms $\Sigma \subseteq mor(\mathcal{A})$ we have a *canonical functor* $Q_\Sigma : \mathcal{A} \to \mathcal{A}[\Sigma^{-1}]$, where $\mathcal{A}[\Sigma^{-1}]$ is called a *category of fractions* of Σ, such that

(a) Q_Σ turns Σ-morphisms to isomorphisms

and

(b) Q_Σ is strictly universal w.r.t. (a), i.e., for every functor $G : \mathcal{A} \to \mathcal{C}$ turning Σ-morphisms to isomorphisms there exists a unique functor $G' : \mathcal{B} \to \mathcal{C}$ with $G = G'Q_\Sigma$.

Now (b) implies that

$$(-)Q_\Sigma : \mathbf{Cat}(\mathcal{B},\mathcal{C}) \to \mathbf{Cat}_{\Sigma^{-1}}(\mathcal{A},\mathcal{C})$$

is an isomorphism of categories, where $\mathbf{Cat}_{\Sigma^{-1}}(\mathcal{A},\mathcal{C})$ is the full subcategory of $\mathbf{Cat}(\mathcal{A},\mathcal{C})$ formed by functors turning members of Σ into isomorphisms.

III.4 Observation. Quotients are, up to natural equivalence, precisely the canonical functors. That is, a morphism

$$F : \mathcal{A} \to \mathcal{B} \quad \text{in} \quad \text{Lex}$$

is a quotient iff there exists $\Sigma \subseteq mor(\mathcal{A})$ and an equivalence

$$E : \mathcal{A}[\Sigma^{-1}] \to \mathcal{B}$$

of categories with $F \cong EQ_\Sigma$.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ F\ } & \mathcal{B} \\ {\scriptstyle Q_\Sigma}\big\downarrow & \nearrow{\scriptstyle E} & \\ \mathcal{A}[\Sigma^{-1}] & & \end{array}$$

In fact, it has been proved in [GZ] that if $\mathcal{A}$ is lex, then

(i) $\mathcal{A}[\Sigma^{-1}]$ is lex and Q_Σ is a lex functor

and

(ii) for every lex functor $G \in \mathbf{Cat}_{\Sigma^{-1}}(\mathcal{A}, \mathcal{C})$ the unique $G' \in \mathbf{Cat}(\mathcal{B}, \mathcal{C})$ with $G = G'Q$ is also lex.

Consequently,

$$F \cong EQ_\Sigma \quad \text{implies } F \text{ is a quotient.}$$

Conversely, suppose that F is a quotient. Then there exits a unique E with $F = EQ_\Sigma$, and by (ii) above, E is lex, and there exists a lex functor $\overline{E} : \mathcal{B} \to \mathcal{A}[\Sigma^{-1}]$ with $Q_\Sigma \cong \overline{E}F$. We are to show that E is an equivalence functor: from $\overline{E}EQ_\Sigma = \overline{E}F \cong Q_\Sigma$ it follows that $\overline{E}E \cong \text{id}$, and from $E\overline{E}F \cong EQ_\Sigma = F$, that $E\overline{E} \cong \text{id}$.

III.5 Gabriel-Ulmer Duality. For the sake of notation we recall briefly the well-known duality between Lex and the 2-category

$$\text{LFP}$$

of LFP-categories, functors preserving limits and filtered colimits, and natural transformations. This duality is a biequivalence Th : LFP $\to$ Lex$^{\text{op}}$ assigning to every LFP-category $\mathcal{L}$ its theory

$$\text{Th}(\mathcal{L}) = (\mathcal{L}_\omega)^{\text{op}}$$

and to every LFP-morphism $E : \mathcal{L} \to \mathcal{L}'$ a functor

$$\text{Th}(E) : \text{Th}(\mathcal{L}') \to \text{Th}(\mathcal{L})$$

called the theory of E, obtained as follows: since E preserves limits and colimits, it has a left adjoint preserving finitely presentable objects, and $\text{Th}(E)^{\text{op}} : \mathcal{L}'_\omega \to \mathcal{L}_\omega$ is the domain-codomain restriction of that left adjoint.

