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LIFTINGS OF STONE'S MONADICITY TO SPACES AND THE DUALITY BETWEEN THE CALCULI OF INVERSE AND DIRECT IMAGES

by Pierre DAMPHOUSSE and René GUITART

RESUME. Dans cet article deux catégories $\underline{\textbf{Qual}}^+$ et $\underline{\textbf{Qual}}^-$ sont introduites, dont la duale de chacune est algébrique (à une équivalence naturelle près) sur l'autre, ceci relevant l'algébricité classique de $\underline{\textbf{Ens}}^{op}$ sur $\underline{\textbf{Ens}}$. De plus $\underline{\textbf{Qual}}^+$ est cartésienne fermée. Le calcul des images inverses (resp. directes) est présenté comme la donnée de $\underline{\textbf{Qual}}^-$ (resp. $\underline{\textbf{Qual}}^+$) et d'une monade sur cette catégorie relevant la "monade de Stone" sur $\underline{\textbf{Ens}}$. La notion de dualité entre catégories est étendue en celle de dualité entre monades, et dans ce sens le calcul des images directes et inverses sont duales.

Comme conséquences, on prouve l'algébricité sur $\underline{\textbf{Qual}}^+$ (à une équivalence naturelle près) de la duale de la catégorie $\underline{\textbf{Top}}$ des espaces topologiques et de la duale de la catégorie des ensembles munis d'une relation d'équivalence.

1 The categories $\underline{\text{Qual}}^+$ and $\underline{\text{Qual}}^-$

Let $f : X \to Y$ be a map. We will write $(\mathcal{P}X, \subseteq)$ for the set of subsets of X ordered with inclusion. Let us recall that we have the following functors (between ordered sets (*see* [5]))

$$\begin{array}{lll} f^* : & (\mathcal{P}Y, \subseteq) \to (\mathcal{P}X, \subseteq) : & B \mapsto f^*B = \{x \in X;\ fx \in B\}, \\ \exists f : & (\mathcal{P}X, \subseteq) \to (\mathcal{P}Y, \subseteq) : & A \mapsto \exists f A = \{y \in Y;\ \exists x(y = fx \wedge x \in A)\} \end{array}$$

where $\exists f$ is left-adjoint to f^*; this fact will be written $\exists f \dashv f^*$, or in a diagram

$$\begin{array}{c} (\mathcal{P}Y, \subseteq) \\ \exists f \Big\uparrow \ \dashv \ \Big\downarrow f^* \\ (\mathcal{P}X, \subseteq) \end{array}$$

Therefore, for each $A \in \mathcal{P}X$ and $B \in \mathcal{P}Y$, we have:

$$\exists f A \subseteq B \iff A \subseteq f^*B$$

We call a pair $(X, \mathcal{X})$, where $\mathcal{X} \in \mathcal{P}\mathcal{P}X \overset{\text{def}}{=} \mathcal{P}^2 X$, a *qualification space*; X is called the *support* of the space, $\mathcal{X}$ its *qualification*, the elements of X its points and those of $\mathcal{X}$ its *qualities*. One shall say that $a \in X$ *has quality* $A \in \mathcal{X}$ when $a \in A$.

A mapping between the supports of two qualification spaces $f : X \to Y$ is said to be

$$\begin{array}{lllll} \textit{open} & \text{when} & (\exists(\exists f))\mathcal{X} \subseteq \mathcal{Y} & \textit{i.e.} & \mathcal{X} \subseteq (\exists f)^* \mathcal{Y} \\ \textit{continuous} & \text{when} & (\exists(f^*))\mathcal{Y} \subseteq \mathcal{X} & \textit{i.e.} & \mathcal{Y} \subseteq f^{**}\mathcal{X}. \end{array}$$

The reader should convince himself, if necessary, that $(\exists f)^*$ and $(\exists(f^*))$ are in general different mappings. This terminology "*open*" and "*continuous*" comes from topology (*see* example 1, section 6).

We write $\underline{\mathbf{Qual}}^+$ (resp. $\underline{\mathbf{Qual}}^-$) the categories of qualification spaces and open maps (resp. continuous) $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$, where $f : X \to Y$ is a morphism in $\underline{\mathbf{Ens}}$, the category of sets and mappings. For a qualification space $(X, \mathcal{X})$, let us set $\underline{\mathbf{S}}(X, \mathcal{X}) = X$, which determines two functors $\underline{\mathbf{S}}^+ : \underline{\mathbf{Qual}}^+ \to \underline{\mathbf{Ens}}$ and $\underline{\mathbf{S}}^- : \underline{\mathbf{Qual}}^- \to \underline{\mathbf{Ens}}$ ("$\underline{\mathbf{S}}$" for "Support"). For any set X, we define two qualification spaces

$$\underline{\mathbf{Tout}}\, X = (X, \mathcal{P}X) \quad \textit{and} \quad \underline{\mathbf{Rien}}\, X = (X, \emptyset)$$

which define four functors

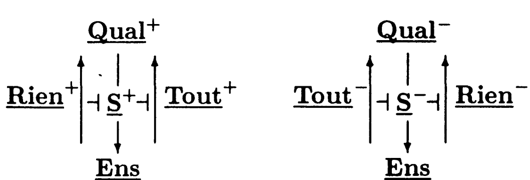
$$\underline{\mathbf{Tout}}^{\pm}, \underline{\mathbf{Rien}}^{\pm} : \underline{\mathbf{Ens}} \to \underline{\mathbf{Qual}}^{\pm}$$

and we have the four natural bijections

$$\begin{array}{ll} \underline{\mathbf{Ens}}(E, X) \simeq \underline{\mathbf{Qual}}^+((E, \emptyset), (X, \mathcal{X})) & \underline{\mathbf{Ens}}(X, E) \simeq \underline{\mathbf{Qual}}^+((X, \mathcal{X}), (E, \mathcal{P}E)) \\ \underline{\mathbf{Ens}}(E, X) \simeq \underline{\mathbf{Qual}}^-((E, \mathcal{P}E), (X, \mathcal{X})) & \underline{\mathbf{Ens}}(X, E) \simeq \underline{\mathbf{Qual}}^-((X, \mathcal{X}), (E, \emptyset)) \end{array}$$

so that:

Proposition 1 *We have the following adjunctions:*



Proposition 2 $\underline{\mathbf{S}}^+$ *and* $\underline{\mathbf{S}}^-$ *are final and initial structure functors. Therefore, a functor* $\underline{\mathbf{G}}$ *in* $\underline{\mathbf{Qual}}^{\pm}$ *has a limit* Λ *(resp. a colimit* Λ*) if and only if* $S^{\pm} \circ \underline{\mathbf{G}}$ *has a limit* L*, and then* $S^{\pm}(\Lambda) = L$*. In particular,* $\underline{\mathbf{Qual}}^+$ *and* $\underline{\mathbf{Qual}}^-$ *are complete and co-complete.*

Proposition 3 *In* $\underline{\mathbf{Qual}}^+$*,* $(\emptyset, \emptyset)$ *is an initial object, and* $(\mathbf{1}, \{\emptyset, \mathbf{1}\})$ *is a final one. In* $\underline{\mathbf{Qual}}^-$*,* $(\emptyset, \{\emptyset\})$ *is an initial object, and* $(\mathbf{1}, \emptyset)$ *is a final one. Moreover,* $(\emptyset, \emptyset)$ *and* $(\emptyset, \{\emptyset\})$ *are isomorphic neither in* $\underline{\mathbf{Qual}}^+$ *nor in* $\underline{\mathbf{Qual}}^-$*.*

Proposition 4 *Let* $(X, \mathcal{X})$ *and* $(Y, \mathcal{Y})$ *be two qualification spaces,* p_X *and* p_Y *the projections from* $X \times Y$ *onto* X *and* Y*, and* i_X *and* i_Y *the canonical injections from* X *and* Y *into* $X + Y$*.*

1. *The product of these two spaces in* $\underline{\mathbf{Qual}}^+$ *is*

   $(X, \mathcal{X}) \times^+ (Y, \mathcal{Y}) \overset{\text{def}}{=} (X \times Y, \mathcal{X} \underset{\sim}{\times}^+ \mathcal{Y})$ *with* $\mathcal{X} \underset{\sim}{\times}^+ \mathcal{Y} = ((\exists p_X)^* \mathcal{X}) \bigcap ((\exists p_Y)^* \mathcal{Y})$.
2. *The product of these two spaces in* $\underline{\mathbf{Qual}}^-$ *is*

   $(X, \mathcal{X}) \times^- (Y, \mathcal{Y}) \overset{\text{def}}{=} (X \times Y, \mathcal{X} \underset{\sim}{\times}^- \mathcal{Y})$ *with* $\mathcal{X} \underset{\sim}{\times}^- \mathcal{Y} = (\exists (p_X^*) \mathcal{X}) \bigcup (\exists (p_Y^*) \mathcal{Y})$.
3. *The sum of these two spaces in* $\underline{\mathbf{Qual}}^+$ *is*

   $(X, \mathcal{X}) +^+ (Y, \mathcal{Y}) \overset{\text{def}}{=} (X + Y, \mathcal{X} \underset{\sim}{+}^+ \mathcal{Y})$ *with* $\mathcal{X} \underset{\sim}{+}^+ \mathcal{Y} = (\exists (\exists i_X) \mathcal{X}) \bigcup (\exists (\exists i_Y) \mathcal{Y})$.
4. *The sum of these two spaces in* $\underline{\mathbf{Qual}}^-$ *is*

   $(X, \mathcal{X}) +^- (Y, \mathcal{Y}) \overset{\text{def}}{=} (X + Y, \mathcal{X} \underset{\sim}{+}^- \mathcal{Y})$ *with* $\mathcal{X} \underset{\sim}{+}^- \mathcal{Y} = (i_X^{**} \mathcal{X}) \bigcap (i_Y^{**} \mathcal{Y})$.

