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ON GALOIS H-OBJECTS AND INVERTIBLE H*-MODULES

by J.N. Alonso ALVAREZ and J.M. Fernández VILABOA*

RESUME. Dans cet article, pour une algèbre de Hopf cocommutative H dans une catégorie fermée symétrique C, les auteurs obtiennent, en généralisant un théorème de L.N. Childs [5], un homomorphisme entre le groupe $Gal_C(H)$ des classes d'isomorphismes des H-objets de Galois et celui $Pic(H^*)$ des classes d'isomorphismes des H^*-modules inversibles.
Finalement, ils montrent que, si $Pic(H^*) = 1$, le groupe de Brauer des H-modules triples d'Azumaya avec action intérieure coïncide avec le groupe de Brauer des H-modules triples d'Azumaya défini par J.M. Fernández Vilaboa dans un article antérieur.

Throughout this paper C denotes a symmetric closed category with equalizers and co-equalizers and with natural isomorphism τ coming from symmetry. We assume the reader is familiar with ordinary Hopf algebras [10] and Galois H-objects [4] and we refer to [1], [8] and [9] for all undefined notions used in the text.

Definition 0.1 *Let H be a commutative Hopf algebra. A left H-module (M, φ_M) is said to be invertible if there exists a left H-module (N, φ_N), and an isomorphism $f : M \otimes_H N \to H$ of left H-modules, where $M \otimes_H N$ is the left H-module defined by the following coequalizer diagram:*

$$M \otimes H \otimes N \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} M \otimes N \xrightarrow{c_{M,N}} M \otimes_H N$$

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($\varphi_1 = (\varphi_M \circ \tau_{MH}) \otimes N$, *and* $\varphi_2 = M \otimes \varphi_N$).

With $Pic(H)$ *we will denote the set of isomorphism classes* $[(M, \varphi_M)]$ *of invertible left* H*-modules.* $Pic(H)$ *is an abelian group under the operation:*

$$[(M, \varphi_M)] * [(N, \varphi_N)] = [(M \otimes_H N, \varphi_{M \otimes_H N})]$$

being the unit element $[(H, \mu_H)]$.

We point out that this group is not the group $Pic(\mathcal{C}\ ,\ H)$ of [9], because the monoidal structures are different.

In what follows, H denotes a finite cocommutative Hopf algebra in $\mathcal{C}$ and H^* the dual commutative Hopf algebra of H. We denote by $Gal_{\mathcal{C}}(H)$ the group of Galois H-objects and by $N_{\mathcal{C}}(H)$ the subgroup of Galois H-objects with a normal basis (see 2.5 of [1]).

Proposition 0.2 *The map* $h : Gal_{\mathcal{C}}(H) \to Pic(H^*)$, *defined by*

$$h[(A; \rho_A)] = [(A^*, \varphi_{A^*})]$$

is a homomorphism, where

$$\varphi_{A^*} = (A^* \otimes [\overline{b_A} \circ (A \otimes \overline{b_H} \otimes A^*) \circ (\rho_A \otimes H^* \otimes A^*)]) \circ (\overline{a_A} \otimes H^* \otimes A^*)$$

Proof. First, note that $\overline{a_A}$ and $\overline{b_A}$ represent the unit and the counit, respectively, of the $\mathcal{C}$-adjunction $A \otimes - \dashv A^* \otimes - : \mathcal{C} \to \mathcal{C}$ wich exists because A is a progenerator.

(A^*, φ_{A^*}) is a left H^*-module:

$$\varphi_{A^*} \circ (\eta_{H^*} \otimes A^*) = A^*$$

$$\begin{aligned}
&\varphi_{A^*} \circ (H^* \otimes \varphi_{A^*}) = \\
&(A^* \otimes [\overline{b_A} \circ (A \otimes \overline{b_H} \otimes A^*) \circ (A \otimes H \otimes \overline{b_H} \otimes H^* \otimes A^*) \\
&\circ (A \otimes \tau_{HH} \otimes H^* \otimes H^* \otimes A^*) \circ (A \otimes \delta_H \otimes H^* \otimes H^* \otimes A^*) \\
&\circ (\rho_A \otimes H^* \otimes H^* \otimes A^*)]) \circ (\overline{a_A} \otimes H^* \otimes H^* \otimes A^*) \\
&= \varphi_{A^*} \circ (\mu_{H^*} \otimes A^*)
\end{aligned}$$

Moreover, if $f : A \to B$ is an isomorphism of Galois H-objects, $f^* = (A^* \otimes (\overline{b_B} \circ (f \otimes B^*))) \circ (\overline{a_A} \otimes B^*) : B^* \to A^*$ is an H^*-module isomorphism.

