CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 40, nº 2 (1999), p. 82-126
<http://www.numdam.org/item?id=CTGDC_1999__40_2_82_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XL-2 (1999)

UNIFORM FILTERS

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RESUME. Le but de cet article est de réunir et développer quelques unes des propriétés principales des filtres uniformes, en insistant sur leur comportement fonctoriel et leur structure quantale.

Introduction.

The notion of uniform (or "topologizing") filter is not new: it has been introduced in the sixties by Gabriel [7], who proved that *idempotent* uniform filters (nowadays referred to as "Gabriel filters") over a ring R correspond bijectively to localizations of the abelian category $R-$**Mod**. At a later stage, Goldman [13] has pointed out that Gabriel filters are also in bijective correspondence with so-called idempotent kernel functors. In view of the importance of localization at idempotent filters, historically the study of general uniform filters has somewhat been neglected.

In the past, except for rather specialized applications given in [2, 10, 15, et al], uniform filters have mainly been considered within the framework of linear topologies, cf. [1, 14, et al] and the monograph [12], which is probably the most complete text on uniform filters and includes a large list of examples.

Recently however, new applications of uniform filters arose, somewhat unexpectedly, in the context of noncommutative algebraic geometry, cf. [9, 19], for example. In particular, these new applications require a deeper study of the functorial properties of uniform filters with respect to change of base ring and thus urged us to reconsider the notion of uniform filter.

*Supported by D.G.U.I. - Gobierno de Canarias.

†Research supported by PB94-0791(D.G.I.C.Y.T.).

‡This note was written while the last author was a guest of the Departamento de Matemática Fundamental of the University of La Laguna. He likes to thank its members for their kind hospitality. He also benefitted from some fruitful discussions with F. Borceux (U.C.L.). Research supported by the F.W.O.

This note is organized as follows. In the first section, we recollect some general results on the lattice of uniform filters, which appear to be somewhat shattered in the literature. In particular we briefly study examples of uniform filters associated to prime and arbitrary twosided ideals. These examples arise naturally in the framework of noncommutative algebraic geometry. In the second section, we show that the lattice of uniform filters possesses a quantale structure, thus allowing the construction of nicely behaving structure sheaves associated to noncommutative rings, à la Borceux-Cruciani [3]. In the last section, we describe how ring homomorphisms between a ring R and a ring S allow, modulo some harmless restrictions, to induce well related uniform filters from R to S, thus providing the tools needed in the geometric context initiated in [9].

Acknowledgement.

The authors wish to thank the referee for some very useful remarks and suggestions.

1. Filters

(1.1) Let R be an associative ring with unit. A *filter* over R is a non-empty set $\mathcal{L}$ of left ideals such that if L and K belong to $\mathcal{L}$, then so does any left ideal $H \supseteq L \cap K$. A left R-module M is said to be $\mathcal{L}$*-torsion*, if for any $m \in M$ there exists some $L \in \mathcal{L}$ with the property that $Lm = 0$, i.e., if $\mathrm{Ann}^l_R(m) \in \mathcal{L}$. The class of all $\mathcal{L}$-torsion left R-modules is denoted by $\mathcal{T}_\mathcal{L}$.
For any pair of filters $\mathcal{L}$ and $\mathcal{H}$, one defines the *composition* $\mathcal{L} \circ \mathcal{H}$ to consist of all left R-ideals L with the property that we may find some $H \in \mathcal{H}$ containing L such that H/L is $\mathcal{L}$-torsion. Since $\mathcal{H}$ is a filter, H may be chosen to consist of all $r \in R$ such that $(L : r) \in \mathcal{L}$. If $\mathcal{L} \subseteq \mathcal{L}'$ and $\mathcal{H}$ are filters, clearly $\mathcal{L} \circ \mathcal{H} \subseteq \mathcal{L}' \circ \mathcal{H}$ and $\mathcal{H} \circ \mathcal{L} \subseteq \mathcal{H} \circ \mathcal{L}'$, and it is easy to prove that $\mathcal{H} \subseteq \mathcal{L} \circ \mathcal{H}$. Note also that $\{R\} \circ \mathcal{L} = \mathcal{L}$ for any filter $\mathcal{L}$.

A filter $\mathcal{L}$ is said to be *uniform*, if it has the property that $\mathcal{L} \subseteq \mathcal{L} \circ \{R\}$, i.e., if for any $L \in \mathcal{L}$ and any $r \in R$, we have that $(L : r) \in \mathcal{L}$ as well. A non-empty set of left ideals of R is a uniform filter if and only if it is the family of left ideals of R, which are open neighbourhoods of 0 for a linear topology on R, cf. [13]. The composition of uniform filters is again a uniform filter, and it is easy to see that this yields an associative operation in the set of uniform filters over R. If $L \in \mathcal{L}$ and $K \in \mathcal{H}$, then $LK \in \mathcal{L} \circ \mathcal{H}$, since $LK \subseteq K \in \mathcal{H}$ and since for every $r \in K$, the left ideal $(LK : r)$ belongs to $\mathcal{L}$, as it contains L. It is also easy to prove that $\mathcal{L} \subseteq \mathcal{L} \circ \mathcal{H}$.

(1.2) Let $\mathcal{L}$ be a uniform filter. For any left R-module M the set $\sigma_{\mathcal{L}} M$ consisting of all elements $m \in M$ annihilated by some $L \in \mathcal{L}$ is easily seen to be a left R-submodule of M. Associating $\sigma_{\mathcal{L}} M$ to any $M \in R-\mathbf{Mod}$ thus defines a kernel functor over R, i.e., a left exact subfunctor $\sigma_{\mathcal{L}}$ of the identity in $R-\mathbf{Mod}$. Conversely, every kernel functor σ in $R-\mathbf{Mod}$ defines a uniform filter $\mathcal{L}$, consisting of all left R-modules L with the property that $\sigma(R/L) = R/L$. This defines a bijective correpondence between uniform filters over R and kernel functors in $R-\mathbf{Mod}$, cf. [13].
If $\mathcal{L}$ and $\mathcal{H}$ are uniform filters then, $\mathcal{L} \circ \mathcal{H}$ is in general not the uniform filter corresponding to the composition $\sigma_{\mathcal{L}}\sigma_{\mathcal{H}}$. Moreover, while $\sigma_{\mathcal{L}}\sigma_{\mathcal{H}} = \sigma_{\mathcal{H}}\sigma_{\mathcal{L}}$, cf. [12], the composition $\mathcal{L} \circ \mathcal{H}$ does not necessarily equal $\mathcal{H} \circ \mathcal{L}$, as shown in [5].

The composition of uniform filters may now also be described as follows:

(1.3) Proposition. *Let $\mathcal{L}$ and $\mathcal{H}$ be uniform filters. For any left R-module M the following assertions are equivalent:*

1. $M \in \mathcal{T}_{\mathcal{L} \circ \mathcal{H}}$;
2. *there exists an exact sequence*
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$
where $M' \in \mathcal{T}_{\mathcal{L}}$ and $M'' \in \mathcal{T}_{\mathcal{H}}$.

Proof. First assume M to be $\mathcal{L}\circ\mathcal{H}$–torsion. Obviously, $M' = \sigma_{\mathcal{L}}M \in \mathcal{T}_{\mathcal{L}}$. Choose $\overline{m} \in M'' = M/M'$. Since $m \in M$, there exists some $I \in \mathcal{L}\circ\mathcal{H}$, such that $Im = 0$. By definition, we may pick some $H \supseteq I$ contained in $\mathcal{H}$, such that H/I is $\mathcal{L}$–torsion. So, for every $h \in H$, we may find $L \in \mathcal{L}$ with $Lh \subseteq I$. But then $Lhm = 0$, proving that $hm \in \sigma_{\mathcal{L}}M = M'$. Hence $h\overline{m} = \overline{0} \in M''$, and as this holds for all $h \in H$, we obtain that $M'' \in \mathcal{T}_{\mathcal{H}}$.
Conversely, assume M fits into an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

where M' is $\mathcal{L}$–torsion and M'' is $\mathcal{H}$–torsion. Consider $m \in M$ with image $\overline{m} \in M''$. We may find $H \in \mathcal{H}$ with $H\overline{m} = \overline{0}$, i.e., $Hm \subseteq M'$. For every $h \in H$, there thus exists some $L \in \mathcal{L}$ with $Lhm = 0$, so $Lh \subseteq I = \mathrm{Ann}_R(m)$. So H/I is $\mathcal{L}$–torsion and as $I \in \mathcal{L}\circ\mathcal{H}$, it follows that $m \in \sigma_{\mathcal{L}\circ\mathcal{H}}(M)$. Since $m \in M$ is arbitrary, this implies that M is $\mathcal{L}\circ\mathcal{H}$–torsion, which finishes the proof. □

(1.4) It is clear that a left R–module M is $\mathcal{L}$–torsion if $\sigma_{\mathcal{L}}M = M$. We say that M is $\mathcal{L}$–torsionfree if $\sigma_{\mathcal{L}}M = 0$, and we denote the class of these by $\mathcal{F}_{\mathcal{L}}$.
If $\mathcal{L}$ and $\mathcal{H}$ are uniform filters, the class of $\mathcal{L}\circ\mathcal{H}$–torsionfree modules is the intersection $\mathcal{F}_{\mathcal{L}} \cap \mathcal{F}_{\mathcal{H}}$. Indeed, as $\mathcal{L}, \mathcal{H} \subseteq \mathcal{L}\circ\mathcal{H}$, we have $\mathcal{F}_{\mathcal{L}\circ\mathcal{H}} \subseteq \mathcal{F}_{\mathcal{L}} \cap \mathcal{F}_{\mathcal{H}}$. Conversely, if M is torsionfree with respect to both $\mathcal{L}$ and $\mathcal{H}$, we claim that $M \in \mathcal{F}_{\mathcal{L}\circ\mathcal{H}}$. Indeed, if $m \in M$ is an $\mathcal{L}\circ\mathcal{H}$–torsion element, then there exists $J \in \mathcal{H}$ containing $I = \mathrm{Ann}^l_R(m)$ such that J/I is $\mathcal{L}$–torsion. Pick r in J. Since $(I : r) \in \mathcal{L}$ and $(I : r)rm \subseteq Im = 0$, we have $rm = 0$, as M is $\mathcal{L}$–torsionfree. So, $Jm = 0$, and we obtain $m = 0$ as M is also $\mathcal{H}$–torsionfree.

Let us now take a look at the properties of the couple $(\mathcal{T}_{\mathcal{L}}, \mathcal{F}_{\mathcal{L}})$.

(1.5) Proposition. *Let $\mathcal{L}$ be a uniform filter. Then:*

1. *$\mathcal{T}_{\mathcal{L}}$ is a hereditary pretorsion class, i.e., it is closed under taking direct sums, submodules and epimorphic images;*

2. *$\mathcal{F}_{\mathcal{L}}$ is closed under taking products, submodules, exact extensions and injective hulls.*

Proof. If N is a left R-submodule of $M \in \mathcal{T}_{\mathcal{L}}$, then all $n \in N$ also belong to M, so $\mathrm{Ann}^l_R(n) \in \mathcal{L}$, implying that $N \in \mathcal{T}_{\mathcal{L}}$. On the other hand, for all $\bar{m} \in M/N$, we have $\mathrm{Ann}^l_R(\bar{m}) \supseteq \mathrm{Ann}^l_R(m)$ and since the latter belongs to $\mathcal{L}$, so does $\mathrm{Ann}_R(\bar{m})$, hence M/N also belongs to $\mathcal{T}_{\mathcal{L}}$.
Consider a family $\{M_i\}_{i\in I}$ of $\mathcal{L}$-torsion left R–modules. For all $m = (m_i)_{i\in I} \in \bigoplus_{i\in I} M_i$, we have

$$\mathrm{Ann}^l_R(m) = \bigcap_{i\in I} \mathrm{Ann}^l_R(m_i).$$

Since $\mathrm{Ann}^l_R(0) = R$, this intersection is taken over at most a finite number of non-trivial left ideals in $\mathcal{L}$. So, $\mathrm{Ann}^l_R(x)$ belongs to $\mathcal{L}$ and thus $\bigoplus_{i\in I} M_i \in \mathcal{T}_{\mathcal{L}}$.
In order to check the second assertion, if N is a left R-submodule of $M \in \mathcal{F}_{\mathcal{L}}$ and if $\mathrm{Ann}^l_R(n) \in \mathcal{L}$ for $n \in N$, then $n = 0$ (as M is $\mathcal{L}$–torsionfree), so $\sigma_{\mathcal{L}}(N) = 0$. If $E(M)$ is an injective envelope of M, then $\sigma_{\mathcal{L}}(E(M)) = 0$, since M is essential in $E(M)$, and $M \cap \sigma_{\mathcal{L}}(E(M)) = \sigma_{\mathcal{L}}(M) = 0$.
If $\{M_i\}_{i\in I} \subseteq \mathcal{F}_{\mathcal{L}}$ and $(m_i)_{i\in I} \in \sigma_{\mathcal{L}}(\prod_{i\in I} M_i)$ then $m_j = 0$ for all $j \in I$, since $\mathrm{Ann}^l_R(m_j)$ belongs to $\mathcal{L}$, as it contains $\mathrm{Ann}^l_R((m_i)_{i\in I}) \in \mathcal{L}$, so $(m_i)_{i\in I} = 0$.
Let the sequence

$$0 \longrightarrow M' \stackrel{f}{\longrightarrow} M \stackrel{g}{\longrightarrow} M'' \longrightarrow 0$$

be exact and assume that $M', M'' \in \mathcal{F}_{\mathcal{L}}$. If $m \in M$ and there exists $I \in \mathcal{L}$ such that $Im = 0$, then $Ig(m) = g(Im) = 0$, hence $g(m) = 0$ as M'' is $\mathcal{L}$–torsionfree. So there exists $m' \in M'$ such that $f(m') = m$, and therefore $Im' = 0$, since $f(Im') = Im = 0$ and f is a monomorphism. Thus $m' = 0$ and $m = 0$, proving that $M \in \mathcal{F}_{\mathcal{L}}$. □

For a given hereditary pretorsion class $\mathcal{T}$, let us define the uniform filter $\mathcal{L}_{\mathcal{T}}$ as consisting of all left ideals I of R such that $R/I \in \mathcal{T}$. Since its class of $\mathcal{L}_{\mathcal{T}}$-torsion modules is exactly $\mathcal{T}$, this yields a bijective correspondence between uniform filters and hereditary pretorsion classes.