Proposition 5 $\underline{\mathbf{Qual}}^+$ *is cartesian closed, but* $\underline{\mathbf{Qual}}^-$*,* $\underline{\mathbf{Qual}}^{+\mathrm{op}}$*,* $\underline{\mathbf{Qual}}^{-\mathrm{op}}$ *are not.*

Indeed:

<u>*Case of* $\mathbf{Qual}^+$</u>: If $(Y, \mathcal{Y})$ and $(Z, \mathcal{Z})$ are two qualification spaces, let us set

$$(Z, \mathcal{Z})^{(Y, \mathcal{Y})} = (Z^Y, \mathcal{Z}^{\mathcal{Y}})$$

where Z^Y is the set of <u>*all*</u> mappings from Y to Z, and where $\mathcal{Z}^{\mathcal{Y}}$ is the qualification on Y^Z such that, $U \in \mathcal{Z}^{\mathcal{Y}}$, for a $U \subseteq Y^Z$, if and only if the *evaluation* map

$$\mathcal{E}_U : U \times Y \to Z : (u, y) \mapsto u(y)$$

is open from $(U, \{U\}) \times^+ (Y, \mathcal{Y}) \overset{\text{def}}{=} (U \times Y, \{U\} \underset{\sim}{\times}^+ \mathcal{Y})$ to $(Z, \mathcal{Z})$, that is to say, if and only if

$$(\exists (\exists \mathcal{E}_U))(\{U\} \underset{\sim}{\times}^+ \mathcal{Y}) \subseteq \mathcal{Z}.$$

This determines an endofunctor $(-)^{(Y, \mathcal{Y})}$ of the category $\underline{\mathbf{Qual}}^+$, right-adjoint to $(-) \times^+ (Y, \mathcal{Y})$; therefore, $\underline{\mathbf{Qual}}^+$ is cartesian closed.

<u>*Case of* $\mathbf{Qual}^-$</u>: This category is not cartesian closed ; indeed, the endofunctor $(-) \times^- (Y, \mathcal{Y})$ does not commute with finite sums, and thus cannot be a left-adjoint.

<u>*Case of* $\mathbf{Qual}^{-\mathrm{op}}$</u>: This category is not cartesian closed because we would then have, for $(Y, \mathcal{Y})$ and $(Z, \mathcal{Z})$, a qualification space $(T, \mathcal{T})$ such that

$$\underline{\mathbf{Qual}}^{-}((Z,\mathcal{Z}),[(\mathbf{1},\emptyset)+^{-}(Y,\mathcal{Y})]) \simeq \underline{\mathbf{Qual}}^{-}((T,\mathcal{T}),(\mathbf{1},\emptyset)).$$

But, the left-hand side is generally infinite, while the right-hand side has exactly one element.

<u>*Case of* $\mathbf{Qual}^{+\mathrm{op}}$</u>: The same argument as for $\underline{\mathbf{Qual}}^{-\mathrm{op}}$ applies here, with

$$\underline{\mathbf{Qual}}^{+}((Z,\mathcal{Z}),[(\mathbf{1},\{\emptyset,\mathbf{1}\})+^{+}(Y,\mathcal{Y})]) \simeq \underline{\mathbf{Qual}}^{+}((T,\mathcal{T}),(\mathbf{1},\{\emptyset,\mathbf{1}\})).$$

2 Stone's monadicity

Given $f : X \to Y$ (in $\underline{\mathbf{Ens}}$), we write $f^{\mathrm{op}} : Y \to X$ (in $\underline{\mathbf{Ens}}^{\mathrm{op}}$) for the corresponding morphism in $\underline{\mathbf{Ens}}^{\mathrm{op}}$, and we write $\underline{\mathbf{C}} : \underline{\mathbf{Ens}}^{\mathrm{op}} \to \underline{\mathbf{Ens}}$ for the *contravariant* "Power Set" functor defined as $\underline{\mathbf{C}}X = \mathcal{P}X$ and $\underline{\mathbf{C}}f^{\mathrm{op}} = f^{*}$. In other words, through the natural bijection $\mathcal{P}X \simeq \underline{\mathbf{Ens}}(X, \mathbf{2})$, where $\mathbf{2} = \{\emptyset, \mathbf{1}\}$, we have $\underline{\mathbf{C}} \simeq \underline{\mathbf{Ens}}(-, \mathbf{2})$. We write $\underline{\mathbf{C}}^{\mathrm{op}} : \underline{\mathbf{Ens}} \to \underline{\mathbf{Ens}}^{\mathrm{op}}$ the functor defined as $\underline{\mathbf{C}}^{\mathrm{op}}X = \mathcal{P}X$ and $\underline{\mathbf{C}}^{\mathrm{op}}f = f^{*\mathrm{op}}$ [1].

For any set X, let us set

$$\eta_X : X \to \mathcal{P}^2X : x \mapsto \{A \in \mathcal{P}X : x \in A\}.$$

Then $\eta : \underline{\mathbf{Id}_{\mathbf{Ens}}} \to \underline{\mathbf{C}}\underline{\mathbf{C}}^{\mathrm{op}}$ is a natural transformation. For any mapping $u : X \to \mathcal{P}Y$, let $r^d : Y \to \mathcal{P}X : y \mapsto \{x : y \in rx\}$. We then have $r = \underline{\mathbf{C}}((r^d)^{\mathrm{op}})\eta_X = (r^d)^{*}\eta_X$ and a natural bijection

$$\begin{array}{ccc} \underline{\mathbf{Ens}}(X,\underline{\mathbf{C}}Y) = \underline{\mathbf{Ens}}(X,\mathcal{P}Y) & \xrightarrow{\sim} & \underline{\mathbf{Ens}}(Y,\mathcal{P}X) = \underline{\mathbf{Ens}}^{\mathrm{op}}(\underline{\mathbf{C}}^{\mathrm{op}}X,Y) \\ r & \mapsto & r^d \end{array}$$

Thus we have the adjunction $\underline{\mathbf{C}}^{\mathrm{op}} \vdash \underline{\mathbf{C}}\ \ [\eta]$:

$$\begin{array}{c} \underline{\mathbf{Ens}}^{\mathrm{op}} \\ \underline{\mathbf{C}}^{\mathrm{op}}\uparrow \ \dashv\ \downarrow\underline{\mathbf{C}} \\ \underline{\mathbf{Ens}} \end{array}$$

Therefore, we recover a monad (triple) $\mathbf{\Pi} = (\underline{\Pi}, \eta, \mu)$ over $\underline{\mathbf{Ens}}$ where

$$\underline{\Pi}f = \underline{\mathbf{C}}\,\underline{\mathbf{C}}^{\mathrm{op}}f \qquad \eta_X = \eta_X \qquad \mu_X = (\eta_{\mathcal{P}X})^{*}$$

It is a classical result that the comparison functor from $\underline{\mathbf{Ens}}^{\mathrm{op}}$ to this category of algebras $\underline{\mathbf{Ens}}^{\mathbf{\Pi}}$ is an equivalence of categories; this is the so called *"Stone's*

[1] We shall enlighten the notation as follows: when no ambiguity may arise, we write $\underline{\mathbf{C}}f = f^{*}$ instead of $\underline{\mathbf{C}}f^{\mathrm{op}} = f^{*}$, or $\underline{\mathbf{C}}^{\mathrm{op}}f = f^{*\mathrm{op}}$.

monadicity" (*see* [7] and [8] for the source of these ideas). We also say that *the functor* $\underline{\mathbf{C}}$ *is algebraic (or tripleable) up to within an equivalence over* $\underline{\mathbf{Ens}}$.

Since we already have the adjunction $\underline{\mathbf{C}}^{\mathrm{op}} \vdash \underline{\mathbf{C}}\ \ [\eta]$, it follows from BECK's criteria, that the algebraicity (up to within an equivalence) is always equivalent (*see* [6], page 151, exercice 6) to the β-condition that we express here for $\underline{\mathbf{C}}$.