The biequivalence forming the inverse of Th is easier to describe: it assigns to every small lex category $\mathcal{A}$ the LFP-category

$$\mathit{Ind}\,\mathcal{A}^{\text{op}} = \mathit{Lex}(\mathcal{A}, \mathbf{Set})$$

of all lex set-valued functors, and to every lex functor $F : \mathcal{A} \to \mathcal{A}'$ it assigns the functor

$$\mathit{Ind}\,F^{\text{op}} = (-)F : \mathit{Lex}(\mathcal{A}', \mathbf{Set}) \to \mathit{Lex}(\mathcal{A}, \mathbf{Set})$$

of precomposition with F. Let us recall that *Ind* denotes a free completion under filtered colimits (see [AGV]): for every category $\mathcal{H}$ we have a category $\mathit{Ind}\,\mathcal{H}$ with filtered colimits and a functor $\eta_{\mathcal{H}} : \mathcal{H} \to \mathit{Ind}\,\mathcal{H}$ with the expected universal property (that each functor F from $\mathcal{H}$ to a category $\mathcal{K}$ with filtered colimits has an essentially unique extension to a functor $F' : \mathit{Ind}\,\mathcal{H} \to \mathcal{K}$ preserving filtered colimits). For small categories with finite colimits, we can describe $\mathit{Ind}\,\mathcal{H}$ as the codomain restriction

$$\eta_{\mathcal{H}} : \mathcal{H} \to \mathit{Ind}\,\mathcal{H} = \mathit{Lex}(\mathcal{H}^{\text{op}}, \mathbf{Set})$$

of the Yoneda embedding.

III.6 Characterization Theorem. *A full subcategory $\mathcal{K}$ of a locally finitely presentable category $\mathcal{L}$ is an ω-orthogonality class iff*

(a) *$\mathcal{K}$ is closed under limits and filtered colimits*

and

(b) *the theory of the embedding $\mathcal{K} \to \mathcal{L}$ is a quotient.*

Remark. As observed above, (a) implies that $\mathcal{K}$ is reflective in $\mathcal{L}$, thus, it is an LFP category. (b) then refers to the lex functor $(Q_\omega)^{\text{op}} : (\mathcal{L}_\omega)^{\text{op}} \to (\mathcal{K}_\omega)^{\text{op}}$ which is the dual of the domain-codomain restriction Q_ω of a reflector $Q : \mathcal{L} \to \mathcal{K}$.

Proof. I. Sufficiency: assuming (a) and (b), we will prove that every $s : X \to X'$ in $Inj_\omega\,\mathcal{K}$ can be prolongated to an element of $Orth_\omega\,\mathcal{K}$ (see II.7). Denote by $Q : \mathcal{L} \to \mathcal{K}$ a reflector. Since QX is injective w.r.t. s, there exists

(1) $$h : X' \to QX \quad \text{with} \quad hs = \eta_X\,.$$

It follows that the unique morphism

$$h^* : QX' \to QX \quad \text{with} \quad h^*\eta_{X'} = h$$

fulfills

(2) $$h^* Qs = \mathrm{id}_{QX}$$

because we have

$$(h^* Qs)\eta_X = h^*\eta_{X'}s = hs = \eta_X\,.$$

Since Q is a left adjoint to the embedding $E : \mathcal{K} \hookrightarrow \mathcal{L}$ which preserves filtered colimits, Q preserves finite presentability of objects. Consequently, h^* is a morphism of $\mathcal{K}_\omega$. Now the theory of E is obtained from the domain-codomain restriction $Q_\omega : \mathcal{L}_\omega \to \mathcal{K}_\omega$ of Q by dualization:

$$\mathrm{Th}(E) = (Q_\omega)^{\text{op}}\,.$$

By dualizing the necessary and sufficient condition of III.2, we conclude that there exists a morphism

$$t : X' \to X'' \quad \text{in} \quad \mathcal{L}_\omega$$

and an isomorphism $i : Q_\omega X'' \to Q_\omega X$ such that the following triangle

$$\begin{array}{ccc} Q_\omega X & \xleftarrow{\;h^*\;} & Q_\omega X' \\ \uparrow{\scriptstyle i} & \swarrow{\scriptstyle Q_\omega t} & \\ Q_\omega X'' & & \end{array}$$

commutes. The proof will be concluded when we show that the prolongation

$$ts : X \to X'' \qquad \text{in} \quad \mathcal{L}_\omega$$

of s lies in $Orth_\omega \mathcal{K}$. In fact

$$Q_\omega(ts) = i^{-1}$$

because i is an isomorphism with

$$iQ_\omega(ts) = h^* Q_\omega s = \mathrm{id}_{QX}$$

(see (2)). For every object $K \in \mathcal{K}$ and every morphism $f : X \to K$ there exists a unique morphism $h : QX'' \to K$ with $Qf = hQ(ts)$, viz, $h = Qfi$. Consequently, there exists a unique morphism $h_0 : X'' \to K$ with $f = h_0(ts)$, viz, $h_0 = h\eta_{X''}$.