(1.6) Let us show that the set of all the uniform filters over a ring R ordered by inclusion has a canonical lattice structure. We define the meet of any family $\{\mathcal{L}_a\}_{a\in A}$ of uniform filters as

$$\bigwedge_{a\in A}\mathcal{L}_a = \bigcap_{a\in A}\mathcal{L}_a.$$

Taking meets is compatible with left composition. Indeed, if $\mathcal{H}$ and $\{\mathcal{L}_a\}_{a\in A}$ are uniform filters, and if $I \in \bigwedge_{a\in A}(\mathcal{H}\circ\mathcal{L}_a)$, then for all $a \in A$ there exists $J_a \in \mathcal{L}_a$ containing I such that J_a/I is $\mathcal{H}$–torsion. Since it contains J_a for all $a \in A$, the left ideal $J = \sum_{a\in A} J_a$ lies in $\bigcap_{a\in A}\mathcal{L}_a$ and $J/I = \sum_{a\in A} J_a/I$ is $\mathcal{H}$–torsion as it is the sum of $\mathcal{H}$–torsion modules. So $I \in \mathcal{H}\circ\bigwedge_{a\in A}\mathcal{L}_a$, proving that

$$\bigwedge_{a\in A}(\mathcal{H}\circ\mathcal{L}_a) \subseteq \mathcal{H}\circ\bigwedge_{a\in A}\mathcal{L}_a.$$

The other inclusion also holds, as $\mathcal{H}\circ\bigwedge_{a\in A}\mathcal{L}_a$ is contained in $\mathcal{H}\circ\mathcal{L}_a$ for all $a \in A$.
On the other hand, as $(\bigwedge_{a\in A}\mathcal{L}_a)\circ\mathcal{H} \subseteq \mathcal{L}_a\circ\mathcal{H}$ for all $a \in A$,

$$(\bigwedge_{a\in A}\mathcal{L}_a)\circ\mathcal{H} \subseteq \bigwedge_{a\in A}(\mathcal{L}_a\circ\mathcal{H}).$$

Moreover, if $\mathcal{H}$ is closed under taking intersections indexed by A (e.g., if A is finite), then for $I \in \bigwedge_{a\in A}(\mathcal{L}_a\circ\mathcal{H})$ the left ideal

$$J_a = \{r \in R\,;(I:r) \in \mathcal{L}_a\}$$

belongs to $\mathcal{H}$. So $I \in (\bigwedge_{a\in A}\mathcal{L}_a)\circ\mathcal{H}$, as

$$\{r \in R\,;(I:r) \in \bigcap_{a\in A}\mathcal{L}_a\} = \bigcap_{a\in A} J_a \in \mathcal{H}.$$

Thus, if $\mathcal{H}$ is closed under taking intersections indexed by A, the inclusion $(\bigwedge_{a\in A}\mathcal{L}_a)\circ\mathcal{H} \supseteq \bigwedge_{a\in A}(\mathcal{L}_a\circ\mathcal{H})$ also holds.

The condition of $\mathcal{H}$ being closed under taking intersections is not necessary, as the following example shows.

(1.7) Example. Let $\mathcal{H}$ be the uniform filter of all non–zero ideals of $\mathbb{Z}$ and for any $n \in \mathbb{N} \setminus \{0,1\}$, let $\mathcal{L}_n$ denote the smallest uniform filter containing $n\mathbb{Z}$. Then we have $\bigwedge_n \mathcal{L}_n = \{\mathbb{Z}\}$, hence $(\bigwedge_n \mathcal{L}_n) \circ \mathcal{H} = \mathcal{H}$. On the other hand, we also have $\mathcal{L}_n \circ \mathcal{H} = \mathcal{H}$, hence $\bigwedge_n(\mathcal{L}_n \circ \mathcal{H}) = \mathcal{H}$.

In order to define the join of a family of uniform filters, consider the following result, whose proof we include for completeness' sake.

(1.8) Lemma. ([2]) *For any family F of left ideals on R, define*

1. $F' = \{(I : r); I \in F, r \in R\}$;
2. $F'' = \{\bigcap_{i=1}^n I_i; I_i \in F', 1 \leq i \leq n\}$;
3. $\mathcal{L} = \{I \leq_l R; \exists J \in F'', J \subseteq I\}$.

Then $\mathcal{L}$ is the smallest uniform filter containing F.

Proof. Since for all $I \in \mathcal{L}$, there exists some $J \in F''$ contained in I as well as in every ideal containing I, obviously $\mathcal{L}$ is a filter. If I and J are left ideals in $\mathcal{L}$, then there exist $I_1, \ldots, I_n$ and $J_1, \ldots, J_m$ in F' such that $\bigcap_{i=1}^n I_i \subseteq I$ and $\bigcap_{j=1}^m J_j \subseteq J$. So $(\bigcap_{i=1}^n I_i) \cap (\bigcap_{j=1}^m J_j) \subseteq I \cap J$ and thus $I \cap J \in \mathcal{L}$. For all $I \in \mathcal{L}$ and $r \in R$, there exist left R–ideals $I_1, \ldots, I_n$ in F and elements $r_1, \ldots, r_n \in R$ such that $\bigcap_{i=1}^n (I_i : r_i) \subseteq I$. So

$$(I : r) \supseteq (\bigcap_{i=1}^{n}(I_i : r_i) : r) = \bigcap_{i=1}^{n}((I_i : r_i) : r) = \bigcap_{i=1}^{n}(I_i : rr_i),$$

and therefore $I \in \mathcal{L}$. This proves that $\mathcal{L}$ is a uniform filter containing F, as $F \subseteq F' \subseteq F'' \subseteq \mathcal{L}$.

On the other hand, if $\mathcal{H}$ is a uniform filter containing F, then F' is contained in $\mathcal{H}$, since $\mathcal{H} \circ \{R\} \subseteq \mathcal{H}$. So $F'' \subseteq \mathcal{H}$, as $\mathcal{H}$ is closed for finite intersections, and since $\mathcal{H}$ is a filter, $\mathcal{L} \subseteq \mathcal{H}$. It follows that $\mathcal{L}$ is the smallest uniform filter containing F, indeed. □

As an easy consequence, let us point out that the smallest uniform filter $\bigvee_{a \in A} \mathcal{L}_a$, which contains every member of a family $\{\mathcal{L}_a; a \in A\}$ of uniform filters, consists exactly of the left R–ideals I with the property that there exist $a_1, \ldots, a_n \in A$ and corresponding $I_i \in \mathcal{L}_{a_i}$ such that $\bigcap_{i=1}^n I_i \subseteq I$.

(1.9) If $\mathcal{H}$ is a uniform filter and $\{\mathcal{L}_a; a \in A\}$ is a family of uniform filters, such that $\bigcup_{a\in A}\mathcal{L}_a$ is also a uniform filter, then

$$\bigcup_{a\in A}\mathcal{H}\circ\mathcal{L}_a \subseteq \mathcal{H}\circ\bigcup_{a\in A}\mathcal{L}_a,$$

since $\mathcal{H}\circ\mathcal{L}_a \subseteq \mathcal{H}\circ\bigcup_{a\in A}\mathcal{L}_a$, for every $a \in A$. Conversely, for all $L \in \mathcal{H}\circ\bigcup_{a\in A}\mathcal{L}_a$, there exists some $a \in A$ and some $K \in \mathcal{L}_a$, containing L, such that K/L is $\mathcal{H}$–torsion. So, $L \in \mathcal{H}\circ\mathcal{L}_a \subseteq \bigcup_{a\in A}\mathcal{H}\circ\mathcal{L}_a$. This proves the equality

$$\bigcup_{a\in A}\mathcal{H}\circ\mathcal{L}_a = \mathcal{H}\circ\bigcup_{a\in A}\mathcal{L}_a.$$

The left-right analogue of this equality is also valid, if we assume R to be left noetherian:

(1.10) Proposition. *Consider uniform filters $\mathcal{H}$ and $\{\mathcal{L}_a; a \in A\}$. Then:*

1. *if $\bigcup_a \mathcal{L}_a$ is a (uniform) filter, then $\bigcup_a(\mathcal{L}_a\circ\mathcal{H}) \subseteq (\bigcup_a\mathcal{L}_a)\circ\mathcal{H}$;*
2. *if $A = \{1,\ldots,n\}$ and if $\bigcup_{i\leq l}\mathcal{L}_i$ is a uniform filter for $2 \leq l \leq n$, then $\bigcup_n(\mathcal{L}_n\circ\mathcal{H}) = (\bigcup_n\mathcal{L}_n)\circ\mathcal{H}$;*
3. *if R is left noetherian and $\{\mathcal{L}_a\}_{a\in A}$ is directed (i.e., if for all a and b in A there exists $c \in A$ such that $\mathcal{L}_a\cup\mathcal{L}_b \subseteq \mathcal{L}_c$), then $\bigcup_a(\mathcal{L}_a\circ\mathcal{H}) = (\bigcup_a\mathcal{L}_a)\circ\mathcal{H}$.*

Proof. The first assertion is trivial, since $\mathcal{L}_a\circ\mathcal{H} \subseteq (\bigcup_a\mathcal{L}_a)\circ\mathcal{H}$ for every positive integer n.
Next, assume $A = \{1, 2\}$ and consider $I \in (\mathcal{L}_1\cup\mathcal{L}_2)\circ\mathcal{H}$. Then

$$J = \{r \in R; (I : r) \in \mathcal{L}_1\cup\mathcal{L}_2\}$$

is a left ideal that belongs to $\mathcal{H}$. So, if for $i = 1, 2$ we put

$$J_i = \{r \in R; (I : r) \in \mathcal{L}_i\},$$

then $J_1 \cup J_2 = J$. Since the join of two left ideals is not a left ideal in general, at least one of them contains the other, i.e., we have $J_1 = J$ or $J_2 = J$. Hence $I \in (\mathcal{L}_1 \circ \mathcal{H}) \cup (\mathcal{L}_2 \circ \mathcal{H})$. If, more generally, $A = \{1, \ldots, n\}$, then

$$(\bigcup_{i=1}^{n} \mathcal{L}_i) \circ \mathcal{H} \subseteq ((\bigcup_{i=1}^{n-1} \mathcal{L}_i) \circ \mathcal{H}) \cup (\mathcal{L}_n \circ \mathcal{H}) \subseteq \cdots \subseteq \bigcup_{i=1}^{n} (\mathcal{L}_i \circ \mathcal{H}).$$

To prove the third assertion, let us consider $L \in (\bigcup_a \mathcal{L}_a) \circ \mathcal{H}$. Then there exists some $H \in \mathcal{H}$ with $L \subseteq H$ and such that H/L is $\bigcup_a \mathcal{L}_a$–torsion. Assume the left R–module H to be generated by the elements $h_1, \ldots, h_r$. For any $1 \leq i \leq r$, we may find some index a_i and some $K_i \in \mathcal{L}_{a_i}$ with $K_i h_i \subseteq L$. Put $K = \bigcap_i K_i$ and let $b \in A$ such that $\mathcal{L}_{a_i} \subseteq \mathcal{L}_b$ for all the indices a_i, then $K \in \mathcal{L}_b$ and $K h_i \subseteq L_i$ for all $1 \leq i \leq r$. If $h = \sum_i^r r_i h_i$ is an arbitrary element of H, then

$$(L : s) \supseteq \bigcap_i (L : r_i h_i) = \bigcap_i ((L : h_i) : r_i) \supseteq \bigcap_i (K : r_i).$$

Since this last intersection is obviously an element of $\mathcal{L}_b$, we find that $(L : s) \in \mathcal{L}_b$ as well, which shows that $L \in \mathcal{L}_b \circ \mathcal{H} \subseteq \bigcup_a (\mathcal{L}_a \circ \mathcal{H})$. This proves the assertion. □

(1.11) If $\mathcal{L}$ is a uniform filter, then, obviously, $\mathcal{L} \subseteq \mathcal{L} \circ \mathcal{L}$. If the other inclusion also holds, i.e., if $\mathcal{L} \circ \mathcal{L} = \mathcal{L}$, then $\mathcal{L}$ is said to be a *Gabriel filter*. Of course, it is easy to see that this definition is equivalent to the usual one, given in [20], for example.

Gabriel filters are closed under taking products of left ideals. Indeed, if $\mathcal{L}$ is a Gabriel filter then the product of two left ideals L and L' in $\mathcal{L}$ is a left ideal contained in L' and L'/LL' is $\mathcal{L}$–torsion, since $(LL' : r)$ contains $L \in \mathcal{L}$ for all $r \in L'$. So, $LL' \in \mathcal{L} \circ \mathcal{L} = \mathcal{L}$.

(1.12) For any uniform filter $\mathcal{L}$ we may find a minimal Gabriel filter $\widetilde{\mathcal{L}}$ containing it — we will refer to $\widetilde{\mathcal{L}}$ as the Gabriel filter *generated by* $\mathcal{L}$. Indeed, since any intersection of Gabriel filters is again a Gabriel filter, as one easily verifies, clearly $\widetilde{\mathcal{L}}$ is just the intersection of all Gabriel filters containing $\mathcal{L}$.

If R is left noetherian, then lemma (1.8) yields a more direct description of $\widetilde{\mathcal{L}}$. Actually, if we denote for any positive integer n by $\mathcal{L}^n$ the n–fold composition $\mathcal{L} \circ \cdots \circ \mathcal{L}$, then it appears that $\widetilde{\mathcal{L}} = \bigcup_n \mathcal{L}^n$. Indeed, it is clear that $\mathcal{H} = \bigcup_n \mathcal{L}^n$ is a uniform filter, which is contained in all Gabriel filters containing $\mathcal{L}$. It thus remains to check that $\mathcal{H} \circ \mathcal{H} \subseteq \mathcal{H}$, and this follows from

$$\mathcal{H} \circ \mathcal{H} \subseteq \bigcup_n \bigcup_m \mathcal{L}^n \circ \mathcal{L}^m \subseteq \bigcup_p \mathcal{L}^p = \mathcal{H}.$$

(1.13) If $\mathcal{L}$ is a Gabriel filter, then it is easy to verify that the corresponding $\sigma_{\mathcal{L}}$ is an idempotent kernel functor [13] (or radical, in the terminology of [6]), i.e., it has the supplementary property that

$$\sigma_{\mathcal{L}}(M/\sigma_{\mathcal{L}}M) = 0$$

for any $M \in R-\mathbf{Mod}$.

Of course, conversely, any idempotent kernel functor σ in $R-\mathbf{Mod}$ defines a Gabriel filter $\mathcal{L}(\sigma)$. This yields a bijective correspondence between Gabriel filters and idempotent kernel functors, cf. [13].

(1.14) If $\mathcal{L}$ is a Gabriel filter, then its associated hereditary pretorsion class is closed under extensions (so it is a hereditary *torsion class*), and the $\mathcal{L}$–torsionfree class may be described as the class of all left R–modules N such that $\mathrm{Hom}_R(M, N) = 0$ for any $\mathcal{L}$–torsion left R–module M.

Conversely, let $\mathcal{F}$ be a class of left R–modules closed under taking submodules, products, extensions and injective hulls. Then one may canonically associate to $\mathcal{F}$ a hereditary torsion class $\mathcal{T}$ consisting of all left R–modules M such that $\mathrm{Hom}_R(M, N) = 0$ for any $N \in \mathcal{F}$, as well as a Gabriel filter $\mathcal{L}_{\mathcal{T}}$ defined through its associated torsion class $\mathcal{T}$, cf. [20].

In contrast with Gabriel filters, uniform filters are in general not determined by their associated class of torsionfree modules.

(1.15) Proposition. *If $\mathcal{L}$ is a uniform filter, then the class of $\mathcal{L}$–torsionfree modules is also the class of $\widetilde{\mathcal{L}}$–torsionfree modules.*

Proof. Let us denote by $\mathcal{H}$ the Gabriel filter with associated torsionfree class $\mathcal{F}_{\mathcal{L}}$, i.e., with $\mathcal{F}_{\mathcal{H}} = \mathcal{F}_{\mathcal{L}}$. Let M be an $\mathcal{L}$–torsion module and consider $f \in \mathrm{Hom}_R(M, N)$, where $N \in \mathcal{F}_{\mathcal{H}}$. Then for all $m \in M$, clearly, $\mathrm{Ann}^l_R(f(m))$ belongs to $\mathcal{L}$, as it contains $\mathrm{Ann}^l_R(m) \in \mathcal{L}$. So $f(m) = 0$ which shows that $f = 0$. This implies M to be $\mathcal{H}$–torsion, hence $\mathcal{L} \subseteq \mathcal{H}$. On the other hand, if $\mathcal{H}'$ is a Gabriel filter that contains $\mathcal{L}$ and if M is $\mathcal{H}'$–torsionfree, then it is also $\mathcal{L}$–torsionfree, since for any $m \in M$ with $\mathrm{Ann}^l_R(m) \in \mathcal{L} \subseteq \mathcal{H}'$, we have $m = 0$. So $\mathcal{F}_{\mathcal{H}'} \subseteq \mathcal{F}_{\mathcal{L}} = \mathcal{F}_{\mathcal{H}}$, and $\mathcal{H} \subseteq \mathcal{H}'$. □

Let us conclude this section, by briefly studying some particular examples of filters associated to ideals.