<u>The β-condition</u> *: If* $X \underset{g^{\mathrm{op}}}{\overset{f^{\mathrm{op}}}{\rightrightarrows}} Y$ *in* $\underline{\mathbf{Ens}}^{\mathrm{op}}$ *is a* $\underline{\mathbf{C}}$*-splittable pair, that is a pair such that there exists in* $\underline{\mathbf{Ens}}$ *a split fork, i.e. a diagram*

$$\underline{\mathbf{C}}X \underset{\underline{\mathbf{C}}\,g^{\mathrm{op}}}{\overset{\underline{\mathbf{C}}\,f^{\mathrm{op}}}{\rightrightarrows}} \underline{\mathbf{C}}Y \underset{s}{\overset{e}{\rightleftarrows}} Z \qquad \underline{\mathbf{C}}X \xleftarrow{\;t\;} \underline{\mathbf{C}}Y$$

where

$$\begin{array}{ll} 1) & e \cdot \underline{\mathbf{C}}\,f^{\mathrm{op}} = e \cdot \underline{\mathbf{C}}\,g^{\mathrm{op}} \\ 2) & e \cdot s = \mathbf{1}_Z \qquad \underline{\mathbf{C}}\,f^{\mathrm{op}} \cdot t = \mathbf{1}_{\mathcal{P}Y} \qquad \underline{\mathbf{C}}\,g^{\mathrm{op}} \cdot t = s \cdot e, \end{array}$$

then $X \underset{g^{\mathrm{op}}}{\overset{f^{\mathrm{op}}}{\rightrightarrows}} Y$ *has a coequalizer in* $\underline{\mathbf{Ens}}^{\mathrm{op}}$*, and* $\underline{\mathbf{C}}$ *preserves and reflects the coequalizers of such pairs, which means here that if* $Y \xrightarrow{u^{\mathrm{op}}} Q$ *satisfies* $u^{\mathrm{op}} \cdot f^{\mathrm{op}} = u^{\mathrm{op}} \cdot g^{\mathrm{op}}$*, then* u^{op} *is a coequalizer of* $X \underset{g^{\mathrm{op}}}{\overset{f^{\mathrm{op}}}{\rightrightarrows}} Y$ *in* $\underline{\mathbf{Ens}}^{\mathrm{op}}$ *if and only if* $\underline{\mathbf{C}}\,u^{\mathrm{op}}$ *is a coequalizer of* $\underline{\mathbf{C}}\,X \underset{\underline{\mathbf{C}}\,g^{\mathrm{op}}}{\overset{\underline{\mathbf{C}}\,f^{\mathrm{op}}}{\rightrightarrows}} \underline{\mathbf{C}}\,Y$ *in* $\underline{\mathbf{Ens}}$.

3 Final liftings of Stone's monadicity

A functor $\underline{\mathbf{U}} : \underline{\mathbf{C}} \to \underline{\mathbf{D}}$ is said to be *cofibring* or a *final-arrow functor in* $\underline{\mathbf{D}}$, if, for each object C of $\underline{\mathbf{C}}$ and each arrow $f : \underline{\mathbf{U}}C \to D$ in $\underline{\mathbf{D}}$, there exists a unique $\overline{f} : C \to \overline{D}$ in $\underline{\mathbf{C}}$, such that (1) $\underline{\mathbf{U}}\overline{f} = f$ and (2) for all $g : C \to E$ and all $h : D \to \underline{\mathbf{U}}E$ with $h \cdot f = \underline{\mathbf{U}}g$, there exists a unique $\tilde{h}$ with $\tilde{h} \cdot \overline{f} = g$ and $U\tilde{h} = h$. The arrow $\overline{f}$ is called *cocartesian arrow* or *final arrow over* f. Then if $\underline{\mathbf{D}}$ has coequalizers, $\underline{\mathbf{C}}$ has also coequalizers: if $B \underset{v}{\overset{u}{\rightrightarrows}} C$ is given in $\underline{\mathbf{C}}$, its coequalizer is $\overline{f}$ where f is a coequalizer (in $\underline{\mathbf{D}}$) of $\underline{\mathbf{U}}B \underset{\underline{\mathbf{U}}v}{\overset{\underline{\mathbf{U}}u}{\rightrightarrows}} \underline{\mathbf{U}}C$.

Proposition 6 *Let us have the following commutative diagram of functors :*

$$\begin{array}{ccc} \underline{A} & \xrightarrow{\underline{S}_A} & \underline{A}' \\ {\scriptstyle \underline{K}}\downarrow & & \downarrow{\scriptstyle \underline{K}'} \\ \underline{X} & \xrightarrow[\underline{S}_X]{} & \underline{X}' \end{array} \quad : \quad \underline{\mathbf{K}}' \cdot \underline{\mathbf{S}}_A = \underline{\mathbf{S}}_X \cdot \underline{\mathbf{K}}$$

where $\underline{\mathbf{K}}'$ *is a functor satisfying the* β*-condition.*

Let us suppose that $\underline{\mathbf{S}}_A$ *and* $\underline{\mathbf{S}}_X$ *are final-arrow functors and that, for any arrow* $f : C \to D$ *in* $\underline{A}$, f *is a final epimorphism if and only if* $\underline{\mathbf{K}}f : \underline{\mathbf{K}}C \to \underline{\mathbf{K}}D$ *in* $\underline{X}$ *is a final epimorphism.*

Then the functor $\underline{\mathbf{K}}$ *satisfies the* β*-condition. Moreover, if* $\underline{\mathbf{K}}$ *has also a left-adjoint,* $\underline{\mathbf{K}}$ *is algebraic up to an equivalence. This is true in particular if* $\underline{\mathbf{K}}'$ *is the contravariant* Power Set *functor* $\underline{\mathbf{C}}$, *of which we already know* (*see* section 2) *that it fulfills the* β*-condition.*

Indeed, if $X \overset{p}{\underset{q}{\rightrightarrows}} Y$ is a $\underline{\mathbf{K}}$-splittable pair, that is a pair for which there is a split fork

$$\underline{\mathbf{K}}X \overset{\underline{\mathbf{K}}p}{\underset{\underline{\mathbf{K}}q}{\rightrightarrows}} \underline{\mathbf{K}}Y \overset{e}{\underset{s}{\rightleftarrows}} Z, \qquad t : \underline{\mathbf{K}}Y \to \underline{\mathbf{K}}X$$

then $\underline{\mathbf{S}}_A X \overset{\underline{S}_A p}{\underset{\underline{S}_A q}{\rightrightarrows}} \underline{\mathbf{S}}_A Y$ is a $\underline{\mathbf{K}}'$-splittable pair, which therefore admits a coequalizer $\underline{\mathbf{S}}_A Y \overset{a}{\to} Q$ and the corresponding final arrow $\overline{a}$ is a coequalizer of $X \overset{p}{\underset{q}{\rightrightarrows}} Y$. Then, given $X \overset{p}{\underset{q}{\rightrightarrows}} Y$ $\underline{\mathbf{K}}$-scindable and $u : Y \to K$, u is a coequalizer if and only if u is final and $\underline{\mathbf{S}}_A u$ is a coequalizer of $\underline{\mathbf{S}}_A X \overset{\underline{S}_A p}{\underset{\underline{S}_A q}{\rightrightarrows}} \underline{\mathbf{S}}_A Y$, and then u is an epimorphism, which is equivalent to u being final and $\underline{\mathbf{K}}'\underline{\mathbf{S}}_A u = \underline{\mathbf{S}}_X \underline{\mathbf{K}}u$ being a coequalizer of $\underline{\mathbf{S}}_X \underline{\mathbf{K}}X \overset{\underline{S}_X \underline{\mathbf{K}}p}{\underset{\underline{S}_X \underline{\mathbf{K}}q}{\rightrightarrows}} \underline{\mathbf{S}}_X \underline{\mathbf{K}}Y$, which in turn is equivalent to $\underline{\mathbf{K}}$ being a final epimorphism and $\underline{\mathbf{S}}_X \underline{\mathbf{K}}u$ being a coequalizer of

$$\underline{\mathbf{S}}_X \underline{\mathbf{K}}X \overset{\underline{S}_X \underline{\mathbf{K}}p}{\underset{\underline{S}_X \underline{\mathbf{K}}q}{\rightrightarrows}} \underline{\mathbf{S}}_X \underline{\mathbf{K}}Y ,$$

that is to say $\underline{\mathbf{K}}u$ a coequalizer of $\underline{\mathbf{K}}X \overset{\underline{\mathbf{K}}p}{\underset{\underline{\mathbf{K}}q}{\rightrightarrows}} \underline{\mathbf{K}}Y$.