We denote by $t : (A \otimes B)^* \to A^* \otimes B^*$ the isomorphism

$$\begin{aligned} t &= [(A^* \otimes B^* \otimes \overline{b_{A\otimes B}}) \circ (A^* \otimes B^* \otimes \tau_{BA} \otimes (A \otimes B)^*) \\ &\circ (A^* \otimes \overline{a_B} \otimes A \otimes (A \otimes B)^*) \circ (\overline{a_A} \otimes (A \otimes B)^*)] \end{aligned}$$

$((A \bullet B)^*, \varphi_{(A\bullet B)^*}) \cong (A^* \otimes_{H^*} B^*, \varphi_{A^*\otimes_{H^*}B^*})$ as H^*-modules. Indeed:

The morphism $r = c_{A^*,B^*} \circ t : (A \otimes B)^* \to A^* \otimes_{H^*} B^*$ factors through the coequalizer $i^*_{AB} : (A \otimes B)^* \to (A \bullet B)^*$:

$$\begin{aligned}
& r \circ (\partial^1_{AB})^* = c_{A^*,B^*} \circ (A^* \otimes B^* \otimes \overline{b_{A\otimes B\otimes H}}) \circ \\
& (A^* \otimes B^* \otimes A \otimes \tau_{HB} \otimes (A \otimes B \otimes H)^*) \\
& \circ (A^* \otimes B^* \otimes \rho_A \otimes B \otimes (A \otimes B \otimes H)^*) \circ \\
& (A^* \otimes B^* \otimes \tau_{BA} \otimes (A \otimes B \otimes H)^*) \\
& \circ (A^* \otimes \overline{a_B} \otimes A \otimes (A \otimes B \otimes H)^*) \circ (\overline{a_A} \otimes (A \otimes B \otimes H)^*) \\
& = c_{A^*,B^*} \circ (A^* \otimes B^* \otimes \overline{b_{A\otimes B\otimes H}}) \circ \\
& (A^* \otimes B^* \otimes \tau_{BA} \otimes H \otimes (A \otimes B \otimes H)^*) \\
& \circ (A^* \otimes B^* \otimes B \otimes \rho_A \otimes (A \otimes B \otimes H)^*) \circ \\
& (A^* \otimes \overline{a_B} \otimes A \otimes (A \otimes B \otimes H)^*) \circ (\overline{a_A} \otimes (A \otimes B \otimes H)^*) \\
& = c_{A^*,B^*} \circ (A^* \otimes B^* \otimes \overline{b_{A\otimes B\otimes H}}) \circ \\
& (A^* \otimes B^* \otimes \tau_{BA} \otimes H \otimes (A \otimes B \otimes H)^*) \\
& \circ (A^* \otimes B^* \otimes B \otimes \tau_{HA} \otimes (A^* \otimes B^* \otimes H)^*) \circ \\
& (A^* \otimes B^* \otimes \rho_B \otimes A \otimes (A^* \otimes B^* \otimes H)^*) \\
& \circ (A^* \otimes \overline{a_B} \otimes A \otimes (A^* \otimes B^* \otimes H)^*) \circ (\overline{a_A} \otimes (A \otimes B \otimes H)^*) = r \circ (\partial^2_{AB})^*
\end{aligned}$$

and then, there exists a morphism $f : (A \bullet B)^* \to A^* \otimes_{H^*} B^*$ such that $f \circ i^*_{AB} = r$. Is not difficult to show that f is a left H^*-module isomorphism with inverse the factorization of the morphism $i^*_{AB} \circ t^{-1}$: $A^* \otimes B^* \to (A \bullet B)^*$ through the coequalizer c_{A^*,B^*} .

Proposition 0.3 $Kerh = N_{\mathcal{C}}(H)$.

Proof. If $[(A;\rho_A)] \in Kerh$, there exists a left H^*-module isomorphism $f : A^* \to H^*$. The morphism $f^* = ([\overline{b_H} \circ (H \otimes f)] \otimes A) \circ (H \otimes \overline{a_A})$ is an isomorphism of right H-comodules and then $[(A;\rho_A)] \in N_{\mathcal{C}}(H)$. Moreover, if $g : H \to A$ is a right H-comodule isomorphism, then g^* is an isomorphism of left H^*-modules.

Note that, if $\mathcal{C}$ is the category of R-modules over a commutative ring R, this proposition already appears in [5].

Let $BM(\mathcal{C}, H)$ be the Brauer group whose elements are equivalence classes of left H-module Azumaya monoids in $\mathcal{C}$ and let $BM_{inn}(\mathcal{C}, H)$ be the subgroup of $BM(\mathcal{C}, H)$ built up with the equivalence classes that can be representated by a left H-module Azumaya monoid with inner action (see 4.4 of [1]).

Proposition 0.4 *If* $Pic(H^*) = 1$, *then* $BM_{inn}(\mathcal{C}, H) \simeq BM(\mathcal{C}, H)$.

Proof: By 4.5 in [1], $BM_{inn}(\mathcal{C}, H) \simeq B(\mathcal{C}) \oplus N_{\mathcal{C}}(H)$. Moreover, by section 2 in [9], $BM(\mathcal{C}, H) \simeq B(\mathcal{C}) \oplus Gal_{\mathcal{C}}(H)$. Since $N_{\mathcal{C}}(H)$ is the kernel of $h : Gal_{\mathcal{C}}(H) \to Pic(H^*)$, if $Pic(H^*) = 1$ then $N_{\mathcal{C}}(H) \simeq Gal_{\mathcal{C}}(H)$ and then $BM_{inn}(\mathcal{C}, H) \simeq BM(\mathcal{C}, H)$.

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