(1.16) For any left ideal L of R, we denote by $\mathcal{L}_L$ the smallest uniform filter containing L. As showed in (1.8), $\mathcal{L}_L$ consists exactly of those left ideals H of R for which there exists some finite subset $F \subseteq R$ with $(L : F) \subseteq H$.
More generally, if L is a left ideal in R and if $\mathcal{H}$ is a uniform filter over R, then, by [12], the smallest uniform filter $\mathcal{H}[L]$ containing L and $\mathcal{H}$ consists of all left ideals K of R such that there exists some $H \in \mathcal{H}$ and some finite subset $F \subseteq R$ with $H \cap (L : F) \subseteq K$.

(1.17) An element i in a lattice $I, \leq$ is said to be *compact* if, whenever $i \leq \bigvee_{a \in A} i_a$, there exists a finite subset of indices $F \subseteq A$ such that $i \leq \bigvee_{a \in F} i_a$.
Since the compact elements in the lattice of uniform filters over R are of the form $\mathcal{L}_L$, for some left ideal L of R (see [12]), the lattice of uniform filters is algebraic, i.e., every uniform filter $\mathcal{L}$ is the join of a family of compact uniform filters, $\{\mathcal{L}_L\}_{L \in \mathcal{L}}$ being such a family.

(1.18) For every twosided ideal I in R, the filter $\mathcal{L}_I$ consists of all left ideals L of R, which contain I. It is easy to see that $\mathcal{L}_I$ is uniform. Indeed, since I is twosided, $I \subseteq (I : r) \subseteq (J : r)$ for all $J \in \mathcal{L}_I$ and $r \in R$, and this implies that $\{r \in R; (J : r) \in \mathcal{L}_I\} = R$. If $\mathcal{H}$ is a uniform filter containing I, then $J \in \mathcal{H}$ for all left ideals $I \subseteq J$ of R, so $\mathcal{L}_I \subseteq \mathcal{H}$.
The filter $\mathcal{L}_I$ is clearly *jansian* ([12]), i.e., it is closed under taking arbitrary intersections. Conversely, if $\mathcal{L}$ is a jansian uniform filter, then

the left ideal $I = \bigcap_{L \in \mathcal{L}} L$ belongs to $\mathcal{L}$. Moreover, if $r \in R$, then $(I : r) \in \mathcal{L}$, so $I \subseteq (I : r)$. Hence, $Ir \subseteq I$, which shows that I is a twosided ideal and thus, clearly, that $\mathcal{L} = \mathcal{L}_I$.

In particular, if R is left fully bounded noetherian and if L is a left ideal in R, then $\mathcal{L}_L$ is jansian. Indeed it easily follows from Gabriel's condition (H) that it coincides with $\mathcal{L}_{L^*}$, where L^* is the largest twosided ideal contained in L.

(1.19) The smallest Gabriel filter which contains $\mathcal{L}_I$ is that associated to $\xi(R/I)$, the smallest radical such R/I is torsion, cf. [12]. So, if R is a left noetherian ring, then $\widetilde{\mathcal{L}_I}$ consists of all left ideals J of R which contain I^n, for some positive integer n.

Let I and J be twosided ideals; a left ideal L belongs to $\mathcal{L}_I \circ \mathcal{L}_J$ if and only if $J \subseteq \{r \in R; I \subseteq (L : r)\}$, and this occurs if and only if $IJ \subseteq L$, i.e., if and only if $L \in \mathcal{L}_{IJ}$. We thus have proved that $\mathcal{L}_I \circ \mathcal{L}_J = \mathcal{L}_{IJ}$.

(1.20) For any uniform filter $\mathcal{L}$ we denote by $\mathcal{L}^{(2)}$ the set of all twosided ideals contained in $\mathcal{L}$. A uniform filter $\mathcal{L}$ is said to be *symmetric*, if every $J \in \mathcal{L}$ contains a twosided ideal belonging to $\mathcal{L}^{(2)}$. Clearly, $\mathcal{L}_I$ is a symmetric uniform filter for any twosided ideal I of R, so every jansian uniform filter is symmetric. If R is a left fully bounded noetherian ring, then every uniform filter $\mathcal{L}$ over R is symmetric, since every left ideal of R verifies Gabriel's condition (H).

If $\mathcal{L}$ is a symmetric uniform filter, then

$$\mathcal{L} = \bigcup_{I \in \mathcal{L}^{(2)}} \mathcal{L}_I.$$

(1.21) Lemma. *Consider symmetric uniform filters $\mathcal{L}$, $\mathcal{H}$ and $\{\mathcal{L}_a\}_{a \in A}$ over R. Then:*

(1.21.1) *$\bigwedge_{a \in A} \mathcal{L}_a$ and $\bigvee_{a \in A} \mathcal{L}_a$ are symmetric uniform filters;*

(1.21.2) *if R is left noetherian, then the uniform filter $\mathcal{L} \circ \mathcal{H}$ is symmetric.*

Proof. First let us show $\bigwedge_{a \in A} \mathcal{L}_a$ is symmetric. If $L \in \bigwedge_{a \in A} \mathcal{L}_a = \bigcap_{a \in A} \mathcal{L}_a$, then for every $a \in A$ there exists a twosided ideal $I_a \in \mathcal{L}_a$

such that $I_a \in L$. So $I = \sum_{a\in A} I_a \subseteq L$ and $I \in \bigcap_{a\in A} \mathcal{L}_a$, proving that $\bigwedge_{a\in A} \mathcal{L}_a$ is symmetric.
Pick $L \in \bigvee_{a\in A} \mathcal{L}_a$. From the comments following lemma (1.8), it follows that there exists $a_1, \ldots, a_n \in A$ and ideals $I_i \in \mathcal{L}_{a_i}$, which can be chosen twosided under our assumptions, such that $I = \bigcap_{i=1}^n I_i \subseteq L$. This proves that $\bigvee_{a\in A} \mathcal{L}_a$ is symmetric, since I is also twosided and belongs to $\bigvee_{a\in A} \mathcal{L}_a$.
Let us now assume R to be left noetherian. For every ideal I in $\mathcal{L} \circ \mathcal{H}$, there exists $J \in \mathcal{H}$ containing I and such that $(I : x) \in \mathcal{L}$ for all $x \in J$. Since $\mathcal{H}$ is symmetric and $(J : R)$ is the largest twosided ideal contained in J, obviously $(J : R)$ also belongs to $\mathcal{H}$ and $(I : R) \subseteq (J : R)$. Moreover, $(I : xr) \in \mathcal{L}$ for all $x \in (J : R)$ and $r \in R$, and since $\mathcal{L}$ is symmetric, $((I : xr) : R) = (I : Rxr)$ belongs to $\mathcal{L}$, for all $r \in R$ and all $x \in J$.
If $x \in (J : R)$ and if RxR is generated as a left R–ideal by $\{xr_1, \ldots, xr_n\}$, then $(I : RxR) = \bigcap_{1\leq i\leq n}(I : Rxr_i) \in \mathcal{L}$ and

$$((I : R) : x) = (I : xR) \supseteq (I : RxR),$$

so $((I : R) : x) \in \mathcal{L}$. This implies that the largest twosided ideal $(I : R)$ contained in I belongs to $\mathcal{L} \circ \mathcal{H}$, hence that $\mathcal{L} \circ \mathcal{H}$ is symmetric. □

From this one easily deduces:

(1.22) Corollary. *If R is left noetherian, then the Gabriel filter $\widetilde{\mathcal{L}}$ generated by any symmetric uniform filter $\mathcal{L}$ is also symmetric.*

Proof. Using the previous lemma, an easy induction argument shows that any finite composition of copies of $\mathcal{L}$ is also a symmetric uniform filter. Since R is left noetherian, $\widetilde{\mathcal{L}} = \bigvee_{i=1}^{\infty} \mathcal{L}^i$ is a join of symmetric uniform filters, so it is symmetric. □

(1.23) One may also canonically associate a uniform filter to any *prime* left ideal of R. Indeed, if P is a prime left ideal of R, then the set $\mathcal{L}_{R\setminus P}$ of all left ideals L of R containing some twosided ideal $J \not\subseteq P$ is a filter which does not contain P. If $L, L' \in \mathcal{L}_{R\setminus P}$, then there exist twosided ideals $J \not\subseteq P$ resp. $J' \not\subseteq P$, contained in L resp. L'. Since $L \cap L'$ contains the twosided ideal $JJ' \not\subseteq P$, it follows that $L \cap L' \in \mathcal{L}_{R\setminus P}$.

On the other hand, if $L \in \mathcal{L}_{R\setminus P}$ and $J \subseteq L$ is a twosided ideal not contained in P, then for all $r \in R$ we have $J \subseteq (J : r) \subseteq (L : r)$, so $(L : r) \in \mathcal{L}_{R\setminus P}$. This shows that $\mathcal{L}_{R\setminus P}$ is a uniform filter, which is obviously symmetric.

Actually, $\mathcal{L}_{R\setminus P}$ is the largest symmetric uniform filter not containing P. Indeed, if $\mathcal{H}$ is a symmetric uniform filter, which does not contain P and if $H \in \mathcal{H}$, then there exists a twosided ideal $J \in \mathcal{H}$ contained in H. If $J \subseteq P$, then also $P \in \mathcal{H}$ — a contradiction. So, $H \in \mathcal{L}_{R\setminus P}$.

In particular, if R is left noetherian, then $\mathcal{L}_{R\setminus P}$ is the largest symmetric Gabriel filter not containing P, and

$$\mathcal{L} = \bigcap_{\substack{P \text{prime} \\ P \notin \mathcal{L}}} \mathcal{L}_{R\setminus P}$$

for every symmetric Gabriel filter $\mathcal{L}$, cf. [6].

2. Quasi–quantales

Let us denote by $R -$ **filt** the lattice of uniform filters over the ring R, partially ordered by the inclusion. The aim of this section is to associate, to any left R–module M, a sheaf over this quasi–quantale with a suitably nice functorial behaviour. In this way, our constructions may be viewed as a generalization of those in [3].

(2.1) Weakening the second condition in the definition of quantale given in [3] by taking just *finite* suprema, let us say that a complete lattice $\mathcal{Q}$ with top element 1 is a *quasi–quantale* if it is equipped with a binary "multiplication"

$$\& \; : \mathcal{Q} \times \mathcal{Q} \longrightarrow \mathcal{Q}$$

satisfying, for any $U, V, W \in \mathcal{Q}$ and any family $\{V_i\}_{i\in I}$ of elements in $\mathcal{Q}$, the following conditions:

(2.1.1) $U \,\&\, (\bigvee_{i\in I} V_i) = \bigvee_{i\in I} U \,\&\, V_i$;

(2.1.2) $(\bigvee_{i\in I} V_i) \,\&\, U \geq \bigvee_{i\in I} V_i \,\&\, U$ with equality if I is finite;

(2.1.3) $(U \mathbin{\&} V) \mathbin{\&} W = U \mathbin{\&} (V \mathbin{\&} W)$;

(2.1.4) $U \mathbin{\&} 1 = 1 \mathbin{\&} U = U$.

If equality holds in (2.1.2) whether $\{V_i\}_{i\in I}$ is a finite family or not, then $\mathcal{Q}$ is a *quantale* in the sense of [3].
If $\mathcal{Q}$ and $\mathcal{Q}'$ are (quasi–)quantales, a lattice morphism $q : \mathcal{Q} \longrightarrow \mathcal{Q}'$ is said to be a *morphism* of (quasi–)quantales if q preserves arbitrary suprema, multiplication and the top element.

(2.2) Example. The complete lattice $R-\mathbf{filt}^{opp}$ with the multiplication $\circ$ is a quasi–quantale. This follows inmediately from the comments in (1.6).

(2.3) Example. Let us mention an example of a quantale exhibited in [3]. In the lattice $Id(R)-\mathbf{filt}$ of jansian uniform filters (i.e., generated by some twosided ideal of R) we have the equalities

$$\mathcal{L}_I \vee \mathcal{L}_J = \mathcal{L}_{I\cap J},$$

$$\bigwedge_{a\in A} \mathcal{L}_{I_a} = \mathcal{L}_{\sum_{a\in A} I_a},$$

and the inclusion $I \subseteq J$ is equivalent to $\mathcal{L}_I \supseteq \mathcal{L}_J$. So we may conclude that the lattice of twosided ideals of R is isomorphic to $Id(R)-\mathbf{filt}^{opp}$, ordered by reverse inclusion. Moreover, since

$$\mathcal{L}_I \circ \mathcal{L}_J = \mathcal{L}_{IJ},$$

and the lattice of twosided ideals of R is a quantale, this isomorphism of lattices is actually an isomorphism of quantales.
The inclusion map $Id(R)-\mathbf{filt}^{opp} \longrightarrow R-\mathbf{filt}^{opp}$ is a morphism of quasi–quantales, hence $Id(R)-\mathbf{filt}^{opp}$ is a subquasi–quantale of $R-\mathbf{filt}^{opp}$.

(2.4) Example. Similarly, let R be left noetherian and consider the complete lattice $R-\mathbf{filt}^{(2)}$ which consists of symmetric uniform filters in R partially ordered by inclusion. The opposite lattice $(R-\mathbf{filt}^{(2)})^{opp}$ is a subquasi–quantale of $R-\mathbf{filt}^{opp}$.

(2.5) Given a uniform filter $\mathcal{L}$, the interval $[\mathcal{L}, \{R\}]$ in $R-\mathbf{filt}^{opp}$ is a subquasi–quantale of $R-\mathbf{filt}^{opp}$ (i.e., with suprema taken in $R-\mathbf{filt}^{opp}$ and multiplication $\circ$) if and only if it is closed under multiplication.

On the other hand, the interval $[\mathcal{L}_0, \mathcal{L}]$, with $\mathcal{L}_0$ the trivial (Gabriel) filter, consisting of all left ideals of R, is a quasi–quantale if and only if for every uniform filter $\mathcal{H}$ containing $\mathcal{L}$, we have $\mathcal{L}\circ\mathcal{H} = \mathcal{H}\circ\mathcal{L} = \mathcal{H}$. This leads us to define a uniform filter $\mathcal{H}$ to be *$\mathcal{L}$–Gabriel* if $\mathcal{L}\circ\mathcal{H} = \mathcal{H}\circ\mathcal{L} = \mathcal{H}$. In particular, $\mathcal{H}$ then contains $\mathcal{L}$. Trivially, a uniform filter $\mathcal{H}$ is a Gabriel filter if and only if $\mathcal{H}$ is $\mathcal{L}$–Gabriel for every uniform filter $\mathcal{L} \subseteq \mathcal{H}$.

(2.6) Example. For every Gabriel filter $\mathcal{L}$, the set

$$\mathcal{Q}_{\mathcal{L}} = \{\mathcal{H} \in R - \mathbf{filt}\ ;\ \mathcal{L} \subseteq \mathcal{H},\ \mathcal{H} \text{ is } \mathcal{L}\text{–Gabriel}\}$$

ordered by reverse inclusion is a quasi–quantale. Indeed, if $\mathcal{H}$ and $\mathcal{H}'$ are $\mathcal{L}$–Gabriel uniform filters, then $\mathcal{L} \circ \mathcal{H} \circ \mathcal{H}' = \mathcal{H} \circ \mathcal{H}' = \mathcal{H} \circ \mathcal{H}' \circ \mathcal{L}$. On the other hand, $\mathcal{Q}_{\mathcal{L}}$ is closed under taking suprema in $R - \mathbf{filt}^{opp}$, since for any family $\{\mathcal{H}_a\}_{a\in A} \subseteq \mathcal{Q}_{\mathcal{L}}$ we have

$$\mathcal{L} \circ \bigcap_{a\in A} \mathcal{H}_a = \bigcap_{a\in A} \mathcal{L} \circ \mathcal{H}_a = \bigcap_{a\in A} \mathcal{H}_a,$$

$$\bigcap_{a\in A} \mathcal{H}_a \subseteq (\bigcap_{a\in A} \mathcal{H}_a) \circ \mathcal{L} \subseteq \bigcap_{a\in A}(\mathcal{H}_a \circ \mathcal{L}) = \bigcap_{a\in A} \mathcal{H}_a.$$

Note that, obviously, $\mathcal{Q}_{\{R\}}$ is the quasi–quantale $R - \mathbf{filt}^{opp}$.