4 Functors lifting $\underline{\mathbf{C}}$

In [4] and [2], the importance in Set Theory (considered from the point of view of *Algebraic Universes* (*see* [4] for references)) is underlined by what we may call the ***pulsative structure*** of the power set construction. This structure is given through the data, for each set X, of the monotonous functions

$$\begin{array}{rcccl} \pi_X & : & \mathcal{P}X & \longrightarrow & \mathcal{P}^2X \\ & & A & \longmapsto & \{B \in \mathcal{P}X;\ \forall x(x \in B \Rightarrow x \in A)\} \\ \psi_X & : & \mathcal{P}X & \longrightarrow & \mathcal{P}^2X \\ & & A & \longmapsto & \{B \in \mathcal{P}X;\ \exists x(x \in B\ \wedge\ x \in A)\} \\ \upsilon_X & : & \mathcal{P}^2X & \longrightarrow & \mathcal{P}X \\ & & \mathcal{A} & \longmapsto & \{x \in X\ ;\ \exists A(x \in A\ \wedge\ A \in \mathcal{A})\} \\ \delta_X & : & \mathcal{P}^2X & \longrightarrow & \mathcal{P}X \\ & & \mathcal{A} & \longmapsto & \{x \in X\ ;\ \forall A(x \in A \Rightarrow A \in \mathcal{A})\} \end{array}$$

together with the data of the functors $\exists$ and $(\ )^*$ (*i.e.* $\underline{\mathbf{C}}$) given in section 1 and section 2. We have the adjunctions :

$$\delta_X \dashv \psi_X \qquad \text{and} \qquad \upsilon_X \dashv \pi_X$$

with the inclusion

$$\delta_X \subset \upsilon_X.$$

Moreover, through the bijections

$$\nu_X : \mathcal{P}X \longrightarrow \mathcal{P}X : A \longmapsto \{x \in X : \neg(x \in A)\},$$

these adjunctions are seen to be "conjugate", *i.e.* $\pi_X = \nu_{CX} \cdot \psi_X \cdot \nu_X$ and $\upsilon_X = \nu_X \cdot \delta_X \cdot \nu_{CX}$.

For a qualification space $(X, \mathcal{X})$, seen as an object in $\underline{\mathbf{Qual}}^{+\mathrm{op}}$, let

$$\widehat{\underline{\mathbf{C}}}(X, \mathcal{X}) = (\underline{\mathbf{C}}X, \widehat{\mathcal{X}}),$$

where

$$\widehat{\mathcal{X}} = \{A^{\psi}\ ;\ A \in \mathcal{X}\} \qquad \text{(cap)}$$

with $A^{\psi} = \{B \in \mathcal{P}X\ ;\ \exists x(x \in A \wedge x \in B)\}$, *i.e.* with A^{ψ} the set of all subsets B of X meeting A. So :

$$\widehat{\mathcal{X}} = (\exists\psi_X)(\mathcal{X})$$

Proposition 7 *The mapping $f : X \to Y$ is open from $(X, \mathcal{X})$ to $(Y, \mathcal{Y})$ if and only if $f^* : \underline{\mathbf{C}}Y \to \underline{\mathbf{C}}X$ is continuous from $(\underline{\mathbf{C}}Y, \widehat{\mathcal{Y}})$ to $(\underline{\mathbf{C}}X, \widehat{\mathcal{X}})$. In* *particular,* $\widehat{\underline{\mathbf{C}}}$ *determines a functor, that we will also write* $\widehat{\underline{\mathbf{C}}}$*, making the following diagram commutative*

$$\begin{array}{ccc} \underline{\mathbf{Qual}}^{+\mathrm{op}} & \xrightarrow{\underline{\mathbf{S}}^{+\mathrm{op}}} & \underline{\mathbf{Ens}}^{\mathrm{op}} \\ {\scriptstyle \widehat{\underline{\mathbf{C}}}}\downarrow & & \downarrow{\scriptstyle \underline{\mathbf{C}}} \\ \underline{\mathbf{Qual}}^{-} & \xrightarrow[\underline{\mathbf{S}}^{-}]{} & \underline{\mathbf{Ens}} \end{array}$$

Indeed, f^* is continuous if and only if $\widehat{\mathcal{X}} \subseteq f^{***}\widehat{\mathcal{Y}}$, which expands into the condition

$$\{\{B' \;:\; f^*B' \in \mathcal{X}'\} \;:\; \mathcal{X}' \in \widehat{\mathcal{X}}\} \subseteq \widehat{\mathcal{Y}},$$

that is

$$\{\{B' \;:\; f^*B' \in A^{\psi}\} \;:\; A \in \mathcal{X}\} \subseteq \{B^{\psi} \;:\; B \in \mathcal{Y}\};$$

therefore, for all $A \in \mathcal{X}$, there exists a $B_A \in \mathcal{Y}$ such that

$$\{B' \;:\; f^*B' \in A^{\psi}\} = B_A^{\psi},$$

which means that for any B' we have the equivalence:

$$f^*B' \bigcap A \neq \emptyset \iff B' \bigcap B_A \neq \emptyset.$$

But $f^*B' \bigcap A \neq \emptyset$ is equivalent to $B' \bigcap fA \neq \emptyset$; this equivalence means therefore that $B_A = f(A)$. Hence, the continuity of f^* amounts to $f(A) \in \mathcal{Y}$ for all $A \in \mathcal{X}$, which is the very expression that f is open.

For each $(X, \mathcal{X})$, object in $\underline{\mathbf{Qual}}^-$, we set $\breve{\underline{\mathbf{C}}}^{\mathrm{op}}(X, \mathcal{X}) = (\underline{\mathbf{C}}^{\mathrm{op}}X, \breve{\mathcal{X}})$, where

$$\breve{\mathcal{X}} = \{\mathcal{X}' \;:\; \bigcup \mathcal{X}' \in \mathcal{X}\}. \qquad \text{(cup)}$$

Therefore we have :

$$\breve{\mathcal{X}} = (\underline{\mathbf{C}}v_X)(\mathcal{X}).$$

Proposition 8

1) For each qualification space $(X, \mathcal{X})$*, the mapping* $\eta_X : X \to \underline{\mathbf{C}}\,\underline{\mathbf{C}}^{\mathrm{op}}X$ *is continuous from* $(X, \mathcal{X})$ *to* $\widehat{\underline{\mathbf{C}}}\,\breve{\underline{\mathbf{C}}}^{\mathrm{op}}(X, \mathcal{X}) = (\underline{\mathbf{C}}\,\underline{\mathbf{C}}^{\mathrm{op}}X, \widehat{\breve{\mathcal{X}}})$.

2) For any mapping $r : X \to \underline{\mathbf{C}}Y$*,* r *is continuous from* $(X, \mathcal{X})$ *to* $\widehat{\underline{\mathbf{C}}}(Y, \mathcal{Y})$ *if and only if the mapping* $r^d : Y \to \underline{\mathbf{C}}^{\mathrm{op}}X$ *is open from* $(Y, \mathcal{Y})$ *to* $\breve{\underline{\mathbf{C}}}^{\mathrm{op}}(X, \mathcal{X})$.

3) It follows that $\breve{\underline{\mathbf{C}}}^{\mathrm{op}}$ *defines a left-adjoint functor to* $\widehat{\underline{\mathbf{C}}}$*, and that this adjunction* $\breve{\underline{\mathbf{C}}}^{\mathrm{op}} \dashv \widehat{\underline{\mathbf{C}}}\ [\eta]$ *is a lifting of the adjunction* $\underline{\mathbf{C}}^{\mathrm{op}} \dashv \underline{\mathbf{C}}\ [\eta]$ *through* $\underline{\mathbf{S}}^{+\mathrm{op}}$ *and* $\underline{\mathbf{S}}^-$*:*

$$\begin{array}{ccc} \underline{\mathbf{Qual}}^{+\mathrm{op}} & \xrightarrow{\underline{\mathbf{S}}^{+\mathrm{op}}} & \underline{\mathbf{Ens}}^{\mathrm{op}} \\ {\scriptstyle \breve{\underline{\mathbf{C}}}^{\mathrm{op}}}\uparrow \dashv \downarrow{\scriptstyle \widehat{\underline{\mathbf{C}}}} & & {\scriptstyle \underline{\mathbf{C}}^{\mathrm{op}}}\uparrow \dashv \downarrow{\scriptstyle \underline{\mathbf{C}}} \\ \underline{\mathbf{Qual}}^{-} & \xrightarrow[\underline{\mathbf{S}}^{-}]{} & \underline{\mathbf{Ens}} \end{array}$$

Indeed, for all $\mathcal{X}' \in \mathcal{P}^2 X$, we have

$$\begin{aligned} \bigcup \mathcal{X}' &= \{x \in X \ ; \ \exists A (A \in \mathcal{X}' \wedge x \in A)\} \\ &= \{x \in X \ ; \ \{A \ ; \ x \in A\} \in \mathcal{X}'^{\psi}\} \\ &= \eta_X^*(\mathcal{X}'^{\psi}), \end{aligned}$$

and hence, for all $\mathcal{X}'$ such that $\bigcup \mathcal{X}' \in \mathcal{X}$, we have $\eta_X^*(\mathcal{X}'^{\psi}) \in \mathcal{X}$, which means that for all $\mathcal{X}'^{\psi} \in \overset{\frown}{\mathcal{X}}$, we have $\eta_X^*(\mathcal{X}'^{\psi}) \in \mathcal{X}$, *i.e.* the continuity of η_X.