$$\begin{array}{ccc} X & \xrightarrow{\;ts\;} & X'' \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle h_0} & \\ K & & \end{array} \qquad\qquad \begin{array}{ccc} QX & \xrightarrow{\;i^{-1}\;} & QX'' \\ {\scriptstyle Qf}\downarrow & \swarrow{\scriptstyle Qfi = h} & \\ K & & \end{array}$$

II. Necessity. Suppose that $\mathcal{K}$ is an ω-orthogonality class in $\mathcal{L}$. Let Σ be any set in $\mathcal{L}_\omega$ admitting a left calculus of fractions such that $\mathcal{K} = \Sigma^\perp$. Denote by

$$E : \mathcal{K} \to \mathcal{L} \qquad \text{and} \qquad Q : \mathcal{L} \to \mathcal{K}$$

the embedding of $\mathcal{K}$, and a reflector, respectively, such that

$$QE = \mathrm{id}_{\mathcal{K}} \; . \tag{3}$$

Observe that Q is a left adjoint preserving finite presentability (shortly: LAFP-functor) which is equivalent to the fact that the corresponding right adjoint E preserves filtered colimits. Let us extend Σ to the class including all reflection arrows $\eta_L : L \to QL$:

$$\widetilde{\Sigma} = \Sigma \cup \{\eta_L; L \in \operatorname{obj} \mathcal{L}\}.$$

It follows from II.3 that

(4) $\widetilde{\Sigma}$ is contained in the closure of Σ under filtered colimits in $\mathcal{L}^{\to}$.

In fact, for a finitely presentable object L we have constructed η_L as a filtered colimit of all elements of Σ with the domain L; for an arbitrary L, η_L is a filtered colimit of reflection arrows of finitely presentable objects.

Now, because $\widetilde{\Sigma}$ contains the reflection morphisms and has all its elements sent into isomorphisms by Q, one concludes from [S], 19.3.5(c) that Q is naturally isomorphic to the canonical functor $Q_{\widetilde{\Sigma}}$. Thus, for every category $\mathcal{H}$, we have an equivalence of categories

$$(-)Q : \mathbf{Cat}(\mathcal{K}, \mathcal{H}) \to \mathbf{Cat}_{\widetilde{\Sigma}^{-1}}(\mathcal{L}, \mathcal{H})$$

(where the index $\widetilde{\Sigma}^{-1}$ denotes the full subcategory of all functors inverting morphisms in $\widetilde{\Sigma}$). Let us verify that the above equivalence of categories restricts to an equivalence

$$(5) \qquad (-)Q : \mathrm{LAFP}(\mathcal{K}, \mathcal{H}) \to \mathrm{LAFP}_{\widetilde{\Sigma}^{-1}}(\mathcal{L}, \mathcal{H})$$

between (full) subcategories of all LAFP-functors. It is clear that if $G : \mathcal{K} \to \mathcal{H}$ is LAFP, then so is

$$H = GQ;$$

let us verify that, conversely, if H is LAFP, then so is G.

(a) $G : \mathcal{K} \to \mathcal{H}$ is a left adjoint. Since $\mathcal{K}$ is an LFP-category, it is sufficient to observe that G preserves colimits. A colimit of a diagram D in $\mathcal{K}$ is, of course, a reflection of a colimit in $\mathcal{L}$, colim $D \cong Q(\text{colim } ED)$, thus

$$\begin{aligned} G(\text{colim } D) &\cong GQ(\text{colim } ED) \\ &= H(\text{colim } ED) \\ &\cong \text{colim } HED \\ &= \text{colim } GQED \\ &= \text{colim } GD \qquad \text{by } (3)\,. \end{aligned}$$

(b) G preserves finitely presentable objects because a right adjoint $\widehat{G}$ of G preserves filtered colimits. In fact, a right adjoint $\widehat{H} \cong E\widehat{G}$ of H preserves filtered colimits, and E reflects them.