(2.7) Example. Similarly, if R is left noetherian and $\mathcal{L}$ is a symmetric Gabriel filter over R, then the lattice $\mathcal{Q}_{\mathcal{L}}^{(2)}$ consisting of all symmetric $\mathcal{L}$–Gabriel filters is a subquasi–quantale of $\mathcal{Q}_{\mathcal{L}}$.

(2.8) A uniform filter $\mathcal{K}$ is said to be *prime**, in the sense of [12], if $\mathcal{K} \subseteq \mathcal{L} \circ \mathcal{H}$ implies $\mathcal{K} \subseteq \mathcal{L}$ or $\mathcal{K} \subseteq \mathcal{H}$ for every $\mathcal{L}$ and $\mathcal{H}$ in $R - \mathbf{filt}$. If $\mathcal{L}$ is a uniform filter, we define its *radical** as the join of all prime* uniform filters contained in $\mathcal{L}$, and we denote it by $\sqrt{\mathcal{L}}$.

(2.9) Proposition. ([3]) *Let $\mathcal{L}$, $\mathcal{H}$, $\mathcal{K}$ and $\{\mathcal{L}_i\}_{i\in I}$ be uniform filters. Then,*

(2.9.1) *if $\mathcal{K}$ is prime* and $\mathcal{K} \subseteq \mathcal{L} \circ \mathcal{H}$, then $\mathcal{K} \subseteq \mathcal{H} \circ \mathcal{L}$;*

(2.9.2) *if $\mathcal{K}$ is prime*, then $\mathcal{K} \subseteq \mathcal{L}$ if and only if $\mathcal{K} \subseteq \sqrt{\mathcal{L}}$;*

(2.9.3) *$\sqrt{\mathcal{L}} = \sqrt{\mathcal{H}}$ if and only if $\mathcal{K} \subseteq \mathcal{L}$ is equivalent to $\mathcal{K} \subseteq \mathcal{H}$ for all prime* $\mathcal{K}$;*

(2.9.4) *if* $\mathcal{L} \subseteq \mathcal{H}$, *then* $\sqrt{\mathcal{L}} \subseteq \sqrt{\mathcal{H}}$;

(2.9.5) $\sqrt{\sqrt{\mathcal{L}}} = \sqrt{\mathcal{L}}$;

(2.9.6) $\sqrt{\mathcal{L}} \subseteq \mathcal{H}$ *if and only if* $\mathcal{K} \subseteq \mathcal{H}$ *for all prime** $\mathcal{K}$ *contained in* $\mathcal{L}$;

(2.9.7) $\sqrt{\bigcap_{i\in I} \sqrt{\mathcal{L}_i}} = \sqrt{\bigcap_{i\in I} \mathcal{L}_i}$;

(2.9.8) $\sqrt{\mathcal{L} \circ \mathcal{H}} = \sqrt{\mathcal{L}} \vee \sqrt{\mathcal{H}}$.

(2.10) Somewhat more generally, an element $U \neq 1$ in a quasi–quantale $\mathcal{Q}$ is said to be *prime* if $U \leq V \,\&\, W$, where $V, W \in \mathcal{Q}$, implies that $U \leq V$ or $U \leq W$. The *radical* of an element U in $\mathcal{Q}$ is defined as the join of the set of all prime elements V such that $U \leq V$, and it is denoted by $\sqrt{U}$.
It is then easy to see that the results in (2.9) trivially generalize to this context.

(2.11) Let $\mathcal{Q}$ be a quasi–quantale. Then the set $\sqrt{\mathcal{Q}}$ of all radical elements of $\mathcal{Q}$, i.e., those $U \in \mathcal{Q}$ such that $\sqrt{U} = U$, is a quantale and $\sqrt{\cdot} : \mathcal{Q} \longrightarrow \sqrt{\mathcal{Q}}$ is a morphism of quasi–quantales.
Indeed, the join of the family $\{U_i\}_{i\in I}$ in $\sqrt{\mathcal{Q}}$ is $\sqrt{\bigvee_{i\in I} U_i}$, where $\bigvee_{i\in I} U_i$ is computed in $\mathcal{Q}$, and the meet of this family coincides with its meet in $\mathcal{Q}$.
For any $U, V \in \sqrt{\mathcal{Q}}$ we define

$$U * V = \sqrt{U \& V} = \sqrt{U} \wedge \sqrt{V} = U \wedge V \ .$$

It is easy to prove that this (commutative!) multiplication satisfies the axioms (2.1.1) – (2.1.4), with equality in (2.1.2), even for an arbitrary family. So $\sqrt{\mathcal{Q}}$ is a quantale.
The top element in $\sqrt{\mathcal{Q}}$ is $\sqrt{1} = 1$, since it is the meet of the empty family of prime elements. From the definition of the multiplication in $\sqrt{\mathcal{Q}}$ and the previous comment on the join in $\sqrt{\mathcal{Q}}$, it follows that $\sqrt{\cdot}$ is a morphism of quasi–quantales.

(2.12) A *weak sheaf* over the quasi–quantale $\mathcal{Q}$ (or *weak $\mathcal{Q}$–sheaf*) is a pair $(A, [\cdot = \cdot])$, where A is a set (think of A as a set of generators) and where the map

$$[\cdot = \cdot] : A \times A \longrightarrow \mathcal{Q}$$

(which may be viewed as describing relations between generators) satisfies:

(2.12.1) $\bigvee_{b\in A}[a=b]\&[b=c]=[a=c]$;

(2.12.2) $\bigvee_{b\in A}[b=c]\&[a=b]=[c=a]$,

for every a and c in A.
A weak $\mathcal{Q}$–sheaf $(A,[\cdot=\cdot])$ is said to be a $\mathcal{Q}$*–sheaf* if $[a=a]=1$ for every $a\in A$.

(2.13) Note. With respect to the analogous definition given in [3], our definition of $\mathcal{Q}$–sheaf has the additional property that $[a=a]=1$ for every $a\in A$. The main example in [3], i.e., the sheaf $(R,[\cdot=\cdot])$ over the quantale $Id(R)$ of twosided ideals of R given by $[r=r']=\mathrm{Ann}^l_R(r-r')^*$, satisfies this condition. So do the examples below.

(2.14) Example. If $\mathcal{L}$ is a Gabriel filter, then every left R–module M is the set of generators of a $\mathcal{Q}_{\mathcal{L}}$–sheaf $(M,[\cdot=\cdot])$ by defining for all m and m' in M,

$$[m=m']=\mathcal{L}\circ\mathcal{L}_{\mathrm{Ann}^l_R(m-m')}\circ\mathcal{L},$$

where, as in (1.16), $\mathcal{L}_{\mathrm{Ann}^l_R(m-m')}$ is the smallest uniform filter containing the left annihilator $\mathrm{Ann}^l_R(m-m')$.
Indeed, if $\mathcal{L}$ is a Gabriel filter, then $[m=m'']\in\mathcal{Q}_{\mathcal{L}}$ for every m and m'' in M, and we have

$$\bigvee_{m'\in M}[m=m']\&[m'=m'']=\bigcap_{m'\in M}[m=m']\circ[m'=m'']$$

and this is contained in

$$[m=m'']\circ[m''=m'']=(\mathcal{L}\circ\mathcal{L}_{\mathrm{Ann}^l_R(m-m'')}\circ\mathcal{L})\circ(\mathcal{L}\circ\{R\}\circ\mathcal{L})=[m=m''].$$

On the other hand, if $m'\in M$, then

$$\mathrm{Ann}^l_R(m-m')\in\mathcal{L}_{\mathrm{Ann}^l_R(m-m')}\subseteq\mathcal{L}_{\mathrm{Ann}^l_R(m-m')}\circ\mathcal{L}\circ\mathcal{L}\circ\mathcal{L}_{\mathrm{Ann}^l_R(m'-m'')},$$

and similarly

$$\mathrm{Ann}^l_R(m'-m'')\in\mathcal{L}_{\mathrm{Ann}^l_R(m-m')}\circ\mathcal{L}\circ\mathcal{L}\circ\mathcal{L}_{\mathrm{Ann}^l_R(m'-m'')},$$

so

$$\mathrm{Ann}^l_R(m-m') \cap \mathrm{Ann}^l_R(m'-m'') \in \mathcal{L}_{\mathrm{Ann}^l_R(m-m')} \circ \mathcal{L} \circ \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(m'-m'')}.$$

But this intersection of ideals is contained in the annihilator

$$\mathrm{Ann}^l_R(m-m'') = \mathrm{Ann}^l_R(m-m'+m'-m''),$$

and thus

$$\mathrm{Ann}^l_R(m-m'') \in \mathcal{L}_{\mathrm{Ann}^l_R(m-m')} \circ \mathcal{L} \circ \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(m'-m'')}.$$

Therefore,

$$\mathcal{L}_{\mathrm{Ann}^l_R(m-m'')} \subseteq \mathcal{L}_{\mathrm{Ann}^l_R(m-m')} \circ \mathcal{L} \circ \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(m'-m'')},$$

and so $[m = m''] \subseteq \bigcap_{m' \in M}[m = m'] \circ [m' = m'']$, since m' has been chosen arbitrarily, whence equality.
The second condition follows from this one, as $[m = n] = [n = m]$ for all m and n in M.
Furthermore, $[m = m] = \mathcal{L} \circ \{R\} \circ \mathcal{L} = \mathcal{L}$ for every $m \in M$.
Note that if $\mathcal{Q}_\mathcal{L}$ is the quasi–quantale $R - \mathbf{filt}^{opp}$, i.e., $\mathcal{L}$ is the filter $\{R\}$, then the equality in the $R - \mathbf{filt}^{opp}$–sheaf $(M, [\cdot = \cdot])$ is given by $[m = m'] = \mathcal{L}_{\mathrm{Ann}^l_R(m-m')}$.

(2.15) Example. Similarly, if R is a left noetherian ring and $\mathcal{L}$ is a symmetric Gabriel filter over R, then every left R–module M defines a $\mathcal{Q}^{(2)}_\mathcal{L}$–sheaf with M as set of generators and equality given by

$$[m = m']^{(2)} = \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(m-m')^*} \circ \mathcal{L},$$

with $\mathcal{L}_{\mathrm{Ann}^l_R(m-m')^*}$ the jansian filter generated by $\mathrm{Ann}^l_R(m-m')^*$, the largest twosided ideal contained in $\mathrm{Ann}^l_R(m-m')$.
If $\mathcal{L} = \{R\}$, then this equality can be considered within the quasi–quantale $Id(R) - \mathbf{filt}^{opp}$ and the sheaf $(M, [\cdot = \cdot]^{(2)})$ is actually an $Id(R) - \mathbf{filt}^{opp}$–sheaf. We thus recover the example given in [3].

(2.16) If $(A,[\cdot=\cdot])$ and $(B,[\cdot=\cdot])$ are $\mathcal{Q}$–sheaves, a *premorphism* f from $(A,[\cdot=\cdot])$ to $(B,[\cdot=\cdot])$ is a pair of maps

$$[f\cdot=\cdot]:A\times B\longrightarrow\mathcal{Q}\quad,\quad[\cdot=f\cdot]:B\times A\longrightarrow\mathcal{Q}$$

satisfying:

(2.16.1) $\bigvee_{a'\in A}[a=a']\ \&\ [fa'=b]=[fa=b]$;

(2.16.2) $\bigvee_{a'\in A}[b=fa']\ \&\ [a'=a]\geq[b=fa]$;

(2.16.3) $\bigvee_{b'\in B}[b=b']\ \&\ [b'=fa]=[b=fa]$;

(2.16.4) $\bigvee_{b'\in B}[fa=b']\ \&\ [b'=b]\geq[fa=b]$;

(2.16.5) $[a=a'']\leq\bigvee_{b\in B}[fa=b]\ \&\ [b=fa'']$,

for all a and a'' in A, and b in B.

(2.17) Note. The definition of premorphism is weaker than the definition of morphism proposed by Borceux and Cruciani ([3]), since neither the equality in (2.16.2) and (2.16.3) nor the inequality

$$\bigvee_{a\in A}[b=fa]\ \&\ [fa=b']\leq[b=b'],\quad\forall b,b'\in B$$

are required. The reason is that these properties are not inherited by the composition of premorphisms, defined as follows.

(2.18) A *precategory* $\mathcal{C}$ consists of a class of objects Obj$\mathcal{C}$ and, for every pair of objects A and B, a set of arrows $\mathcal{C}(A,B)$ such that for any objects A, B and C there exists a map $\mathcal{C}(A,B)\times\mathcal{C}(B,C)\longrightarrow\mathcal{C}(A,C)$, and for any A in Obj$\mathcal{C}$ there exists a distinguished element $id_A\in\mathcal{C}(A,A)$.
It is easy to see that $\mathcal{Q}$–sheaves $(A,[\cdot=\cdot])$ and premorphisms of $\mathcal{Q}$–sheaves $f:(A,[\cdot=\cdot])\longrightarrow(B,[\cdot=\cdot])$ with the property that for every $a\in A$ there exists b_a and b'_a in B such that $[fa=b_a]=[b'_a=fa]=1$ may be made into a precategory.
Indeed, define for any $\mathcal{Q}$–sheaf $(A,[\cdot=\cdot])$ the distinguished premorphism

$$[id_Aa=a']=[a=id_Aa']=[a=a']$$

for any a and a' belonging to A, and for any pair of premorphisms $f : (A, [\cdot = \cdot]) \longrightarrow (B, [\cdot = \cdot])$ and $g : (B, [\cdot = \cdot]) \longrightarrow (C, [\cdot = \cdot])$, define the composition gf by

$$[gfa = c] = \bigvee_{b \in B} [fa = b] \ \& \ [gb = c],$$

and

$$[c = gfa] = \bigvee_{b \in B} [c = gb] \ \& \ [b = fa]$$

for all $a \in A$ and all $c \in C$.

(2.19) Every precategory $\mathcal{C}$ has an *enveloping category* $\mathcal{C}'$ whose objects are those of the precategory and whose sets of morphisms are

$$\mathrm{Mor}_{\mathcal{C}'}(A, B) = \frac{\mathcal{C}(A, B)}{\sim},$$

where $\sim$ is the transitive closure (i.e., $g \sim h \Leftrightarrow g \approx \cdots \approx h$) of the relation given by $f \approx f'$ if and only if there exists a chain of arrows

$$A = A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n = B$$

in $\mathcal{C}$ such that both f and f' are the composition of $id_A, f_1, id_{A_1}, f_2, \ldots,$ f_n and id_B, in this order, distinguished arrows being optional in each decomposition and possibly associating in different ways when calculating the composition.