Then "r is continuous" means that for all $B \in \mathcal{Y}$, $r^*(B^{\psi}) \in \mathcal{X}$, that is to say $\{x \in \mathcal{X} \ ; \ rx \bigcap B \neq \emptyset\} \in \mathcal{X}$, or else that

$$\{x \in X \ ; \ \exists y \, (y \in B \wedge y \in rx)\} = \bigcup_{y \in B} \{x \ ; \ y \in rx\} \in \mathcal{X},$$

which means that $\{\{x \in X \ ; \ y \in rx\} \ ; \ y \in B\} \in \breve{\mathcal{X}}$, or $\{r^d y \ ; \ y \in B\} \in \breve{\mathcal{X}}$, *i.e.* $r^d B \in \breve{\mathcal{X}}$; this is the very expression that r^d is open.

Proposition 9 *Let* $(X, \mathcal{X}) \xrightarrow{f} (Y, \mathcal{Y})$ *be in* $\underline{\mathbf{Qual}}^+$, *and* $(\underline{\mathbf{C}}Y, \widehat{\mathcal{Y}}) \xrightarrow{f^*} (\underline{\mathbf{C}}X, \widehat{\mathcal{X}})$ *be the image of* f^{op} *through* $\widehat{\underline{\mathbf{C}}}$ *in* $\underline{\mathbf{Qual}}^-$. *Then* f *is an initial injection if and only if* f^* *is a final surjection.*

Indeed, f is initial if and only if

$$\forall A, A \in \mathcal{P}X \ : \ (\exists f A \in \mathcal{Y} \Longrightarrow A \in \mathcal{X}); \tag{1}$$

on the other hand, "f^* *is final*" may be written

$$\forall \mathcal{A}, \mathcal{A} \in \mathcal{P}^2 X \ : \ (f^{**}\mathcal{A} \in \widehat{\mathcal{Y}} \Longrightarrow \mathcal{A} \in \widehat{\mathcal{X}}),$$

that is:

$$\begin{aligned} \forall \mathcal{A} \in \mathcal{P}^2 X \Bigg[\Big(\exists B \in \mathcal{Y}, \ \forall B' \in \mathcal{P}Y \ \ ([f^* B' \in \mathcal{A}] \Longleftrightarrow [B' \textstyle\bigcap B \neq \emptyset]) \Big) \\ \Longrightarrow \Big(\exists A \in \mathcal{X}, \ \forall A' \in \mathcal{P}X \ \ ([A' \in \mathcal{A}] \Longleftrightarrow [A' \textstyle\bigcap A \neq \emptyset]) \Big) \Bigg]. \end{aligned} \tag{2}$$

Let us suppose first that f is an initial injection. To check (2), *i.e.* to produce an A from a given B, we first observe that for any B' we have $f^* B' = f^*(B' \bigcap \mathrm{Im}\, f)$, so that B is such that for any B' the condition $B' \bigcap B \neq \emptyset$ is equivalent to $(B' \bigcap \mathrm{Im}\, f) \bigcap B \neq \emptyset$, which implies that $B \subset \mathrm{Im}\, f$.

Let us set $A = f^* B$; since f is injective and $B \subset \mathrm{Im}\, f$, $(\exists f)A = (\exists f) f^* B = B \in \mathcal{Y}$, and therefore, f being initial, it follows that $A \in \mathcal{X}$. Then, for any $A' \in \mathcal{P}X$, $A' \bigcap A \neq \emptyset$ is equivalent to $((\exists f)A') \bigcap B \neq \emptyset$ (since f is injective), which, from the hypothesis, is equivalent to $f^* \exists f A' \in \mathcal{A}$; and since $f^* \exists f A' = A'$ (once more because f is injective), this is finally equivalent to $A' \in \mathcal{A}$. Moreover, f^* is surjective because f is injective.

Let us suppose next that f^* is a final surjection. Then, since f^* is surjective, f is injective. To check (1), we apply (2) to

$$\mathcal{A} = \{f^*B' \ ; \ B' \bigcap \exists f A \neq \emptyset\}.$$

From (2), there is an $A_1 \in \mathcal{X}$ such that for all $A' \in \mathcal{X}$, $A' \bigcap A_1 \neq \emptyset$ is equivalent to the existence of a $B' \in \mathcal{P}Y$ with $B' \bigcap \exists f A \neq \emptyset$ and $A' = f^*B'$. But $B' \bigcap fA \neq \emptyset$ is equivalent to $f^*B' \bigcap A \neq \emptyset$, *i.e.* $A' \bigcap A \neq \emptyset$. Since f is injective, for all A' we have $A' = f^*\exists f A'$, and thus $A' \bigcap A_1 \neq \emptyset$ is finally equivalent to $A' \bigcap A \neq \emptyset$, which implies $A = A_1 \in \mathcal{X}$.

We know from Proposition 2 that $\underline{\mathbf{S}}^{\pm}$ are final and initial structure functors. In particular, $\underline{\mathbf{S}}^{+\mathrm{op}}$ is a final structure functor, and $\underline{\mathbf{S}}^{-}$ and $\underline{\mathbf{S}}^{+\mathrm{op}}$ are initial arrow functors (*see* section 3). We also know (*see* Section 2) that $\underline{\mathbf{C}}$ satisfies the β-condition, and (*see* Proposition 6 and 7) that we have a functor $\widehat{\underline{\mathbf{C}}}$ such that $\underline{\mathbf{C}}\,\underline{\mathbf{S}}^{+\mathrm{op}} = \underline{\mathbf{S}}^{-}\,\widehat{\underline{\mathbf{C}}}$.

Finally, Proposition 9 tells us that $\widehat{\underline{\mathbf{C}}}$ satisfies the given condition for $\underline{\mathbf{K}}$ in proposition 6; this proposition 6 tells us therefore that $\widehat{\underline{\mathbf{C}}}$ satisfies the β-condition. Since moreover (*see* Proposition 8) $\widehat{\underline{\mathbf{C}}}$ has a left-adjoint $\breve{\underline{\mathbf{C}}}$, it follows from BECK's criteria mentionned in section 2 that:

Proposition 10 *The functor* $\widehat{\underline{\mathbf{C}}} : \underline{\mathbf{Qual}}^{+\mathrm{op}} \longrightarrow \underline{\mathbf{Qual}}^{-}$ *is algebraic up to an equivalence.*

Proposition 11 *Let* $(X, \mathcal{X}) \to (Y, \mathcal{Y})$ *be in* $\underline{\mathbf{Qual}}^{-}$ *and let*

$$(\underline{C}Y, \breve{\mathcal{Y}}) \xrightarrow{f^*} (\underline{C}X, \breve{\mathcal{X}})$$

be the image of f^{op} *through* $(\breve{\underline{\mathbf{C}}}^{\mathrm{op}})^{\mathrm{op}}$ *in* $\underline{\mathbf{Qual}}^{+}$. *Then f is an initial injection if and only if f^* is a final surjection.*

Indeed f is initial if and only if

$$\forall A \in \mathcal{P}X \ \ (A \in \mathcal{X} \Rightarrow \exists B \in \mathcal{Y} \ (A = f^*B)), \tag{3}$$

and f^* is final may be written:

$$\forall \mathcal{A} \in \mathcal{P}^2X(\mathcal{A} \in \breve{\mathcal{X}} \Rightarrow \exists \mathcal{Y}' \in \breve{\mathcal{Y}} \ \mathcal{A} = \exists(f^*)\mathcal{Y}'),$$

that is:

$$\forall \mathcal{A} \in \mathcal{P}^2X \left[\left(\bigcup \mathcal{A} \in \mathcal{X}\right) \Longrightarrow \left(\exists \mathcal{Y}'[(\bigcup \mathcal{Y}' \in \mathcal{Y}) \wedge (\mathcal{A} = \{f^*B' \ : \ B' \in \mathcal{Y}'\})]\right)\right]. \tag{4}$$

Let us first suppose that f is a final injective arrow. To check (4) above, let $\mathcal{A}$ be such that $\bigcup\mathcal{A} = A \in \mathcal{X}$. Then, from (3), $A = f^*B$ for a certain $B \in \mathcal{Y}$. For each $A' \in \mathcal{A}$, let $B' = \exists f A' \subset \exists f A \subset B$, and let

$$\mathcal{Y}' = \{\exists f A' = B' \ ; \ A' \in \mathcal{A}\} \bigcup \{B\}.$$

We have $\bigcup \mathcal{Y}' = B$, and since f is injective, so that $f^*\exists f A' = A'$, we have $\mathcal{A} = \{f^*B' \ ; \ B' \in \mathcal{Y}'\}$. Finally, the injectivity of f implies the surjectivity of f^*.