We are ready to prove that $\text{Th}(E)$ is a quotient. Recall that $(\text{Th}(E))^{\text{op}}$ is simply the dual of the domain-codomain restriction

$$Q_\omega : \mathcal{L}_\omega \to \mathcal{K}_\omega$$

of Q. Let $\mathcal{C}$ be a small, lex category. We are to establish the equivalence

$$(-)Q_\omega : \mathit{Lex}(\mathcal{K}_\omega^{\text{op}}, \mathcal{C}) \to \mathit{Lex}_{\Sigma^{-1}}(\mathcal{L}_\omega^{\text{op}}, \mathcal{C})\,.$$

Now Gabriel-Ulmer duality yields an equivalence of categories

$$E^{\mathcal{K}} : \mathit{Lex}(\mathcal{K}_\omega^{\text{op}}, \mathcal{C}) \to \text{LFP}(\mathit{Ind}\mathcal{C}^{\text{op}}, \mathcal{K})$$

and we compose it with the equivalence

$$I^{\mathcal{K}} : \text{LFP}(\mathit{Ind}\mathcal{C}^{\text{op}}, \mathcal{K}) \to \text{LAFP}(\mathcal{K}, \mathit{Ind}\mathcal{C}^{\text{op}})$$

assigning to each functor (a right adjoint preserving filtered colimits) a left adjoint (which is LAFP, of course). This yields an equivalence

$$I^{\mathcal{K}}E^{\mathcal{K}} : \mathit{Lex}(\mathcal{K}_\omega^{\text{op}}, \mathcal{C}) \cong \text{LAFP}(\mathcal{K}, \mathit{Ind}\mathcal{C}^{\text{op}})\,.$$

Analogously, we have

$$I^{\mathcal{L}}E^{\mathcal{L}} : \mathit{Lex}(\mathcal{L}_\omega^{\text{op}}, \mathcal{C}) \cong \text{LAFP}(\mathcal{L}, \mathit{Ind}\mathcal{C}^{\text{op}})\,.$$

We observe that the latter can be restricted to an equivalence

$$J : \mathit{Lex}_{\Sigma^{-1}}(\mathcal{L}_\omega^{\mathrm{op}}, \mathcal{C}) \cong \mathrm{LAFP}_{\widetilde{\Sigma}^{-1}}(\mathcal{L}, \mathit{Ind}\mathcal{C}^{\mathrm{op}}).$$

In fact, given a lex functor $F : \mathcal{L}_\omega^{\mathrm{op}} \to \mathcal{C}$ which inverts Σ, then the corresponding left adjoint $I^{\mathcal{L}}E^{\mathcal{L}}(F) : \mathcal{L} \to \mathit{Ind}\mathcal{C}^{\mathrm{op}}$ also inverts Σ (since its restriction to finitely presentable objects is equivalent to F^{op}) , and moreover, it preserves colimits. Since morphisms in $\widetilde{\Sigma}$ are filtered colimits of morphisms in Σ, see (4), the latter functor inverts $\widetilde{\Sigma}$. Conversely, whenever a left adjoint inverts $\widetilde{\Sigma}$, then its domain-codomain restriction to finitely presentable objects inverts Σ.

Denote by J^- an equivalence-inverse of J, then we have obtained from (5) and the above an equivalence of categories

$$\begin{array}{ccccc} \mathit{Lex}(\mathcal{K}_\omega^{\mathrm{op}}, \mathcal{C}) & \xrightarrow{I^{\mathcal{K}}E^{\mathcal{K}}} & \mathrm{LAFP}(\mathcal{K}, \mathit{Ind}\mathcal{C}^{\mathrm{op}}) & \xrightarrow{(-)Q} & \mathrm{LAFP}_{\widetilde{\Sigma}^{-1}}(\mathcal{K}, \mathit{Ind}\mathcal{C}^{\mathrm{op}}) \\ & & & & \downarrow{\scriptstyle J^-} \\ & & & & \mathit{Lex}_{\Sigma^{-1}}(\mathcal{L}_\omega^{\mathrm{op}}, \mathcal{C}) \end{array}$$

which is, obviously, naturally equivalent to the functor of precomposition with Q_ω^{op}. □