Indeed, the equivalence $\sim$ is compatible with the composition in $\mathcal{C}$, so it induces a composition in $\mathcal{C}'$. Moreover, for any objects A, B, C and D, and any $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Mor}_{\mathcal{C}}(B, C)$ and $h \in \mathrm{Mor}_{\mathcal{C}}(C, D)$,

$$h \circ (g \circ f) \sim (h \circ g) \circ f \ , \quad id_B \circ f \sim f \ , \quad f \circ id_A \sim f.$$

A functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ (between precategories) maps objects into objects and for every pair of objects A and B of $\mathcal{C}$, it defines a map $\mathcal{F} : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(\mathcal{F}A, \mathcal{F}B)$ preserving the composition and the distinguished arrow. If $\mathcal{C}$ is a precategory then there exists a functor $\mathcal{C} \xrightarrow{\mathcal{P}} \mathcal{C}'$, where $\mathcal{C}'$ is the enveloping category of $\mathcal{C}$, which maps an object A onto itself and such that $\mathcal{C}(A, B) \xrightarrow{\mathcal{P}} \mathrm{Mor}_{\mathcal{C}'}(A, B)$ is the projection map.

The pair $(\mathcal{C}', \mathcal{P})$ satisfies the following universal property: if $\mathcal{D}$ is a category and $\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{D}$ is a functor, then there exists a unique functor $\mathcal{C}' \xrightarrow{\mathcal{F}'} \mathcal{D}$ such that $\mathcal{F}'\mathcal{P} = \mathcal{F}$. As a consequence, if $\mathcal{C}$ and $\mathcal{D}$ are precategories and $\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{D}$ is a functor, then there exists a unique functor $\mathcal{C}' \xrightarrow{\mathcal{F}'} \mathcal{D}'$ such that

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{F}} & \mathcal{D} \\ {\scriptstyle \mathcal{P}}\downarrow & & \downarrow{\scriptstyle \mathcal{P}} \\ \mathcal{C}' & \xrightarrow[\mathcal{F}']{} & \mathcal{D}' \end{array}$$

commutes.

If $\mathcal{Q}$ is a quasi–quantale, then the enveloping category of the precategory of $\mathcal{Q}$–sheaves and premorphisms of $\mathcal{Q}$–sheaves $(A, [\cdot = \cdot]) \xrightarrow{f} (B, [\cdot = \cdot])$ such that for any $a \in A$ there exists $b_a, b'_a \in B$ with $[fa = b_a] = [b'_a = fa] = 1$ will be referred to as the *category of $\mathcal{Q}$–sheaves.*

(2.20) Example. Let $\mathcal{L}$ be a Gabriel filter and consider a morphism of left R–modules $f : M \longrightarrow N$. Then we can define a premorphism between the $\mathcal{Q}_{\mathcal{L}}$–sheaves $(M, [\cdot = \cdot])$ and $(N, [\cdot = \cdot])$ by taking

$$[fm = n] \; = \; [n = fm] = \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(f(m)-n)} \circ \mathcal{L},$$

where $\mathcal{L}_{\mathrm{Ann}^l_R(f(m)-n)}$ is the smallest uniform filter containing the annihilator $\mathrm{Ann}^l_R(f(m) - n)$, as in (1.16). Indeed, in order to prove (2.16.1) for every $m \in M$ and $n \in N$, pick m' from M and consider the annihilator

$$\mathrm{Ann}^l_R(f(m) - n) = \mathrm{Ann}^l_R(f(m) - f(m') + f(m') - n).$$

This ideal contains $\mathrm{Ann}^l_R(m - m') \cap \mathrm{Ann}^l_R(f(m') - n)$, which belongs to

$$\mathcal{L}_{\mathrm{Ann}^l_R(m-m')} \circ \mathcal{L} \circ \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(f(m')-n)}.$$

So $\mathrm{Ann}^l_R(f(m) - n)$ also belongs to $\mathcal{L}_{\mathrm{Ann}^l_R(m-m')} \circ \mathcal{L} \circ \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(f(m')-n)}$, which implies

$$\mathcal{L}_{\mathrm{Ann}^l_R(f(m)-n)} \subseteq \mathcal{L}_{\mathrm{Ann}^l_R(m-m')} \circ \mathcal{L} \circ \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(f(m')-n)}.$$

Therefore $[fm = n] \subseteq [m = m'] \circ [fm' = n]$ and since m' was chosen arbitrarily,

$$[fm = n] \subseteq \bigcap_{m' \in M} [m = m'] \circ [fm' = n].$$

Actually, we have equality, as the other inclusion trivially follows by taking $m' = m$ in the intersection.
Properties (2.16.2), (2.16.3) and (2.16.4) (even with equality in (2.16.2) and in (2.16.4)!) are proved in a similar way.
To verify (2.16.5), note that we have

$$\bigcap_{n \in N} [fm = n] \circ [n = fm'] = [f(m) = f(m')]$$

since $(N, [\cdot = \cdot])$ is a sheaf. Now, $\mathrm{Ann}^l_R(m - m') \subseteq \mathrm{Ann}^l_R(f(m) - f(m'))$, so

$$\mathcal{L}_{\mathrm{Ann}^l_R(f(m)-f(m'))} \subseteq \mathcal{L}_{\mathrm{Ann}^l_R(m-m')}$$

and thus $[f(m) = f(m')] \geq [m = m']$, indeed.
In the particular case that $\mathcal{Q}_{\mathcal{L}} = R - \mathbf{filt}^{opp}$, the premorphism of $R-\mathbf{filt}^{opp}$–sheaves $f : (M, [\cdot = \cdot]) \longrightarrow (N, [\cdot = \cdot])$ is given by

$$[fm = n] = [n = fm] = \mathcal{L}_{\mathrm{Ann}^l_R(f(m)-n)}.$$

Finally, let us consider two morphisms $f : M \longrightarrow N$ and $g : N \longrightarrow P$ of left R–modules. Then for any $m \in M$ and $p \in P$, the uniform filter

$$[gfm = p] = \bigcap_{n \in N} [fm = n] \circ [gn = p] = \bigcap_{n \in N} [f(m) = n] \circ [g(n) = p]$$

is contained in $[g \circ f(m) = p]$ (take $n = f(m)$ in the intersection).
Conversely, pick $n \in N$ and consider the left ideal

$$\mathrm{Ann}^l_R(g \circ f(m) - p) = \mathrm{Ann}^l_R(g \circ f(m) - g(n) + g(n) - p).$$

Since it contains the left ideal $\mathrm{Ann}^l_R(f(m) - n) \cap \mathrm{Ann}^l_R(g(n) - p)$ and this belongs to the uniform filter $\mathcal{L}_{\mathrm{Ann}^l_R(f(m)-n)} \circ \mathcal{L} \circ \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(g(n)-p)}$, we have

$$\mathcal{L}_{\mathrm{Ann}^l_R(g \circ f(m)-p)} \subseteq \mathcal{L}_{\mathrm{Ann}^l_R(f(m)-n)} \circ \mathcal{L} \circ \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(g(n)-p)},$$

and so $[g \circ f(m) = p] \subseteq [fm = n] \circ [gn = p]$. As n is chosen arbitrarily,

$$[g \circ f(m) = p] \subseteq \bigcap_{n \in N} [fm = n] \circ [gn = p],$$

hence $[g \circ f(m) = p] = [gfm = p]$.
Using a symmetric argument, we can prove $[p = g \circ f(m)] = [p = gfm]$, and, since $[p' = p''] = [p'' = p']$ for p' and p'' in P, we conclude that

$$[gfm = p] = [g \circ f(m) = p] = [p = gfm]$$

for all $m \in M$ and all $p \in P$.
Since the premorphism of $\mathcal{Q}_{\mathcal{L}}$–sheaves defined by the identity morphism in the left R–module M is the distinguished arrow corresponding to the sheaf $(M, [\cdot = \cdot])$, we thus obtain a functor from the category $R-\mathbf{Mod}$ into the category of $\mathcal{Q}_{\mathcal{L}}$–sheaves.

(2.21) Example. Similarly, if R is left noetherian and $\mathcal{L}$ is a symmetric Gabriel filter, then every morphism of left R–modules defines a morphism of $\mathcal{Q}_{\mathcal{L}}^{(2)}$–sheaves $f : (M, [\cdot = \cdot]^{(2)}) \longrightarrow (N, [\cdot = \cdot]^{(2)})$ given by

$$[fm = n] = [n = fm] = \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(f(m)-n)^*} \circ \mathcal{L}$$

for all $m \in M$ and $n \in N$. Just as in the previous example, this is compatible with composition, so we obtain a functor from the category $R - \mathbf{Mod}$ into the category of $\mathcal{Q}_{\mathcal{L}}^{(2)}$–sheaves.

Let $\mathcal{Q}$ be a quasi–quantale, and let us consider the "trivial" $\mathcal{Q}$–sheaf

$$\# = (\{*\}, [\cdot = \cdot])$$

with the singleton as set of generators and $[* = *] = 1$.

(2.22) Proposition. ([3]) *The sheaf # is a terminal object in the category of $\mathcal{Q}$–sheaves.*

Proof. Let $(A, [\cdot = \cdot])$ be a $\mathcal{Q}$–sheaf. It is easy to prove that the pair of maps

$$[f\cdot = \cdot] : A \times \{*\} \longrightarrow \mathcal{Q}\ , \quad [fa = *] = \bigvee_{a' \in A} [a = a']$$

$$[\cdot = f\cdot] : \{*\} \times A \longrightarrow \mathcal{Q}\ , \quad [* = fa] = \bigvee_{a' \in A} [a' = a]$$

is a premorphism from $(A, [\cdot = \cdot])$ to $\#$.
Moreover, if the pair of maps $[h\cdot = \cdot] : A \times \{*\} \longrightarrow \mathcal{Q}$ and $[\cdot = h\cdot] : \{*\} \times A \longrightarrow \mathcal{Q}$ defines a premorphism of $\mathcal{Q}$–sheaves, then by (2.16.1),

$$[ha = *] = \bigvee_{a' \in A} [a = a'] \& [ha' = *] \leq \bigvee_{a' \in A} [a = a'] \& 1 = \bigvee_{a' \in A} [a = a'],$$

and conversely, by (2.16.5),

$$\bigvee_{a' \in A} [a = a'] \leq \bigvee_{a' \in A} [ha = *] \& [* = ha'] \leq [ha = *],$$

proving that $[ha = *] = \bigvee_{a' \in A} [a = a'] = [fa = *]$. By using (2.16.2) and (2.16.5), one proves similarly that $[* = ha] = [* = fa]$ for all $a \in A$, so f and h are the same arrow in the precategory of $\mathcal{Q}$–sheaves, thus they define the same morphism in the category of $\mathcal{Q}$–sheaves. □

(2.23) Definition. Let $\mathcal{Q}$ be a quasi-quantale and let $(A, [\cdot = \cdot])$ be a $\mathcal{Q}$–sheaf. A *global section* of $(A, [\cdot = \cdot])$ is a premorphism of $\mathcal{Q}$–sheaves $f : \# \longrightarrow (A, [\cdot = \cdot])$ such that the equality holds in (2.16.2) and in (2.16.4), and $[a = f*]\ \&\ [f* = a'] \leq [a = a']$ for every a and a' in A.

(2.24) Let $(A, [\cdot = \cdot])$ be a $\mathcal{Q}$–sheaf and fix a generator $a \in A$. Then the pair of maps

$$[f_a\cdot = \cdot] : \{*\} \times A \longrightarrow \mathcal{Q}\ , \quad [f_a* = a'] = [a = a']$$

$$[\cdot = f_a\cdot] : A \times \{*\} \longrightarrow \mathcal{Q}\ , \quad [a' = f_a*] = [a' = a]$$

defines a global section of $(A, [\cdot = \cdot])$. Indeed, as properties (2.16.1)-(2.16.4) are trivially satisfied and as

$$[a' = f_a*]\ \&\ [f_a* = a''] = [a' = a]\ \&\ [a = a'']$$

$$\leq \bigvee_{b \in A} [a' = b]\ \&\ [b = a''] = [a' = a'']$$

for all $a', a'' \in A$, it remains to check that f_a verifies (2.16.5). Since $[a = a] = 1$, it follows that

$$\bigvee_{a' \in A} [f_a * = a'] \& [a' = f_a *] = \bigvee_{a' \in A} [a = a'] \& [a' = a]$$
$$= [a = a] = 1 \geq [* = *],$$

so f_a verifies (2.16.5).

The next result shows that the set of global sections of the structure $\mathcal{Q}_{\mathcal{L}}$–sheaf associated to a left R–module M contains the module $M/\sigma_{\mathcal{L}} M$.

(2.25) Proposition. *Let M be a left R–module and let $\mathcal{L}$ be a Gabriel filter (resp. a symmetric Gabriel filter). Then for all $m \in M$, the premorphism of $\mathcal{Q}_{\mathcal{L}}$–sheaves $f_m : \# \longrightarrow (M, [\cdot = \cdot])$ (resp. the premorphism of $\mathcal{Q}_{\mathcal{L}}^{(2)}$–sheaves $f_m : \# \longrightarrow (M, [\cdot = \cdot]^{(2)})$) is a global section. Moreover, for m and m' in M, the premorphisms f_m and $f_{m'}$ are equal if and only if $\overline{m} = \overline{m'}$ in $M/\sigma_{\mathcal{L}} M$.*

Proof. From (2.24), it follows that f_m is a global section.
On the other hand, if $f_m = f'_m$ then

$$\mathcal{L}_{\mathrm{Ann}^l_R(m-m')} \subseteq \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(m-m')} \circ \mathcal{L}$$
$$= [f_m * = m'] = [f'_m * = m'] = \mathcal{L},$$

so $\mathrm{Ann}^l_R(m - m') \in \mathcal{L}$, i.e., $\overline{m} = \overline{m'}$.
Conversely, if $\mathrm{Ann}^l_R(m - m') \in \mathcal{L}$ and $m'' \in M$ then

$$\mathrm{Ann}^l_R(m' - m'') \supseteq \mathrm{Ann}^l_R(m - m') \cap \mathrm{Ann}^l_R(m - m'') \in \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(m-m'')},$$

and therefore $\mathcal{L}_{\mathrm{Ann}^l_R(m'-m'')} \subseteq \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(m-m'')}$. So,

$$[f_{m'} * = m''] = \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(m'-m'')} \circ \mathcal{L}$$
$$\subseteq \mathcal{L} \circ \mathcal{L}_{\mathrm{Ann}^l_R(m-m'')} \circ \mathcal{L} = [f_m * = m''],$$

and since the other inclusion follows using the same argument,

$$[m'' = f_{m'} *] = [f_{m'} * = m''] = [f_m * = m''] = [m'' = f_m *].$$

As m'' was chosen arbitrarily, these equalities yield that $f_m = f_{m'}$.
The proof in the symmetric case follows in a similar way. □

The following lemma will be useful in the proof of the representation theorem.

(2.26) Lemma. ([3, Prop. 2.9]) *Assume $\mathcal{Q}$ to be a quasi–quantale and let $(A, [\cdot = \cdot])$ be a $\mathcal{Q}$–sheaf. Then, for any a and a' in A and any global section f of $(A, [\cdot = \cdot])$, we have:*

(2.26.1) $\sqrt{[a = a']} = \sqrt{[a' = a]}$;

(2.26.2) $\sqrt{[f* = a]} = \sqrt{[a = f*]}$.