Let us now suppose that f^* is a final surjective arrow. Then, since f^* is surjective, f is injective. To check (1), let then $A \in \mathcal{X}$ and let us set

$$\mathcal{A} = \{A' \in \underline{C}X \ ; \ A' \subset A\}.$$

Then $\bigcup \mathcal{A} = A \in \mathcal{X}$, and therefore, from (4), we have a $\mathcal{Y}'$ such that $\bigcup \mathcal{Y}' = B \in \mathcal{Y}$ and $\mathcal{A} = \{f^*B' \quad B' \in \mathcal{Y}'\}$. We have

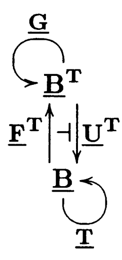
$$\begin{aligned} A &= \bigcup\{A' \in \underline{C}X \ ; \ A' \subset A\} = \bigcup\{f^*B' \ ; \ B' \in \mathcal{Y}'\} \\ &= \quad f^*(\bigcup\{B' \in \underline{C} \ ; \ B' \in \mathcal{Y}'\}) = f^*(\bigcup \mathcal{Y}') = f^*B. \end{aligned}$$

Using the same arguments as in proposition 10 we obtain:

Proposition 12 *The functor* $(\underline{\breve{C}}^{\mathrm{op}})^{\mathrm{op}} = \underline{\breve{C}} : \underline{\mathbf{Qual}}^{-\mathrm{op}} \longrightarrow \underline{\mathbf{Qual}}^{+}$ *is algebraic up to an equivalence.*

5 Duality of $(\underline{\mathbf{Qual}}^{-}, \widehat{\mathbf{\Pi}})$ and $(\underline{\mathbf{Qual}}^{+}, \breve{\mathbf{\Pi}})$

Definition *Let* $\mathbf{T} = (\underline{\mathbf{T}}, \eta, \mu)$ *be a monad* <u>over</u> *a category* $\underline{\mathbf{B}}$ *(to be short, we say that the pair* $(\underline{\mathbf{B}}, \mathbf{T})$ *is a monad). Let* $\underline{\mathbf{B}}^{\mathbf{T}}$ *be the category of algebras over* $\mathbf{T}$ *and* $\underline{\mathbf{F}}^{\mathbf{T}} \dashv \underline{\mathbf{U}}^{\mathbf{T}}$ *be the universal pair of adjoints such that* $\underline{\mathbf{T}} = \underline{\mathbf{U}}^{\mathbf{T}} \cdot \underline{\mathbf{F}}^{\mathbf{T}}$. *Let* $\mathbf{G} = (\underline{\mathbf{G}}, \epsilon, \delta)$ *be the comonad over* $\underline{\mathbf{B}}^{\mathbf{T}}$ *associated with* $\underline{\mathbf{G}} = \underline{\mathbf{F}}^{\mathbf{T}} \cdot \underline{\mathbf{U}}^{\mathbf{T}}$.



This comonad over $\underline{\mathbf{B}}^{\mathbf{T}}$ *defines a monad over* $(\underline{\mathbf{B}}^{\mathbf{T}})^{\mathrm{op}}$, *which we write* $\mathbf{G}^{\mathrm{op}} = (\underline{\mathbf{G}}^{\mathrm{op}}, \epsilon^{\mathrm{op}}, \delta^{\mathrm{op}})$. *We set*

$$(\underline{\mathbf{B}}, \mathbf{T})^{\star} = ((\underline{\mathbf{B}}^{\mathbf{T}})^{\mathrm{op}}, \mathbf{G}^{\mathrm{op}}).$$

We shall say that the monad $(\underline{\mathbf{B}}, \mathbf{T})^{\star}$ *is the dual of* $(\underline{\mathbf{B}}, \mathbf{T})$ *and that it is* reflexive *when canonically* $(\underline{\mathbf{B}}, \mathbf{T})^{\star\star} \simeq (\underline{\mathbf{B}}, \mathbf{T})$. (The terminology "reflexive" is by analogy with duality in linear algebra).

Of course, *reflexive* means that $\underline{\mathbf{F}}^{\mathbf{T}}$ is comonadic. In [1], the cases where $\underline{\mathbf{B}}$ is the category of sets, the category of pointed sets or the category of vector spaces over a fixed commutative field K are explicitely described. However, the emphasis here is on the operator $(-)^\star$.

In particular, $(\underline{\mathbf{B}}, \underline{\mathbf{Id}})^\star = (\underline{\mathbf{B}}^{\mathrm{op}}, \underline{\mathbf{Id}})$, so that the notion of duality $(-)^\star$ for monads extends the notion of duality $(-)^{\mathrm{op}}$ for categories, and, since $\underline{\mathbf{B}}^{\mathrm{op\,op}} = \underline{\mathbf{B}}$, categories do appear like (trivial) reflexive monads. In general, a duality theorem $\underline{\mathbf{B}}^{\mathrm{op}} \simeq \underline{\mathbf{C}}$ constructs the dual of a concrete category $\underline{\mathbf{B}}$ as a concrete category $\underline{\mathbf{C}}$. Here, if $\underline{\mathbf{B}}^{\mathbf{T}} \simeq \underline{\mathbf{C}}$, we have a "realization" $\underline{\mathbf{F}}^{\mathbf{T}} : \underline{\mathbf{B}}^{\mathrm{op}} \to \underline{\mathbf{C}}$ of the dual of a concrete category $\underline{\mathbf{B}}$ with a concrete category $\underline{\mathbf{C}}$.

This paper gives an example of a significant and non-trivial reflexive monad, which presents as dual (in the sense of monad duality defined here) the "algebras of the calculi" of inverse images and of direct images.

Using again notation and results of Propositions 10 and 12 of section 4, and extending notations of section 2, we set

$$\underline{\widehat{\Pi}}(X, \mathcal{X}) = (\underline{\mathbf{C}}^2 X, \overset{\frown}{\widehat{\mathcal{X}}}) \qquad \underline{\widehat{\Pi}} f = \underline{\widehat{\mathbf{C}}}\,\underline{\breve{\mathbf{C}}}^{\mathrm{op}} f$$

$$\underline{\breve{\Pi}}(X, \mathcal{X}) = (\underline{\mathbf{C}}^2 X, \overset{\smile}{\breve{\mathcal{X}}}) \qquad \underline{\breve{\Pi}} g = \underline{\breve{\mathbf{C}}}\,\underline{\widehat{\mathbf{C}}}^{\mathrm{op}} g$$

for f a morphism in $\underline{\mathbf{Qual}}^-$ and g a morphism in $\underline{\mathbf{Qual}}^+$. Then, $\underline{\widehat{\Pi}}$ and $\underline{\breve{\Pi}}$ are the endofunctors of two monads over $\underline{\mathbf{Qual}}^-$ and $\underline{\mathbf{Qual}}^+$, which we write $\widehat{\mathbf{\Pi}}$ and $\breve{\mathbf{\Pi}}$.

Explicitely, after (cap) and (cup) (*see* pages 7, 8), we have (1) and (2) below :

$$\left.\begin{aligned} \overset{\frown}{\widehat{\mathcal{X}}} &= \{\mathcal{R}^{\psi}\ ;\ \mathcal{R} \in \breve{\mathcal{X}}\} = \{\{\mathcal{X}\ ;\ \mathcal{X} \cap \mathcal{R} \neq \emptyset\}\ ;\ \cup\mathcal{R} \in \mathcal{X}\} \\ &= \bigcup_{A\in\mathcal{X}} \{\{\mathcal{X}\ ;\ \mathcal{X} \cap \mathcal{R} \neq \emptyset\}\ ;\ \cup\mathcal{R} = A\}. \end{aligned}\right\} \tag{1}$$

i.e. each element $\mathbb{X} \in \overset{\frown}{\widehat{\mathcal{X}}}$ is obtained as follows : we choose a quality $A \in \mathcal{X}$ and a covering $\mathcal{R}$ of A, and $\mathbb{X}$ is made of all qualifications $\mathcal{X}$ on X containing an element of $\mathcal{R}$;

$$\left.\begin{aligned} \overset{\smile}{\breve{\mathcal{X}}} &= \{\mathbb{X}\ ;\ \cup\mathbb{X} \in \widehat{\mathcal{X}}\} = \{\mathbb{X}\ ;\ \exists A \in \mathcal{X}\ ;\ \cup\mathbb{X} = A^{\psi}\} \\ &= \bigcup_{A\in\mathcal{X}} \{\mathbb{X}\ ;\ \cup\mathbb{X} = A^{\psi}\} \end{aligned}\right\} \tag{2}$$

i.e. each element $\mathbb{X} \in \overset{\smile}{\breve{\mathcal{X}}}$ is obtained as follows : we choose a quality $A \in \mathcal{X}$; an element $\mathbb{X} \in \overset{\smile}{\breve{\mathcal{X}}}$ is a covering of the set of all subsets meeting A.

Proposition 13 *We have the following dualities :*

$$(\underline{\mathbf{Qual}}^-, \widehat{\mathbf{\Pi}})^\star \simeq (\underline{\mathbf{Qual}}^+, \breve{\mathbf{\Pi}}) \qquad (\underline{\mathbf{Qual}}^+, \breve{\mathbf{\Pi}})^\star \simeq (\underline{\mathbf{Qual}}^-, \widehat{\mathbf{\Pi}})$$

and therefore, $(\underline{\mathbf{Qual}}^-, \widehat{\mathbf{\Pi}})$ *and* $(\underline{\mathbf{Qual}}^+, \breve{\mathbf{\Pi}})$ *are reflexive.*

Thus, with this proposition, the *pulsative structure* of set theory, that is the conjugated adjunctions $\psi_X \dashv \delta_X$ and $\upsilon_X \dashv \pi_X$, the inclusion $\delta_X \subset \upsilon_X$, and the adjunction $\underline{\mathbf{C}}^{\mathrm{op}} \dashv \underline{\mathbf{C}}$, are encapsulated within a unique principle of duality.