IV. A Description of Categories of Fractions

IV.1 Remark. In the following theorem we apply the above results to describing the category $\mathcal{A}[\Sigma^{-1}]$, for all small lex categories $\mathcal{A}$ and all sets Σ admitting a right calculus of fractions. We use the fact that $\mathcal{A}$ is equivalent to the theory of the LFP category $\mathit{Lex}\,\mathcal{A}^{\mathrm{op}}$ and consider Σ as a set of morphisms in the latter. Observe that objects of $\mathcal{A}$ are finitely presentable in $\mathit{Lex}\,\mathcal{A}^{\mathrm{op}}$, thus, $\Sigma^\perp$ (in $\mathit{Lex}\,\mathcal{A}^{\mathrm{op}}$) is an LFP category and the embedding

$$E : \Sigma^\perp \to \mathit{Ind}\,\mathcal{A}^{\mathrm{op}}$$

is a morphism of LFP. We are going to verify that the theory of E is a canonical functor of Σ. Recall that $\Sigma^\perp$ is reflective in $\mathit{Ind}\,\mathcal{A}^{\mathrm{op}}$ and a reflector $Q : \mathit{Ind}\,\mathcal{A}^{\mathrm{op}} \to \Sigma^\perp$ preserves finitely presentable objects, i.e., restricts to a morphism $\mathcal{A} \to (\Sigma^\perp)_\omega$; a dual of this restriction is $\mathrm{Th}(E) : \mathcal{A} \to \mathrm{Th}\,\Sigma^\perp$.

IV.2 Theorem. *Let Σ be a set of morphisms of a small lex category $\mathcal{A}$ admitting a right calculus of fractions. Then the orthogonality class $\Sigma^\perp$ in $Ind\,\mathcal{A}^{\rm op}$ is an* LFP *category, its embedding E is an* LFP *functor, and the theory of E is canonical for Σ. Shortly:*

$$\mathcal{A}[\Sigma^{-1}] = \mathrm{Th}(\Sigma^\perp)\,.$$

Proof. Apply Theorem III.6 to the locally finitely presentable category

$$\mathcal{L} = Ind\,\mathcal{A}^{\rm op}$$

(whose theory is equivalent to $\mathcal{A}$) and the set

$$\overline{\Sigma} = \{f \in \mathcal{L}_\omega\,;\ \text{every object of } \Sigma^\perp \text{ is orthogonal to } f\},$$

which admits a left calculus of fractions in $\mathcal{A}$ by II.2. We have, of course,

$$\Sigma^\perp = \overline{\Sigma}^\perp,$$

and by III.6 the theory

$$\mathrm{Th}(E) : \mathcal{A} \to \mathrm{Th}(\Sigma^\perp)$$

is a quotient. Following III.4, this implies that $\mathrm{Th}(E)$ is naturally isomorphic to the canonical functor for the set Γ of all morphisms mapped by $\mathrm{Th}(E)$ to an isomorphism. Now $(\mathrm{Th}(E))^{\rm op}$ is the domain-codomain restriction of a reflector $Q : \mathcal{L} \to \Sigma^\perp$; it is clear that Q maps morphisms of $\overline{\Sigma}$ to isomorphisms, thus, $\overline{\Sigma} \subseteq \Gamma$. On the other hand, every object of $\Sigma^\perp$ is orthogonal to Γ, since Q maps morphisms of Γ to isomorphisms (and is a reflector of $\Sigma^\perp$), thus, $\Gamma \subseteq \overline{\Sigma}$. Consequently, $\mathrm{Th}(E)$ is naturally isomorphic to the canonical functor of $\Gamma = \overline{\Sigma}$, which is to say

$$\mathcal{A}[\Sigma^{-1}] = \mathrm{Th}(\overline{\Sigma}^\perp) = \mathrm{Th}(\Sigma^\perp)\,. \qquad \square$$

V. A Generalization of the Characterization Theorem

V.1 We have observed above that our orthogonality construction works even for finitely accessible categories $\mathcal{L}$, provided that the class

Σ admits a left calculus of fractions in $\mathcal{L}_\omega$. Thus, we obtain a reflective subcategory $\mathcal{K} = \Sigma^\perp$ of $\mathcal{L}$. Here we are going to prove that, moreover, the embedding $\mathcal{K} \to \mathcal{L}$ is a quotient, at least when $\mathcal{L}$ has coequalizers. The method of our proof is substantially different from that in III.6. Let us first explain our concepts. If $\mathcal{K}$ is a reflective, full subcategory of a finitely accessible category $\mathcal{L}$, and if it is closed under filtered colimits in $\mathcal{L}$, then a reflector $Q : \mathcal{L} \to \mathcal{K}$ preserves finitely presentable objects. Consequently, like in the LFP case, we have a domain-codomain restriction $Q_\omega : \mathcal{L}_\omega \to \mathcal{K}_\omega$ of Q and the dual functor $(Q_\omega)^{\mathrm{op}}$ is called the *theory* of the embedding $\mathcal{K} \to \mathcal{L}$. (Here, no claim is made that $\mathcal{L}_\omega^{\mathrm{op}}$ is a lex category, of course.) We also extend the terminology of M. Makkai and call a (not necessarily lex) functor a *quotient* provided that it is essentially onto in the sense of III.2.