(2.27) Lemma. *Let R be a left noetherian ring and let $\mathcal{L}$ be a Gabriel filter on R (resp. a symmetric Gabriel filter on R). If $\mathcal{H}$ belongs to $\mathcal{Q}_{\mathcal{L}}$ (resp. to $\mathcal{Q}_{\mathcal{L}}^{(2)}$) and $\sqrt{\mathcal{H}} = \mathcal{L}$, then $\mathcal{H} = \mathcal{L}$.*

Proof. Let us start with $\mathcal{Q}_{\mathcal{L}}$. Suppose $\mathcal{L}$ is strictly contained in $\mathcal{H}$ and pick an element L of $\mathcal{H}$ not belonging to $\mathcal{L}$. Then, the set of all left ideals containing L which do not belong to $\mathcal{L}$ is inductive since R is left noetherian, so there exists a left ideal K containing L which is maximal with respect to the property of not belonging to $\mathcal{L}$. The $\mathcal{L}$–Gabriel filter $\mathcal{L} \circ \mathcal{L}_K \circ \mathcal{L}$ is a prime element of $\mathcal{Q}_{\mathcal{L}}$. Indeed, if $\mathcal{H}'$ and $\mathcal{H}''$ are elements of $\mathcal{Q}_{\mathcal{L}}$ such that

$$\mathcal{L} \circ \mathcal{L}_K \circ \mathcal{L} \subseteq \mathcal{H}' \circ \mathcal{H}'',$$

then K belongs to $\mathcal{H}' \circ \mathcal{H}''$. So there exists an element $H \in \mathcal{H}''$ containing K, and such that $(K : r) \in \mathcal{H}'$ for any $r \in H$. If H belongs to $\mathcal{L}$, then $K \in \mathcal{H}' \circ \mathcal{L} = \mathcal{H}'$ and

$$\mathcal{L} \circ \mathcal{L}_K \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{H}' \circ \mathcal{L} = \mathcal{H}'.$$

On the other hand, if H does not belong to $\mathcal{L}$, then, by the maximality of K, we have $K = H$ and $\mathcal{L}_K \subseteq \mathcal{H}''$, so

$$\mathcal{L} \circ \mathcal{L}_K \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{H}'' \circ \mathcal{L} = \mathcal{H}''.$$

Therefore, since $\mathcal{L}$ is strictly contained in

$$\mathcal{L}_K \subseteq \mathcal{L} \circ \mathcal{L}_K \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{H} \circ \mathcal{L} = \mathcal{H},$$

the radical of $\mathcal{H}$ cannot be $\mathcal{L}$.

Similarly, if $\mathcal{L}$ is a symmetric Gabriel filter and $\mathcal{H}$ is an element of $\mathcal{Q}_{\mathcal{L}}^{(2)}$ strictly containing $\mathcal{L}$, then there exists an element L in $\mathcal{H}$ not belonging to $\mathcal{L}$. Since R is left noetherian, the set of all twosided ideals containing $L^* \in \mathcal{H}$ and not belonging to $\mathcal{L}$ is inductive and nonempty, so it has a maximal element P. The ideal P is prime. Indeed, pick a pair of twosided ideals I and J such that $IJ \subseteq P$. If both $I + L^*$ and $J + L^*$ belong to $\mathcal{L}$, then $P \supseteq (I+L^*)(J+L^*) \in \mathcal{L} \circ \mathcal{L} = \mathcal{L}$ — a contradiction. So one of $I + L^*$ and $J + L^*$ does not belong to $\mathcal{L}$, and, by maximality, $I \subseteq I + L^* \subseteq P$ or $J \subseteq J + L^* \subseteq P$.
Hence $\mathcal{L}_P$ is a prime element of $(R-\mathbf{filt}^{(2)})^{opp}$, and this implies $\mathcal{L}\circ\mathcal{L}_P\circ\mathcal{L}$ is a prime element of $\mathcal{Q}_{\mathcal{L}}^{(2)}$. Indeed, if the composition of $\mathcal{H}', \mathcal{H}'' \in \mathcal{Q}_{\mathcal{L}}^{(2)}$ contains $\mathcal{L}_P \subseteq \mathcal{L}\circ\mathcal{L}_P\circ\mathcal{L}$, then $\mathcal{L}_P \subseteq \mathcal{H}'$ or $\mathcal{L}_P \subseteq \mathcal{H}''$, so $\mathcal{L}\circ\mathcal{L}_P\circ\mathcal{L} \subseteq \mathcal{H}'$ or $\mathcal{L} \circ \mathcal{L}_P \circ \mathcal{L} \subseteq \mathcal{H}''$.
Therefore $\sqrt{\mathcal{H}} \neq \mathcal{L}$, since $\mathcal{L} \neq \mathcal{L} \circ \mathcal{L}_P \circ \mathcal{L} \subseteq \mathcal{H}$, as $L^* \subseteq P \in \mathcal{H}$. □

Note that in case $\mathcal{L} = \{R\}$, the assertion for $\mathcal{Q}_{\mathcal{L}} = R - \mathbf{filt}^{opp}$ is true even without the hypothesis of R being left noetherian.

Throughout the rest of this section, $\mathcal{Q}$ will denote one of the quasi–quantales $R-\mathbf{filt}^{opp}$, $(R-\mathbf{filt}^{(2)})^{opp}$ and $Id(R)-\mathbf{filt}^{opp}$, and R is assumed to be left noetherian in case $\mathcal{Q} = (R - \mathbf{filt}^{(2)})^{opp}$.

(2.28) Theorem. *For any left R–module M, the map $m \longmapsto f_m$ is an injection of M in the set of global sections of the $\mathcal{Q}$–sheaf associated to M. If the uniform filter $\{R\}$ is a compact element of $\mathcal{Q}$, then this map is a bijection.*

Proof. Since the first assertion follows from (2.25) with $\mathcal{L} = \{R\}$, it only remains to prove the surjectivity, when $\{R\}$ is compact in $\mathcal{Q}$.
Let f be a global section of $(M, [\cdot = \cdot])$. As (2.16.5) holds,

$$\bigcap_{m\in M} [f* = m] \circ [m = f*] = [* = *] = \{R\},$$

and since $\{R\}$ is assumed to be compact, there exist $m_1, \ldots, m_n \in M$ such that

$$\{R\} = \bigcap_{i=1}^{n} [f* = m_i] \circ [m_i = f*].$$

As f is a global section, for every pair of indices i and j,

$$[m_i = m_j] \subseteq [m_i = f*] \circ [f* = m_j]$$

$$\subseteq [m_i = f*] \circ [f* = m_j] \circ [m_j = f*],$$

so there exists a left ideal $I_{ij} \in [f* = m_j] \circ [m_j = f*]$ such that

$$(\mathrm{Ann}^l_R(m_i - m_j) : r) = \mathrm{Ann}^l_R(r(m_i - m_j)) \in [m_i = f*]$$

for all $r \in I_{ij}$, as $\mathrm{Ann}^l_R(m_i - m_j) \in [m_i = m_j]$. Then

$$I_j = \bigcap_{i=1}^{n} I_{ij} \in [f* = m_j] \circ [m_j = f*]$$

and $I_1 + \cdots + I_n = R$, since $\bigcap_{i=1}^{n}[f* = m_i] \circ [m_i = f*] = \{R\}$. So $1 = r_1 + \cdots + r_n$ for some $r_j \in I_j$.
Put $m = r_1 m_1 + \cdots + r_n m_n$. Then

$$\begin{aligned} m_i - m &= (r_1 + \cdots + r_n)m_i - (r_1 m_1 + \cdots + r_n m_n) \\ &= r_1(m_i - m_1) + \cdots + r_n(m_i - m_n), \end{aligned}$$

and therefore $m_i - m$ is an $[m_i = f*]$–torsion element since it is the sum of $[m_i = f*]$–torsion elements. So $\mathrm{Ann}^l_R(m_i - m) \in [m_i = f*]$ and this implies $[m_i = m] \subseteq [m_i = f*]$. Hence

$$[f* = m] = \bigcap_{m' \in M} [f* = m'] \circ [m' = m] \subseteq \bigcap_{i=1}^{n}[f* = m_i] \circ [m_i = m]$$

$$\subseteq \bigcap_{i=1}^{n}[f* = m_i] \circ [m_i = f*] = \{R\}$$

and therefore $[f* = m] = \{R\}$.
By definition, $\sqrt{[f* = m]} \subseteq [f* = m]$ and $\sqrt{[f* = m]} = \sqrt{[m = f*]}$ by lemma (2.26), so

$$\sqrt{[f* = m]} = \sqrt{[m = f*]} = \{R\}.$$

From (2.27), it follows that $[m = f*] = \{R\}$.

Therefore

$$[f_m* = m'] = [m = m'] \subseteq [m = f*] \circ [f* = m']$$
$$= [f* = m'] \subseteq [f* = m] \circ [m = m'] = [m = m'],$$

resp.

$$[m' = f_m*] = [m' = m] \subseteq [m' = f*] \circ [f* = m]$$
$$= [m' = f*] \subseteq [m' = m] \circ [m = f*] = [m' = m],$$

for every $m' \in M$, and this yields $f = f_m$. □

(2.29) Note. Let us point out some basic facts about the hypothesis that $\{R\}$ is a compact element in the quasi–quantale $\mathcal{Q}$.

First, let us stress the fact that $\{R\}$ is *always* a compact element in the quantale of jansian uniform filters $Id(R) - \mathbf{filt}^{opp}$, as mentioned in [3].

On the other hand, if R has a finite number of maximal left ideals (resp. of maximal twosided ideals) then $\{R\}$ is a compact element in the quasi–quantale $R - \mathbf{filt}^{opp}$ (resp. in $(R - \mathbf{filt}^{(2)})^{opp}$).

Indeed, if $\{M_1, \ldots, M_n\}$ is the set of maximal left ideals and if $\bigcap_{a\in A} \mathcal{L}_a = \{R\}$, then for every $1 \leq i \leq n$ there exists $a_i \in A$ such that $M_i \notin \mathcal{L}_{a_i}$. So if $L \in \bigcap_{i=1}^{n} \mathcal{L}_{a_i}$ and $L \neq R$, then there exists some maximal left ideal, M_i say, containing L, and therefore $M_i \in \bigcap_{i=1}^{n} \mathcal{L}_{a_i} \subseteq \mathcal{L}_{a_i}$ — a contradiction. Thus $\bigcap_{i=1}^{n} \mathcal{L}_{a_i} = \{R\}$, which yields that $\{R\}$ is a compact element in $R - \mathbf{filt}^{opp}$. The proof for the quasi–quantale $(R - \mathbf{filt}^{(2)})^{opp}$ is similar.

Conversely, if $\{R\}$ is a compact element in $(R - \mathbf{filt}^{(2)})^{opp}$, then the set $\mathbf{Max}^{(2)}R$ of all maximal twosided ideals is finite. Indeed, since every maximal twosided ideal M is prime, we may consider the largest symmetric uniform filter $\mathcal{L}_{R\setminus M}$ not containing M (see (1.23)). If the proper left ideal L belongs to $\bigcap\{\mathcal{L}_{R\setminus M} \ ; \ M \in \mathbf{Max}^{(2)}R\}$, then there exists a maximal twosided ideal M containing L^*, the bound of L, and hence $L \notin \mathcal{L}_{R\setminus M}$ — a contradiction. So,

$$\bigcap\{\mathcal{L}_{R\setminus M} \ ; \ M \in \mathbf{Max}^{(2)}R\} = \{R\},$$

and since $\{R\}$ is compact in $(R - \mathbf{filt}^{(2)})^{opp}$, there exist a finite family $\{M_1, \ldots, M_n\} \subseteq \mathbf{Max}^{(2)}R$ such that $\bigcap_{i=1}^{n} \mathcal{L}_{R\setminus M_i} = \{R\}$. Actually we have that $\{M_1, \ldots, M_n\} = \mathbf{Max}^{(2)}R$. Indeed, if M is a maximal twosided ideal, then $M \notin \bigcap_{i=1}^{n} \mathcal{L}_{R\setminus M_i}$, so $M \subseteq M_i$ for some i, and therefore $M = M_i$ by maximality.

3. Functoriality

Throughout this section, R and S will be rings and $\varphi : R \longrightarrow S$ a (unitary) ring homomorphism.

(3.1) For any uniform filter $\mathcal{L}$ over R, denote by $\mathcal{T}_{\mathcal{L}}$ its associated hereditary pretorsion class and consider

$$\mathcal{T}' = \{N \in S-\mathbf{Mod}; {}_RN \in \mathcal{T}_{\mathcal{L}}\},$$

where ${}_RN$ denotes the image of N by the restriction of scalars functor $\varphi_* : S-\mathbf{Mod} \longrightarrow R-\mathbf{Mod}$. As this functor is exact and preserves direct sums, it is easy to see that $\mathcal{T}'$ is closed under taking submodules, epimorphic images and direct sums in $S-\mathbf{Mod}$. So $\mathcal{T}'$ is a hereditary pretorsion class in $S-\mathbf{Mod}$, and therefore there exists a uniform filter over S, which will be denoted by $\overline{\mathcal{L}}$, such that $\mathcal{T}_{\overline{\mathcal{L}}} = \mathcal{T}'$.

We are interested in relating $\mathcal{L}$ and $\overline{\mathcal{L}}$, as well as the kernel functors $\sigma_{\mathcal{L}}$ and $\sigma_{\overline{\mathcal{L}}}$, in a more direct form.
As a first result, we have:

(3.2) Lemma. *Let $\mathcal{L}$ be a uniform filter over R. If $\overline{\mathcal{L}}$ consists of the left ideals I of S with the property that $\varphi^{-1}(I) \in \mathcal{L}$, then the following statements are true:*

(3.2.1) $\sigma_{\overline{\mathcal{L}}}(N) = \sigma_{\mathcal{L}}({}_RN)$ *for any left S-module N;*

(3.2.2) $\mathcal{F}_{\overline{\mathcal{L}}} = \{N \in S-\mathbf{Mod}; {}_RN \in \mathcal{F}_{\mathcal{L}}\}$.

Proof. (1) Let N be a left S-module. Without any extra assumption, we know that $\sigma_{\overline{\mathcal{L}}}(N) \subseteq \sigma_{\mathcal{L}}({}_RN)$, since the latter is $\mathcal{L}$-torsion, as $\mathcal{T}_{\overline{\mathcal{L}}}$ consists of all $M \in S-\mathbf{Mod}$ such that ${}_RM \in \mathcal{T}_{\mathcal{L}}$.
Conversely, if $x \in \sigma_{\mathcal{L}}({}_RN)$, then there exists some $I \in \mathcal{L}$ such that $Ix = 0$. By assumption, the left ideal $S\varphi(I)$ belongs to $\overline{\mathcal{L}}$, since $\varphi^{-1}(S\varphi(I))$

contains $I \in \mathcal{L}$. So, $x \in \sigma_{\overline{\mathcal{L}}}(N)$, as $S\varphi(I)x = 0$, which proves the other inclusion.

(2) Since $\sigma_{\overline{\mathcal{L}}}(N) = \sigma_{\mathcal{L}}({}_RN)$, a left S-module N is $\overline{\mathcal{L}}$-torsionfree if and only if ${}_RN$ is $\mathcal{L}$-torsionfree. □

If a uniform filter $\mathcal{L}$ over R has the property that $\overline{\mathcal{L}}$ consists exactly of the left ideals I of S with the property that $\varphi^{-1}(I) \in \mathcal{L}$, then $\mathcal{L}$ is said to be *compatible with* φ.

Let us denote by $N_R(S)$ the set of all elements $s \in S$ which are *normalizing* over R, i.e., with the property that $s\varphi(R) = \varphi(R)s$. Then we have:

(3.3) Lemma. *Let $\varphi : R \longrightarrow S$ be a ring morphism and let $\mathcal{L}$ and $\mathcal{H}$ be uniform filters over R satisfying the following properties:*

(3.3.1) *$\mathcal{H}$ is compatible with φ;*

(3.3.2) *S is generated as a left R-module by a subset $N \subseteq N_R(S)$;*

(3.3.3) *there exists a filter basis $\mathcal{L}'$ for $\mathcal{L}$ such that the left ideal $L_s^* = \{r \in R; rs \in sL\}$ belongs to $\mathcal{L}$ for all $L \in \mathcal{L}'$ and all $s \in N$.*

Then $\mathcal{L} \circ \mathcal{H}$ is also compatible with φ.