6 The algebraicity of topogeneses over $\underline{\mathbf{Qual}}^+$

Let us call *topogenesis* (*see* [3]) a full coreflexive subcategory $\underline{\mathbf{Top}_\Gamma}$ of $\underline{\mathbf{Qual}}^-$ such that the inclusion $\underline{\mathbf{I}} : \underline{\mathbf{Top}_\Gamma} \hookrightarrow \underline{\mathbf{Qual}}^-$ and its right-adjoint $\underline{\mathbf{\Gamma}}$ (that we write as a subscript in $\underline{\mathbf{Top}_\Gamma}$) commute with the forgetful functors $\underline{\mathbf{U}}$ and $\underline{\mathbf{S}}^-$ to $\underline{\mathbf{Ens}}$, and such that $\underline{\mathbf{S}}^-$ sends the counit $\gamma : \underline{\mathbf{I}} \cdot \underline{\mathbf{\Gamma}} \to \underline{\mathbf{Id}}_{\underline{\mathbf{Qual}}^-}$ of the adjunction $\underline{\mathbf{I}} \dashv \underline{\mathbf{\Gamma}}$ on $\underline{\mathbf{Id}}_{\underline{\mathbf{S}}^-} : \underline{\mathbf{S}}^- \to \underline{\mathbf{S}}^-$; that is to say :

$$\begin{array}{ccc} \underline{\mathbf{Top}_\Gamma} & \underset{\underline{\mathbf{\Gamma}}}{\overset{\underline{\mathbf{I}}}{\rightleftarrows}} & \underline{\mathbf{Qual}}^- \\ & {\scriptstyle \underline{\mathbf{U}}}\searrow \quad \swarrow{\scriptstyle \underline{\mathbf{S}}^-} & \\ & \underline{\mathbf{Ens}} & \end{array} \quad : \quad \begin{array}{rcl} \underline{\mathbf{\Gamma}} \cdot \underline{\mathbf{I}} & = & \underline{\mathbf{Id}}_{\underline{\mathbf{Top}_\Gamma}} \\ \underline{\mathbf{S}}^- \cdot \underline{\mathbf{I}} & = & \underline{\mathbf{U}} \\ \underline{\mathbf{U}} \cdot \underline{\mathbf{\Gamma}} & = & \underline{\mathbf{S}}^- \\ \underline{\mathbf{S}}^- \cdot \gamma & = & \underline{\mathbf{Id}}_{\underline{\mathbf{S}}^-} \end{array}$$

We write $\underline{\mathbf{\Gamma}}(X, \mathcal{X})$ under the form $\underline{\mathbf{\Gamma}}(X, \mathcal{X}) = (X, \mathcal{X}^\gamma)$, and

$$\gamma_{(X,\mathcal{X})} : (X, \mathcal{X}^\gamma) \longrightarrow (X, \mathcal{X}).$$

The upper subscript γ in $\mathcal{X}^\gamma$ corresponds to $\underline{\mathbf{\Gamma}}$; for each particular case, we shall use, from now on and in the same spirit, the lower case letter corresponding to the uppercase letter designating a right-adjoint functor

Proposition 14 *The functor* $(\underline{\breve{\mathbf{C}}}^{\mathrm{op}})^{\mathrm{op}} \cdot \underline{\mathbf{I}}^{\mathrm{op}} : \underline{\mathbf{Top}_\Gamma}^{\mathrm{op}} \longrightarrow \underline{\mathbf{Qual}}^+$ *is algebraic up to an equivalence for each topogenesis* $\underline{\mathbf{Top}_\Gamma}$.

Indeed, this functor has an adjoint, namely $\underline{\mathbf{\Gamma}}^{\mathrm{op}} \cdot \underline{\widehat{\mathbf{C}}}^{\mathrm{op}}$. Let us check the β-condition; for the sake of simplicity, let us write $\underline{\mathbf{J}}$ for the functor $(\underline{\breve{\mathbf{C}}}^{\mathrm{op}})^{\mathrm{op}} \cdot \underline{\mathbf{I}}^{\mathrm{op}}$.

— First, if $(X, \mathcal{X}) \overset{p}{\underset{q}{\rightrightarrows}} (Y, \mathcal{Y})$ is a $\underline{\mathbf{J}}$-splittable pair, then the pair

$$(X, \mathcal{X}) \overset{\underline{\mathbf{I}}^{\mathrm{op}} p}{\underset{\underline{\mathbf{I}}^{\mathrm{op}} q}{\rightrightarrows}} (Y, \mathcal{Y})$$

in $\underline{\mathbf{Qual}}^-$ is $(\underline{\breve{\mathbf{C}}}^{\mathrm{op}})^{\mathrm{op}}$-splittable, and from the β-condition for $(\underline{\breve{\mathbf{C}}}^{\mathrm{op}})^{\mathrm{op}}$, we have a coequalizer $(X, \mathcal{X}) \overset{\underline{\mathbf{I}}^{\mathrm{op}} p}{\underset{\underline{\mathbf{I}}^{\mathrm{op}} q}{\rightrightarrows}} (Y, \mathcal{Y}) \xrightarrow{e} (Z, \mathcal{Z})$ in $\underline{\mathbf{Qual}}^{-\mathrm{op}}$. Due to the adjunction $\underline{\mathbf{I}} \vdash \underline{\mathbf{\Gamma}}$ $[\gamma]$, this morphism e determines a morphism $(Y,\mathcal{Y}) \xrightarrow{e} (Z,\mathcal{Z}^{\gamma})$ which is in fact a coequalizer in $(X,\mathcal{X}) \underset{\underline{\mathbf{I}}^{\mathrm{op}}q}{\overset{\underline{\mathbf{I}}^{\mathrm{op}}p}{\rightrightarrows}} (Y,\mathcal{Y})$ in $\underline{\mathbf{Top}_{\Gamma}}^{\mathrm{op}}$; indeed, if $(Y,\mathcal{Y}) \xrightarrow{h} (Q,\mathcal{Q})$ in $\underline{\mathbf{Qual}}^{-\mathrm{op}}$ equalizes p and q, $\underline{\mathbf{I}}^{\mathrm{op}}h$ equalizes $\underline{\mathbf{I}}^{\mathrm{op}}p$ and $\underline{\mathbf{I}}^{\mathrm{op}}q$, and therefore we have a factorisation in $\underline{\mathbf{Qual}}^{-}$, say $\overline{h} \cdot e = h$, $(Z,\mathcal{Z}) \xrightarrow{h} (Q,\mathcal{Q})$. A fortiori, $(Z,\mathcal{Z}^{\gamma}) \xrightarrow{\overline{h}} (Q,\mathcal{Q})$ factorizes in $\underline{\mathbf{Top}_{\Gamma}}^{\mathrm{op}}$, and this $\overline{h}$ is unique since $Y \xrightarrow{e} Z$ is an injection.

— <u>Second</u>, for this $\underline{\mathbf{J}}$-splittable pair, let $(Y,\mathcal{Y}) \xrightarrow{k} (Z,\mathcal{Z})$ be an arrow in $\underline{\mathbf{Top}_{\Gamma}}^{\mathrm{op}}$.

◇ if k is a coequalizer of the pair $X \underset{q}{\overset{p}{\rightrightarrows}} Y$, then $\underline{\mathbf{J}}k$ is a coequalizer of the pair $\underline{\mathbf{J}}(X,\mathcal{X}) \underset{\underline{\mathbf{J}}q}{\overset{\underline{\mathbf{J}}p}{\rightrightarrows}} \underline{\mathbf{J}}(Y,\mathcal{Y})$; from the β-condition for $(\underline{\breve{\mathbf{C}}}^{\mathrm{op}})^{\mathrm{op}}$, $\underline{\mathbf{J}}k$ is a coequalizer if and only if $\underline{\mathbf{I}}^{\mathrm{op}}k$ is a coequalizer of $(X,\mathcal{X}) \underset{\underline{\mathbf{I}}^{\mathrm{op}}q}{\overset{\underline{\mathbf{I}}^{\mathrm{op}}p}{\rightrightarrows}} (Y,\mathcal{Y})$. And indeed, if $(Y,\mathcal{Y}) \xrightarrow{l} (K,\mathcal{K})$ equalizes $\underline{\mathbf{I}}^{\mathrm{op}}p$ and $\underline{\mathbf{I}}^{\mathrm{op}}q$ in $\underline{\mathbf{Qual}}^{-\mathrm{op}}$, then l determines an arrow $(Y,\mathcal{Y}) \to (K,\mathcal{K}^{g})$ in $\underline{\mathbf{Top}_{\Gamma}}^{\mathrm{op}}$, which equalizes p and q; this l factors therefore into $l = \overline{l} \cdot k$ for a certain $(Y,\mathcal{Y}) \xrightarrow{\overline{l}} (K,\mathcal{K}^{g})$ in $\underline{\mathbf{Top}_{\Gamma}}^{\mathrm{op}}$, which determines a $(Z,\mathcal{Z}) \xrightarrow{\overline{l}} (K,\mathcal{K})$. And this latter $\overline{l}$ is unique since, if $(Z,\mathcal{Z}) \xrightarrow{\tilde{l}} (K,\mathcal{K})$ satisfies $\tilde{l} \cdot \underline{\mathbf{I}}^{\mathrm{op}}k = l$, then $\tilde{l}$ determines $(Y,\mathcal{Y}) \xrightarrow{\tilde{l}} (K,\mathcal{K}^{g})$ satisfying $\tilde{l} \cdot k = l$, and hence $\tilde{l} = \overline{l}$.