V.2 Theorem. *Let $\mathcal{L}$ be a finitely accessible category with coequalizers.*

(a) *Every ω-orthogonality class $\mathcal{K} = \Sigma^\perp$ where Σ admits a left calculus of fractions in $\mathcal{L}_\omega$, is reflective in $\mathcal{L}$, and the theory of the embedding $\mathcal{K} \hookrightarrow \mathcal{L}$ is a quotient.*

(b) *Every ω-injectivity class $\mathcal{K}$ which is reflective and such that the theory of the embedding $\mathcal{K} \hookrightarrow \mathcal{L}$ is a quotient is an ω-orthogonality class*

Proof. Statement (b) has an analogous proof to the sufficiency in III.6 above: we choose a set Σ of morphisms of $\mathcal{L}_\omega$ such that $\mathcal{K}$ is the corresponding injectivity class. As in III.6, each $s \in \Sigma$ is prolongated to a morphisms $t_s s$ of $\mathcal{L}_\omega$ to which all objects of $\mathcal{K}$ are orthogonal. Then $\mathcal{K} = \{t_s s;\ s \in \Sigma\}^\perp$.

We are going to prove Statement (a).

(i) The embedding

$$E : \mathcal{K} \hookrightarrow \mathcal{L}$$

is finitely accessible. In fact, since E preserves filtered colimits and has a left adjoint, say,

$$Q : \mathcal{L} \to \mathcal{K},$$

it is obvious that $\mathcal{K}$ is a finitely accessible category, thus, E is a finitely accessible functor.

(ii) Before proving that Th(E) is a quotient, we show that every finitely presentable object K of $\Sigma^{\perp}$ is a reflection of some object of $\mathcal{L}_\omega$.

Consider the canonical filtered diagram

$$U : \mathcal{L}_\omega \downarrow K \to \mathcal{L}$$

whose canonical colimit is $\mathcal{L}_\omega \downarrow K$. For every object $l : L \to K$ of $\mathcal{L}_\omega \downarrow K$ denote by $l^* : QL \to K$ the unique morphism with

$$l^* \eta_L = l \qquad (l \in \mathcal{L}_\omega \downarrow K),$$

where $\eta_L : L \to QL$ is a reflection of L. Since Q preserves colimits, the morphisms l^* form a colimit cocone of the filtered diagram QU in $\mathcal{K}$. Now K is finitely presentable in $\mathcal{K}$, therefore, id_K factors through one of the colimit maps. Thus, we have $l : L \to K$ in $\mathcal{L}_\omega \downarrow K$ and $s : K \to QL$ with

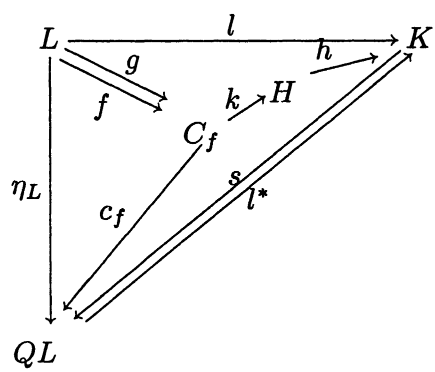
(1) $$l^* s = \mathrm{id}_K \ .$$

We now use the orthogonal-reflection construction II.3: we can assume that QL is a colimit of the filtered diagram D_L with colimit cocone $c_f : C_f \to QL$ (for all $f \in L \downarrow \Sigma$), and we have

(2) $$\eta_L = c_f f \qquad (f \in L \downarrow \Sigma).$$

Since D_L is filtered, the morphism $sl : L \to QL$ with L finitely presentable factors through some of the colimit maps c_f. That is, we have $f : L \to C_f$ in Σ and $g : L \to C_f$ such that