Proof. Even without the above assumptions, $\overline{\mathcal{L} \circ \mathcal{H}}$ consists of left ideals I of S, with $\varphi^{-1}(I) \in \mathcal{L} \circ \mathcal{H}$, since for all $I \in \overline{\mathcal{L} \circ \mathcal{H}}$, the left R-module $R/\varphi^{-1}(I)$ is a submodule of the $\mathcal{L} \circ \mathcal{H}$-torsion R-module S/I.

Conversely, let I be a left ideal of S with $\varphi^{-1}(I) \in \mathcal{L} \circ \mathcal{H}$. Then there exists $J \in \mathcal{H}$ containing $\varphi^{-1}(I)$, such that $J/\varphi^{-1}(I)$ is a $\mathcal{L}$-torsion module. Since $\varphi^{-1}(S\varphi(J))$ contains $J \in \mathcal{H}$ and since $\mathcal{H}$ is compatible with φ, the left ideal $S\varphi(J)$ belongs to $\overline{\mathcal{H}}$, so $S/S\varphi(J)$ is $\mathcal{H}$-torsion. As the sequence

$$0 \longrightarrow S\varphi(J)/I \longrightarrow S/I \longrightarrow S/S\varphi(J) \longrightarrow 0$$

is exact (in $S-$**Mod**, and also in $R-$**Mod**), it suffices to prove that every (generating) element in $S\varphi(J)/I = (RN\varphi(J))/I$ is $\mathcal{L}$-torsion.

Pick an element $s\varphi(r)$, with $s \in N$ and $r \in J$. Since $J/\varphi^{-1}(I)$ is $\mathcal{L}$–torsion and $\mathcal{L}'$ is a filter basis for $\mathcal{L}$, there exists $L \in \mathcal{L}'$ such that $Lr \subseteq \varphi^{-1}(I)$. By assumption, $L_s^* \in \mathcal{L}$, and

$$L_s^* s\varphi(r) \subseteq sL\varphi(r) \subseteq I,$$

so $S\varphi(J)/I$ is an $\mathcal{L}$–torsion module. Therefore S/I is an $\mathcal{L} \circ \mathcal{H}$–torsion left R–module and thus $I \in \overline{\mathcal{L} \circ \mathcal{H}}$. □

With notations as in the previous lemma, we may now prove:

(3.4) Proposition. *Let $\varphi : R \longrightarrow S$ be a ring morphism and let $\mathcal{L}$ be a uniform filter over R such that there exists $N \subseteq N_R(S)$ generating S as a left R–module and such that $\mathcal{L}$ admits a filter basis $\mathcal{L}'$ with $L_s^* \in \mathcal{L}$ for all $L \in \mathcal{L}'$ and all $s \in N$. Then the following assertions hold true:*

(3.4.1) *$\mathcal{L}$ is compatible with φ;*

(3.4.2) *$\sigma_{\overline{\mathcal{L}}}(M) = \sigma_{\mathcal{L}}(_RM)$ for all $M \in S-$***Mod***;*

(3.4.3) *$\mathcal{F}_{\overline{\mathcal{L}}} = \{M \in S-$***Mod***$; {}_RM \in \mathcal{F}_{\mathcal{L}}\}$;*

(3.4.4) *if $M \in R-$***Mod** *is $\mathcal{L}$–torsionfree, then so is* $\mathrm{Hom}_R(S, M)$*;*

(3.4.5) *if $M \in R-$***Mod** *is $\mathcal{L}$–torsion, then so is $S \otimes_R M$.*

Proof. In order to prove (2) and (3), it suffices to verify (1). Since $\mathcal{H} = \{R\}$ is compatible with φ, as $\overline{\mathcal{H}} = \{S\}$, the previous lemma shows that $\mathcal{L} \circ \mathcal{H} = \mathcal{L}$ is also compatible with φ.
On the other hand, let M be an $\mathcal{L}$–torsionfree R–module and consider $f \in \sigma_{\mathcal{L}}(\mathrm{Hom}_R(S, M))$. Then $I = \mathrm{Ann}_R^l(f) \in \mathcal{L}$ and $f(sr) = (rf)(s) = 0$ for all $s \in S$ and $r \in I$.
Moreover, there exists $J \in \mathcal{L}'$ contained in I. Pick $s = \sum_{i=1}^l r_i n_i \in S = RN$, where $r_i \in R$ and $n_i \in N$ for all $1 \leq i \leq l$. Since $J_{n_i}^* \in \mathcal{L}$ for every i, the left ideal $L = \bigcap_{i=1}^l (J_{n_i}^* : r_i)$ also belongs to $\mathcal{L}$, as every $(J_{n_i}^* : r_i)$ does and as $\mathcal{L}$ is closed under taking finite intersections. So,

$$Lf(s) \subseteq f(\sum_{i=1}^l (J_{n_i}^* : r_i) r_i n_i) \subseteq f(\sum_{i=1}^l J_{n_i}^* n_i) \subseteq f(\sum_{i=1}^l n_i J) \subseteq f(SI) = 0,$$

hence $f(s)$ is an $\mathcal{L}$–torsion element in M. Since M is $\mathcal{L}$–torsionfree, $f(s) = 0$, and this implies $f = 0$, as s is arbitrary. Therefore,

$$\sigma_{\mathcal{L}}(\mathrm{Hom}_R(S, M)) = 0$$

and $\mathrm{Hom}_R(S, M)$ is $\mathcal{L}$–torsionfree.

Let us now assume M to be an $\mathcal{L}$–torsion R–module. In order to prove (5), it suffices to check every generator in $S \otimes_R M$ to be an $\mathcal{L}$–torsion element. Let $s \in S$ be an element in N and let $m \in M$. As M is $\mathcal{L}$ torsion, there exists $J \in \mathcal{L}'$ such that $J \subseteq \mathrm{Ann}^l_R(m) \in \mathcal{L}$ and, as $s \in N$, we have $J^*_s \in \mathcal{L}$. Since for all $r \in J^*_s$ there exists $r' \in J$ such that $rs = sr'$, we have

$$r(s \otimes m) = sr' \otimes m = s \otimes r'm = 0$$

for all $r \in J^*_s$, so $J^*_s(s \otimes m) = 0$ and $s \otimes m \in \sigma_{\mathcal{L}}(S \otimes_R M)$. □

As a corollary to the previous result, let us first note that if φ is a *centralizing* extension, i.e., if

$$C_R(S) = \{s \in S \ ; \ \forall r \in R, \ s\varphi(r) = \varphi(r)s\}$$

is a set of generators of S as a left R–module, and $\mathcal{L}$ is a uniform filter, then $\mathcal{L}$ is compatible with φ. Indeed, take $N = C_R(S)$ and $\mathcal{L}' = \mathcal{L}$ in the proposition.

Somewhat more generally, let us denote by $N^s_R(S)$ the set of all elements of $s \in S$ such that $s\varphi(I) = \varphi(I)s$, for any twosided ideal I of R, and let us say φ is *strongly normalizing* ([16, 17]) if S is generated by $N^s_R(S)$ as a left R–module. If $\mathcal{L}$ is a symmetric uniform filter and φ is strongly normalizing then $N = N^s_R(S)$ and $\mathcal{L}' = \mathcal{L}^{(2)}$ satisfy the hypotheses of the proposition. Thus $\mathcal{L}$ is compatible with φ.

(3.5) If $\mathcal{L}$ and $\mathcal{H}$ are uniform filters over R, then $\overline{\mathcal{L}} \circ \overline{\mathcal{H}}$ is included in $\overline{\mathcal{L} \circ \mathcal{H}}$ (or equivalently, $\mathcal{T}_{\overline{\mathcal{L}} \circ \overline{\mathcal{H}}} \subseteq \mathcal{T}_{\overline{\mathcal{L} \circ \mathcal{H}}}$). Indeed, if N is an $\overline{\mathcal{L}} \circ \overline{\mathcal{H}}$–torsion left S–module, then there exists an exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

in S–**Mod** where N' resp. N'' is $\overline{\mathcal{L}}$–torsion resp. $\overline{\mathcal{H}}$–torsion as a left S–module, and thus $\mathcal{L}$–torsion resp. $\mathcal{H}$–torsion as a left R–module. Since this sequence is also exact in R–**Mod**, it follows that N is $\mathcal{L}\circ\mathcal{H}$–torsion over R, so N is also $\overline{\mathcal{L}\circ\mathcal{H}}$–torsion over S.

Conversely:

(3.6) Lemma. *Let $\mathcal{L}$ and $\mathcal{H}$ be uniform filters compatible with the ring morphism $\varphi : R \longrightarrow S$. Then*

$$\overline{\mathcal{L}\circ\mathcal{H}} = \overline{\mathcal{L}}\circ\overline{\mathcal{H}}.$$

Proof. In order to prove the remaining inclusion, pick $I \in \overline{\mathcal{L}\circ\mathcal{H}}$. Let us verify that

$$J = \{s \in S; (I : s) \in \overline{\mathcal{L}}\}$$

belongs to $\overline{\mathcal{H}}$, or equivalently, as $\mathcal{H}$ is compatible with φ, that $\varphi^{-1}(J) \in \mathcal{H}$. For all $r \in R$, we have $\varphi^{-1}(I : \varphi(r)) = (\varphi^{-1}(I) : r)$. So, as $\mathcal{L}$ is compatible with φ,

$$\begin{aligned}\varphi^{-1}(J) &= \{r \in R; (I : \varphi(r)) \in \overline{\mathcal{L}}\} = \{r \in R; \varphi^{-1}(I : \varphi(r)) \in \mathcal{L}\}\\ &= \{r \in R; (\varphi^{-1}(I) : r) \in \mathcal{L}\}\end{aligned}$$

and this belongs to $\mathcal{H}$ since $\varphi^{-1}(I) \in \mathcal{L}\circ\mathcal{H}$. □

(3.7) As an easy application of the previous lemma, let us note that if $\varphi : R \longrightarrow S$ is a centralizing extension and $\{\mathcal{L}_i\}_{i=1}^n$ and $\{\mathcal{L}'_j\}_{j=1}^m$ are uniform filters, resp. if $\varphi : R \longrightarrow S$ is a strongly normalizing extension and $\{\mathcal{L}_i\}_{i=1}^n$ and $\{\mathcal{L}'_j\}_{j=1}^m$ are *symmetric* uniform filters, with $\mathcal{L}_1\circ\cdots\circ\mathcal{L}_n = \mathcal{L}'_1\circ\cdots\circ\mathcal{L}'_m$, then

$$\overline{\mathcal{L}_1}\circ\cdots\circ\overline{\mathcal{L}_n} = \overline{\mathcal{L}_1\circ\cdots\circ\mathcal{L}_n} = \overline{\mathcal{L}'_1\circ\cdots\circ\mathcal{L}'_m} = \overline{\mathcal{L}'_1}\circ\cdots\circ\overline{\mathcal{L}'_m}.$$

In order to consider the functorial properties of structure sheaves, let us mention the following result from [3]. We include the proof for completeness' sake.

(3.8) Proposition. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be quasi–quantales. If $q : \mathcal{Q} \longrightarrow \mathcal{Q}'$ is a morphism of lattices which preserves the top element, the multiplication and arbitrary suprema, then q induces by composition a functor from the category of $\mathcal{Q}$–sheaves to the category of $\mathcal{Q}'$–sheaves.*

Proof. It is fairly easy to prove that if $(A,[\cdot = \cdot])$ and $(B,[\cdot = \cdot])$ are $\mathcal{Q}$–sheaves and $f : (A,[\cdot = \cdot]) \longrightarrow (B,[\cdot = \cdot])$ is a premorphism of $\mathcal{Q}$–sheaves, then $(A, q([\cdot = \cdot]))$ and $(B, q([\cdot = \cdot]))$ are $\mathcal{Q}'$–sheaves and

$$q(f) : (A, q([\cdot = \cdot])) \longrightarrow (B, q([\cdot = \cdot]))$$

given by

$$q([f\cdot = \cdot]) : A \times B \longrightarrow \mathcal{Q}'$$

and

$$q([\cdot = f\cdot]) : B \times A \longrightarrow \mathcal{Q}'$$

is a premorphism of $\mathcal{Q}'$–sheaves.
Let $f : (A,[\cdot = \cdot]) \longrightarrow (B,[\cdot = \cdot])$ and $g : (B,[\cdot = \cdot]) \longrightarrow (C,[\cdot = \cdot])$ be premorphisms of $\mathcal{Q}$–sheaves. The composition of $q(f)$ and $q(g)$ is given by the pair of maps

$$[q(g)q(f)\cdot = \cdot] : A \times C \longrightarrow \mathcal{Q}' \ ;$$

$$(a,c) \longrightarrow \bigvee_{b\in B} q([fa = b]) \ \& \ q([gb = c]) = q(\bigvee_{b\in B} [fa = b] \ \& \ [gb = c]),$$

and

$$[\cdot = q(g)q(f)\cdot] : C \times A \longrightarrow \mathcal{Q}' \ ;$$

$$(c,a) \longrightarrow \bigvee_{b\in B} q([c = gb]) \ \& \ q([b = fa]) = q(\bigvee_{b\in B} [c = gb] \ \& \ [b = fa]),$$

so $q(g)q(f) = q(gf)$.
On the other hand, if $(A,[\cdot = \cdot])$ is a $\mathcal{Q}$–sheaf and id_A is its distinguished arrow, then

$$q([id_A a = a']) = q([a = a']) = q([a = id_A a')],$$

for all $a, a' \in A$, so $q(id_A)$ is the distinguished arrow of $(A, q([\cdot = \cdot]))$.
Se we obtain a functor from the precategory of $\mathcal{Q}$–sheaves to the precategory of $\mathcal{Q}'$–sheaves, which induces a functor from the category of $\mathcal{Q}$–sheaves to the category of $\mathcal{Q}'$–sheaves. □

(3.9) Let $\varphi : R \longrightarrow S$ be a ring morphism and let $\mathcal{L}$ be a Gabriel filter of left ideals in R. In (3.6) it was proved that if $\mathcal{H}$ and $\mathcal{H}'$ are uniform filters compatible with φ, then $\overline{\mathcal{H}} \circ \overline{\mathcal{H}'} = \overline{\mathcal{H} \circ \mathcal{H}'}$. So, if every uniform filter over R is compatible with φ (e.g., if φ is a centralizing extension), then

$$q_\varphi^{\mathcal{L}} : \mathcal{Q}_{\mathcal{L}} \longrightarrow \mathcal{Q}_{\overline{\mathcal{L}}} \quad : \quad \mathcal{H} \longrightarrow \overline{\mathcal{H}}$$

is a morphism of quasi–quantales. Indeed, even without the assumption of φ–compatibility, for every $\mathcal{L}$–Gabriel filter $\mathcal{H}$ we have

$$\overline{\mathcal{H}} \subseteq \overline{\mathcal{H}} \circ \overline{\mathcal{L}} \subseteq \overline{\mathcal{H} \circ \mathcal{L}} = \overline{\mathcal{H}}$$

and

$$\overline{\mathcal{H}} \subseteq \overline{\mathcal{L}} \circ \overline{\mathcal{H}} \subseteq \overline{\mathcal{L} \circ \mathcal{H}} = \overline{\mathcal{H}},$$

so $\overline{\mathcal{L}}$ is a Gabriel filter and $q_\varphi^{\mathcal{L}}$ is a lattice morphism.
Let $\{\mathcal{L}_a;\ a \in A\}$ be a non–empty family of uniform filters on R. Since every uniform filter is compatible with φ, we have

$$\overline{\bigcap_{a\in A} \mathcal{L}_a} = \{L \leq_l S\ ;\ \varphi^{-1}(L) \in \bigcap_{a\in A} \mathcal{L}_a\}$$

$$= \bigcap_{a\in A} \{L \leq_l S\ ;\ \varphi^{-1}(L) \in \mathcal{L}_a\} = \bigcap_{a\in A} \overline{\mathcal{L}_a}.$$