◇ if $\underline{\mathbf{I}}^{\mathrm{op}}k$ is a coequalizer of $(X,\mathcal{X}) \underset{\underline{\mathbf{I}}^{\mathrm{op}}q}{\overset{\underline{\mathbf{I}}^{\mathrm{op}}p}{\rightrightarrows}} (Y,\mathcal{Y})$ in $\underline{\mathbf{Qual}}^{-\mathrm{op}}$, k is clearly a coequalizer of $(X,\mathcal{X}) \underset{q}{\overset{p}{\rightrightarrows}} (Y,\mathcal{Y})$ in $\underline{\mathbf{Top}_{\Gamma}}^{\mathrm{op}}$.

<u>Example 1</u>: The category $\underline{\mathbf{Top}}$ of topological spaces and continuous maps may be canonically embedded fully and faithfully in two ways in $\underline{\mathbf{Qual}}^{-}$.

The first embedding $\underline{\mathcal{O}}$: $\underline{\mathbf{Top}} \to \underline{\mathbf{Qual}}^{-}$ associates with a topological space X on a set X_0 the qualification space $\underline{\mathcal{O}}(X) = (X_0, \mathcal{O}(X))$, where $\mathcal{O}(X)$ is the set of *open sets* of X. If $(Y,\mathcal{Y})$ is a qualification space, we write $\underline{\Omega}(Y,\mathcal{Y})$ the topological space on Y generated by $\mathcal{Y}$, that is with the coarsest topology on Y having among its open sets the elements of $\mathcal{Y}$, which topology we note $\mathcal{Y}^{\omega}$. Then $\underline{\mathcal{O}} \dashv \underline{\Omega}$, and we identify $\underline{\mathbf{Top}}$ with $\underline{\mathbf{Top}_{\Omega}}$.

The second embedding $\underline{\mathcal{F}}$: $\underline{\mathbf{Top}} \to \underline{\mathbf{Qual}}^{-}$ associates with a topological space X on a set X_0 the qualification space $\underline{\mathcal{F}}(X) = (X_0, \mathcal{F}(X))$, where $\mathcal{F}(X)$ is the set of *closed sets* of X. If $(Y,\mathcal{Y})$ is a qualification space, we write $\underline{\Phi}(Y,\mathcal{Y})$

the topological space on Y generated by $\mathcal{Y}$, that is the coarsest topology on Y having among its closed sets the elements of $\mathcal{Y}$, which topology, given through giving its closed sets, we note $\mathcal{Y}^{\phi}$. Then $\underline{\mathcal{F}} \dashv \underline{\Phi}$, and we identify **Top** with $\underline{\mathbf{Top}}_{\Phi}$.

Then, in two different (but isomorphic) ways, we have :

Proposition 15 $\underline{\mathbf{Top}}^{\mathrm{op}}$ *is algebraic up to within an equivalence over the cartesian closed category* $\underline{\mathbf{Qual}}^{+}$.

Example 2: Let **Equiv** be the category with objects pairs (E, r) where E is a set and r an equivalence relation on E, and with a morphism $f : (E_1, r_1) \to (E_2, r_2)$ mappings $f : E_1 \to E_2$ such that

$$\forall x, y \in E_1 \ (x \equiv y \ [\mathrm{mod}\ r_1] \Longrightarrow fx \equiv fy \ [\mathrm{mod}\ r_2]),$$

that is to say, such that there exists an $\overline{f}$ making the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow & & \downarrow \\ E_1/r_1 & \xrightarrow{\overline{f}} & E_2/r_2 \end{array}$$

commutative. One may see **Equiv** up to within an equivalence as the full subcategory of the topos $\underline{\mathbf{Ens}}^{\cdot \to \cdot}$ whose objects are the epimorphisms of **Ens**.

If r is an equivalence relation on E, let us write $\theta(E, r)$ for the topological space on the set E with the topology generated by the classes of equivalence, that is the topology with open sets any union of equivalence classes for r (and therefore with closed sets also any union of equivalence classes). In this way, **Equiv** is isomorphic to the full subcategory of **Top** with objects these spaces X for which $\mathcal{O}(X) = \mathcal{F}(X)$, and θ is the equalizer of $\underline{\mathcal{O}}$ and $\underline{\mathcal{F}}$. We set $\underline{\Theta} = \underline{\mathcal{O}} \cdot \theta = \underline{\mathcal{F}} \cdot \theta$.

$$\underline{\mathbf{Equiv}} \xrightarrow{\theta} \underline{\mathbf{Top}} \underset{\underline{\mathcal{F}}}{\overset{\underline{\mathcal{O}}}{\rightrightarrows}} \underline{\mathbf{Qual}}^{-}, \qquad \underline{\Theta} : \underline{\mathbf{Equiv}} \to \underline{\mathbf{Qual}}^{-}$$

If $(X, \mathcal{X})$ is a qualification space, and r is an equivalence relation on X, we say that r is finer than $\mathcal{X}$ if $\forall A \in \mathcal{X}, \forall x \in X \ (x \in A \Rightarrow rx \subset A)$, and we set $\underline{\mathbf{E}}(X, \mathcal{X}) = (X, \mathcal{X}^e)$ with

$$\mathcal{X}^e = \Big\langle \{r \subset E \times E \ ; \ r \text{ is an equivalence relation finer than } \mathcal{X}\} \Big\rangle$$

where "$\langle - \rangle$" means "*the equivalence relation generated by* $-$". Thus $\mathcal{X}^e$ is finer than $\mathcal{X}$ and is the coarsest of all equivalence relations over X which are finer than $\mathcal{X}$. In fact, $x\mathcal{X}^e x'$ if and only if x and x' share exactly the same qualities of $\mathcal{X}$ (*i.e.* we cannot distinguish x from x' on the basis of their qualities). Given an equivalence relation s on a set Y, a mapping $f : Y \to X$ is compatible with s and $\mathcal{X}^e$ if and only if it is continuous (*i.e.* in $\underline{\mathbf{Qual}}^-$) from $\underline{\Theta}(Y,s)$ to $(X,\mathcal{X})$, *i.e.* we have the adjunction $\underline{\Theta} \dashv \underline{\mathbf{E}}$. Indeed, f is continuous if and only if fs is finer than $\mathcal{X}$, where fs is the equivalence relation on X generated by the relation $f[s]$:

$$xf[s]x' \Leftrightarrow \exists y, y'[y\, s\, y' \wedge fy = x \wedge fy' = x']$$

Then $\underline{\mathbf{Id}}_X$ is compatible with fr' and $\mathcal{X}^e$, and therefore, f is compatible with r' and $\mathcal{X}^e$. $\underline{\mathbf{Equiv}}$ is isomorphic to a full subcategory $\underline{\mathbf{Top}}_{\underline{\mathbf{E}}}$ of $\underline{\mathbf{Qual}}^-$ which is a topogenesis.

Proposition 16 $\underline{\mathbf{Equiv}}^{\mathrm{op}}$ *is algebraic up to within an equivalence over the* concrete *cartesian closed category* $\underline{\mathbf{Qual}}^+$. *More precisely,* $\underline{\mathbf{Equiv}}^{\mathrm{op}}$ *is equivalent to the category of algebras over the monad* $\breve{\mathbf{\Pi}}_{\underline{\mathrm{Equiv}}}$ *on* $\underline{\mathbf{Qual}}^+$, *which is a lifting of the monad of Stone* $\mathbf{\Pi}$ *on* $\underline{\mathbf{Ens}}$, *and which is given by*

$$\underline{\breve{\mathbf{\Pi}}}_{\underline{\mathrm{Equiv}}}(X,\mathcal{X}) = (\mathcal{P}^2X, \breve{\Pi}_{\underline{\mathrm{Equiv}}}\mathcal{X})$$

where $\mathcal{X} \subset \mathcal{P}X$, *and where* $\breve{\Pi}_{\underline{\mathrm{Equiv}}}\mathcal{X} \subset \mathcal{P}^3X$ *is defined through :*

$\mathbb{X} \in \breve{\Pi}_{\underline{\mathrm{Equiv}}}\mathcal{X}$ *iff* $\left\{\right.$ *For all* $P, Q \subset X$, *if* P *and* Q *meet the same elements of* $\mathcal{X}$, *and if* P *belongs to an element of* $\mathbb{X}$, *then* Q *also belongs to this element of* $\mathbb{X}$.

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