(3) $$sl = c_f g,$$



We conclude that

(4) $$\eta_{C_f} = Qfc_f$$

because QL is orthogonal to f (recall that $f \in \Sigma$) and due to naturality of η we have

$$\eta_{C_f} f = Qf\eta_L = Qfc_f f\,.$$

Furthermore,

(5) $$l^* c_f \quad \text{merges } f \text{ and } g$$

because

$$\begin{aligned} l^* c_f f &= l^* \eta_L \\ &= l \\ &= l^* s l \qquad \text{by (1)} \\ &= l c_f g \qquad \text{by (3)}\,. \end{aligned}$$

Let us form a coequalizer $k : C_f \to H$ of f and g; by (5) there is

(6) $$h : H \to K \quad \text{in} \quad \mathcal{L}_\omega \downarrow K \quad \text{with} \quad l^* c_f = hk$$

(in fact, H is finitely presentable in $\mathcal{L}$ because $f, g : L \to C_f$ are morphisms of $\mathcal{L}_\omega$), and of course

(7) $$kf = kg \quad \text{and } k \text{ is a regular epimorphism in } \mathcal{L}\,.$$

To finish the proof of (ii), we will show that

$$h^* : QH \to K \quad \text{is invertible}$$

(thus, K is a reflection of the finitely presentable object H). Put

(8) $$w = Q(kf)s : K \to QH;$$

we prove that w is an inverse to h^*. On the one hand,

$$\begin{aligned} h^*w &= h^*Q(kf)s && \text{by (8)} \\ &= (hkf)^*s \\ &= (l^*c_f f)^*s && \text{by (6)} \\ &= (l^*\eta_L)^*s && \text{by (2)} \\ &= l^*s \\ &= \mathrm{id}_K && \text{by (1)} \end{aligned}$$

To prove $wh^* = \mathrm{id}_{QH}$, we use the fact that $Q(kf)$ is an epimorphism: Qk is an epimorphism by (7) since Q preserves coequalizers, and Qf is an isomorphism because $f \in \Sigma$. Now we show that $Q(kf)\eta_L$ merges wh^* and id_{QH} which concludes the proof of $wh^* = \mathrm{id}_{QH}$ (because then $Q(kf)$ merges wh^* and id_{QH} too):

$$\begin{aligned} wh^*(Q(kf)\eta_L) &= wh^*\eta_H(kf) \\ &= whkf \\ &= Q(kf)shkf && \text{by (8)} \\ &= Q(kf)shkg && \text{by (7)} \\ &= Q(kf)sl^*c_f g && \text{by (6)} \\ &= Q(kf)sl^*sl && \text{by (3)} \\ &= Q(kf)sl && \text{by (1)} \\ &= Q(kf)c_f g && \text{by (3)} \\ &= Q(k)\eta_{C_f} g && \text{by (4)} \\ &= \eta_H kg \\ &= \eta_H kf && \text{by (7)} \\ &= Q(kf)\eta_L\,. \end{aligned}$$

(iii) Th(E) is a quotient: we verify that for every morphism

$$h : QL \to K \quad \text{with} \quad L \in \mathcal{L}_\omega \text{ and } \quad K \in (\Sigma^\perp)_\omega$$

there exists $L' \in \mathcal{L}_\omega$, an isomorphism $i : QL' \to K$ and a morphism $g : L \to L'$ with $h = iQg$ (then Th(E) satisfies the condition of III.2).

To do this, use (ii) to find an object $L' \in \mathcal{L}_\omega$ with a reflection $\eta_{L'} : L' \to K$. By II.3 this reflection is a filtered colimit of all $f : L' \to C_f$ in $L' \downarrow \Sigma$, and if $c_f : C_f \to K$ denotes the colimit cocone, then $h\eta_L : L \to K$ factors (since L is finitely presentable) through some c_f:

(9) $\quad h\eta_L = c_f g \quad$ for some $\quad f : L \to C_f$ in Σ, $g : L \to C_f$.

Observe that Qf is an isomorphism (since $f \in \Sigma$) and $Q\eta_L$ is, of course, the identity (since $QE = \mathrm{id}$) – thus, from $\eta_L = c_f f$ it follows that Qc_f is an isomorphism. Thus, applying Q to the equality (9) yields (due to $Qh = h$ and $Q\eta_L = \mathrm{id}$)

$$h = Qc_f g \qquad (Qc_f \quad \text{an isomorphism}). \qquad \square$$

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