This proves the assertion, since $q_\varphi^{\mathcal{L}}$ preserves the top element by definition.
On the other hand, if φ is a strongly normalizing extension and $\mathcal{L}$ is a symmetric Gabriel filter, then $\overline{\mathcal{L}}$ is also symmetric. Indeed, if

$$L \in \overline{\mathcal{L}} = \{H \leq_l S\ ;\ \varphi^{-1}(H) \in \mathcal{L}\},$$

then there exists a twosided ideal $I \in \mathcal{L}$ with $I \subseteq \varphi^{-1}(L)$. Hence, since φ is strongly normalizing, $S\varphi(I)$ is a twosided ideal of S which belongs to $\overline{\mathcal{L}}$, as $I \subseteq \varphi^{-1}(S\varphi(I))$, and

$$S\varphi(I) \subseteq S\varphi(\varphi^{-1}(L)) \subseteq L.$$

In a similar way, for any symmetric Gabriel filter $\mathcal{L}$ over the left noetherian ring R and any strongly normalizing extension $\varphi : R \longrightarrow S$, we can define a morphism of lattices

$$q_\varphi^{\mathcal{L}} : \mathcal{Q}_{\mathcal{L}}^{(2)} \longrightarrow \mathcal{Q}_{\overline{\mathcal{L}}}^{(2)} \ ; \ q_\varphi^{\mathcal{L}}(\mathcal{H}) = \overline{\mathcal{H}}$$

which is actually a morphism of quasi–quantales.
From this and the previous result, it now easily follows:

(3.10) Proposition. *Let $\varphi : R \longrightarrow S$ a ring morphism and let $\mathcal{L}$ be a Gabriel filter resp. a symmetric Gabriel filter on R. If every element in $\mathcal{Q}_{\mathcal{L}}$ is compatible with φ resp. if φ is a strongly normalizing extension, then inducing filters defines a functor from the category of $\mathcal{Q}_{\mathcal{L}}$–sheaves to the category of $\mathcal{Q}_{\overline{\mathcal{L}}}$–sheaves resp. from the category of $\mathcal{Q}_{\mathcal{L}}^{(2)}$–sheaves to the category of $\mathcal{Q}_{\overline{\mathcal{L}}}^{(2)}$–sheaves.*

(3.11) Note. In particular, if $\mathcal{Q}_{\mathcal{L}} = R - \mathbf{filt}^{opp}$, then $\mathcal{L} = \{R\}$ and

$$\overline{\mathcal{L}} \subseteq \{L \leq_l S \ ; \ \varphi^{-1}(L) \in \mathcal{L}\} = \{L \leq_l S \ ; \ 1 \in \varphi^{-1}(L)\} = \{S\}.$$

So, $\overline{\mathcal{L}} = \{S\}$ and if every uniform filter is compatible with φ, then

$$q_\varphi : R - \mathbf{filt}^{opp} \longrightarrow S - \mathbf{filt}^{opp} \ , \ \ q_\varphi(\mathcal{H}) = \overline{\mathcal{H}}$$

induces by composition a functor from the category of $R-\mathbf{filt}^{opp}$–sheaves to the category of $S - \mathbf{filt}^{opp}$–sheaves.
Similarly, if R and S are left noetherian rings and if $\varphi : R \longrightarrow S$ is a strongly normalizing extension, then q_φ induces a functor from the category of $(R - \mathbf{filt}^{(2)})^{opp}$–sheaves to that of $(S - \mathbf{filt}^{(2)})^{opp}$–sheaves.

(3.12) Let $\mathcal{Q}$ be a quasi–quantale. As we have already pointed out in (2.11), $\sqrt{\cdot} : \mathcal{Q} \longrightarrow \sqrt{\mathcal{Q}}$ is a morphism of quasi–quantales, so it induces by composition a functor from the category of $\mathcal{Q}$–sheaves into the category of $\sqrt{\mathcal{Q}}$–sheaves. Since $\sqrt{\cdot}$ preserves the top element, if $(A, [\cdot = \cdot])$ is a $\mathcal{Q}$–sheaf and f is a global section, the pair of maps given by

$$[\sqrt{f}* = a] = \sqrt{[f* = a]} \ \ , \ \ [a = \sqrt{f}*] = \sqrt{[a = f*]}$$

defines a global section $\sqrt{f}$ of the $\sqrt{\mathcal{Q}}$–sheaf $(A, \sqrt{[\cdot = \cdot]})$.

In particular, if $\mathcal{Q}$ is one of the quasi-quantales $R-\mathbf{filt}^{opp}$, $(R-\mathbf{filt}^{(2)})^{opp}$ and $Id(R)-\mathbf{filt}^{opp}$ then for every element $r \in R$, the morphism $\sqrt{f_r}$ given by

$$[\sqrt{f_r}* = r'] = [r' = \sqrt{f_r}*] = \sqrt{[r = r']}$$

is a global section in $(R, \sqrt{[\cdot = \cdot]})$. If r and r' are elements in R such that $\sqrt{f_r} = \sqrt{f_{r'}}$, then

$$\sqrt{[r = r']} = [\sqrt{f_r}* = r'] = [\sqrt{f_{r'}}* = r'] = \sqrt{[r' = r']} = \{R\},$$

and by lemma (2.27), this implies $[r = r'] = \{R\}$, so $r = r'$. One is interested in looking for conditions on R, implying the map from R to the set of global section of $(R, \sqrt{[\cdot = \cdot]})$ to be surjective, and thus to obtain a representation theorem for R by means of sections on a $\sqrt{\mathcal{Q}}$-sheaf.

Throughout the rest of this section, let $\mathcal{Q}$ be one of the quasi-quantales $R-\mathbf{filt}^{opp}$, $(R-\mathbf{filt}^{(2)})^{opp}$ and $Id(R)-\mathbf{filt}^{opp}$ and assume R to be left noetherian in case $\mathcal{Q} = (R-\mathbf{filt}^{(2)})^{opp}$.
The morphism of quasi-quantales $\sqrt{\cdot} : \mathcal{Q} \longrightarrow \sqrt{\mathcal{Q}}$ and the lattice morphism $i : \sqrt{\mathcal{Q}} \longrightarrow \mathcal{Q}$ may be viewed as functors between preordered categories. Let us collect some properties about them.

(3.13) Lemma. *The maps* $\sqrt{\cdot}$ *and* i *satisfy the following assertions:*

(3.13.1) *the composition* $\sqrt{\cdot} \circ i$ *is the identity morphism in* $\sqrt{\mathcal{Q}}$, *and* $i(\sqrt{\mathcal{L}}) \subseteq \mathcal{L}$ *for all* $\mathcal{L} \in \mathcal{Q}$;

(3.13.2) *the functor* i *is right adjoint to* $\sqrt{\cdot}$, *i.e.,* $\mathcal{L} \supseteq i(\mathcal{H})$ *if and only if* $\sqrt{\mathcal{L}} \supseteq \mathcal{H}$ *for* $\mathcal{L} \in \mathcal{Q}$ *and* $\mathcal{H} \in \sqrt{\mathcal{Q}}$;

(3.13.3) *the top element* $\{R\}$ *is preserved by* i *and reflected by* $\sqrt{\cdot}$;

(3.13.4) $i(\mathcal{H}) \circ i(\mathcal{H}') \supseteq i(\mathcal{H} \wedge \mathcal{H}')$ *for every* $\mathcal{H}$ *and* $\mathcal{H}'$ *in* $\sqrt{\mathcal{Q}}$;

(3.13.5) $\mathcal{H} \wedge \mathcal{H}' = \sqrt{i(\mathcal{H}) \circ i(\mathcal{H}')}$ *for every* $\mathcal{H}$ *and* $\mathcal{H}'$ *in* $\sqrt{\mathcal{Q}}$;

(3.13.6) $i(\sqrt{\mathcal{L}}) \circ i(\sqrt{\mathcal{L}'}) \supseteq i(\sqrt{\mathcal{L} \circ \mathcal{L}'})$ *for every* $\mathcal{L}$ *and* $\mathcal{L}'$ *in* $\mathcal{Q}$;

(3.13.7) *for any family* $\{\mathcal{H}_a\}_{a \in A}$ *in* $\sqrt{\mathcal{Q}}$,

$$\sqrt{\bigcap_{a \in A} i(\mathcal{H}_a)} = \bigvee_{a \in A} \mathcal{H}_a.$$

Proof. The assertion (3) is lemma (2.27), and (1), (4) and (5) follow from the definitions. Since $\sqrt{\cdot}$ is order-preserving, if $\mathcal{L} \in \mathcal{Q}$ contains the radical element $\mathcal{H}$, then $\sqrt{\mathcal{L}}$ contains $\sqrt{\mathcal{H}} = \mathcal{H}$. Conversely, if $\mathcal{H} \subseteq \sqrt{\mathcal{L}}$, then $\mathcal{H} \subseteq \mathcal{L}$ since $\sqrt{\mathcal{L}} \subseteq \mathcal{L}$, and this proves (2).
The assertion in (6) is an easy consequence of (2.9.8), since $\sqrt{\mathcal{L}} \wedge \sqrt{\mathcal{L}'} \subseteq \sqrt{\mathcal{L}} \circ \sqrt{\mathcal{L}'}$.
If $\{\mathcal{H}_a\}_{a\in A} \subseteq \sqrt{\mathcal{Q}}$ then

$$\bigvee_{a\in A} \mathcal{H}_a = \sqrt{i(\bigvee_{a\in A} \mathcal{H}_a)} \subseteq \sqrt{\bigcap_{a\in A} i(\mathcal{H}_a)}$$

$$\subseteq \bigvee_{a\in A} \sqrt{i(\mathcal{H}_a)} = \bigvee_{a\in A} \mathcal{H}_a,$$

so (7) follows. □

Let us now conclude with the following representation theorem:

(3.14) Theorem. *If $\{R\}$ is a compact element in $\mathcal{Q}$ and $\sqrt{\mathcal{L}_0} = \mathcal{L}_0$, with $\mathcal{L}_0$ the set of all left R–ideals, then the map $r \longrightarrow \sqrt{f_r}$ is a bijection between R and the set of global sections of $(R, \sqrt{[\cdot = \cdot]})$.*

Proof. Injectivity follows from (3.12), so let us prove surjectivity. Let f be a global section of $(R, \sqrt{[\cdot = \cdot]})$. Since (2.16.5) and (3.13.7) hold,

$$\{R\} = [* = *] = \bigvee_{r\in R} [f* = r] \wedge [r = f*] = \sqrt{\bigcap_{r\in R} i([f* = r] \wedge [r = f*])},$$

and by (3.13.3),

$$\{R\} = \bigcap_{r\in R} i([f* = R] \wedge [r = f*]).$$

So there exist $r_1, \ldots, r_n \in R$ such that

$$\{R\} = \bigcap_{j=1}^{n} i([f* = r_j] \wedge [r_j = f*]),$$

as $\{R\}$ is a compact element in $\mathcal{Q}$. Hence, as $\sqrt{\cdot}$ preserves the top element,

$$\{R\} = \sqrt{\bigcap_{j=1}^{n} i([f* = r_j] \wedge [r_j = f*])} = \bigvee_{j=1}^{n} [f* = r_j] \wedge [r_j = f*]$$

$$= \bigvee_{j=1}^{n} \sqrt{i([f* = r_j] \wedge [r_j = f*])} = \bigvee_{j=1}^{n} \sqrt{i(\sqrt{i([f* = r_j])} \circ i([r_j = f*]))}$$

$$= \sqrt{\bigcap_{j=1}^{n} i([f* = r_j]) \circ i([r_j = f*])},$$

and, again by (3.13.3), $\bigcap_{j=1}^{n} i([f* = r_j]) \circ i([r_j = f*]) = \{R\}$.
Pick a pair of different indices $1 \leq j, k \leq n$ and let us denote by J_{jk} the twosided ideal $R(r_j - r_k)R$. Then

$$i([r_j = f*]) \circ i([f* = r_k]) \circ \mathcal{L}_{J_{jk}} \supseteq i([r_j = f*] \wedge [f* = r_k]) \circ i(\sqrt{\mathcal{L}_{J_{jk}}})$$

and the later contains $i(\sqrt{[r_j = r_k]}) \circ i(\sqrt{\mathcal{L}_{Jjk}})$ since f is a global section. Hence (3.13.6) and the assumptions on $\mathcal{L}_0$ imply that

$$i([r_j = f*]) \circ i([f* = r_k]) \circ \mathcal{L}_{J_{jk}} \supseteq i(\sqrt{[r_j = r_k] \circ \mathcal{L}_{Jjk}}) = i(\sqrt{\mathcal{L}_0}) = \mathcal{L}_0,$$

so there exists a left ideal in $\mathcal{L}_{J_{jk}}$ such that for every element r belonging to it, $\mathrm{Ann}_R^l(r) \in i([r_j = f*]) \circ i([f* = r_k])$. Since every left ideal in $\mathcal{L}_{J_{jk}}$ contains J_{jk}, in particular $\mathrm{Ann}_R^l(r_j - r_k) \in i([r_j = f*]) \circ i([f* = r_k])$. So there exists a left ideal

$$I_{jk} \in i([f* = r_k]) \subseteq i([f* = r_k]) \circ i([r_k = f*])$$

such that

$$\mathrm{Ann}_R^l(r(r_j - r_k)) = (\mathrm{Ann}_R^l(r_j - r_k) : r) \in i([r_j = f*])$$

for any $r \in I_{jk}$. Put

$$I_k = \bigcap_{1 \leq j \leq n} I_{jk} \in i([f* = r_k]) \circ i([r_k = f*]).$$

Then

$$I_1 + \cdots + I_n \in \bigcap_{k=1}^{n} i([f* = r_k]) \circ i([r_k = f*]) = \{R\},$$

so there exist $s_k \in I_k$ with $s_1 + \cdots + s_n = 1$. Let $s = s_1 r_1 + \cdots + s_n r_n$. The element $r_j - s$ is a $i([r_j = f*])$–torsion element for every j, since

$$r_j - s = (s_1 + \cdots + s_n) r_j - (s_1 r_1 + \cdots + s_n r_n)$$

$$= s_1(r_j - r_1) + \cdots + s_k(r_j - r_n),$$

so $[r_j = s] \subseteq i([r_j = f*])$ and hence

$$\sqrt{[r_j = s]} \subseteq \sqrt{i([r_j = f*])} = [r_j = f*].$$

Therefore,

$$[f* = s] = \bigvee_{r \in R} [f* = r] \wedge [r = s] \subseteq \bigvee_{j=1}^{n} [f* = r_j] \wedge [r_j = s]$$

$$\subseteq \bigvee_{j=1}^{n} [f* = r_j] \wedge [r_j = f*] = \{R\},$$

and by (2.26),

$$[s = f*] = \sqrt{i([s = f*])} = \sqrt{i([f* = s])} = [f* = s].$$

Hence $f = \sqrt{f_s}$, since for all $r \in R$,

$$[\sqrt{f_s}* = r] = \sqrt{[s = r]} \subseteq [s = f*] \wedge [f* = r]$$

$$= [f* = r] \subseteq [f* = s] \wedge \sqrt{[s = r]} = \sqrt{[s = r]}$$

and

$$[r = \sqrt{f_s}*] = \sqrt{[r = s]} \subseteq [r = f*] \wedge [f* = s]$$

$$= [r = f*] \subseteq \sqrt{[r = s]} \wedge [s = f*] = \sqrt{[r = s]}.$$

□

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