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ON THE GEOMETRIC BAR CONSTRUCTION AND THE BROWN REPRESENTABILITY THEOREM

by Peter I. BOOTH

Résumé

Cet article poursuit le développement d'une théorie des fibrations enrichies et de leurs espaces classificateurs. Plus précisément, les quasi-fibrations enrichies produites à partir de la construction "barre" de May sont reliées aux fibrations enrichies dérivées du théorème de représentabilité de Brown. Ceci permet d'élaborer une nouvelle approche pour construire les espaces classificateurs des fibrations enrichies, tout en conservant les principaux avantages de l'une ou l'autre des deux approches mentionnées plus haut quant à leur simplicité, généralité et calculabilité.

1 Introduction

This is one of a series of papers following on from [Bo2]; we normally use the terminology introduced there. Our overall objective is to develop a very smooth theory of enriched fibrations and their classifying spaces. Such a theory should combine and incorporate many individual theories, including the "classical" ones of principal, Hurewicz and sectioned fibrations.

In this paper we relate *universal enriched quasi-fibrations* - produced using the *two-sided geometric bar construction* of [Ma2] - to *universal enriched fibrations* obtained via the *Brown Representability Theorem* in [Bo3] and [Bo4]. These complementary methods of producing classifying spaces are blended together, thereby producing *a classification theory with the advantages of both approaches and the disadvantages of neither.*

We will always assume that $\mathcal{E}$ is a category of *enriched topological spaces* (see [Bo2, p.129], [Bo3, ch.2]). Thus there is a faithful *underlying space functor*, from $\mathcal{E}$ to the underlying topological category, that forgets any additional structure on the objects of $\mathcal{E}$. Further, if P is a space, Q is an $\mathcal{E}$-space and there is a homeomorphism from P onto the underlying space of Q, then there is a unique $\mathcal{E}$-space structure on P such that this homeomorphism is the underlying map of an *$\mathcal{E}$-homeomorphism* (= $\mathcal{E}$-isomorphism).

We recall the concept of an *$\mathcal{E}$-overspace*, i.e. a map $q: Y \to C$ together with an associated $\mathcal{E}$-space structure on the fibre $Y|c = q^{-1}(c)$, for each $c \in C$. For example, if Q is an $\mathcal{E}$-space and C is a space, then the projection $Q \times C \to C$ is an $\mathcal{E}$-overspace in an obvious way.

If $q: Y \to C$ is a map and S is a subspace of C, then $Y|S$ will denote the subspace $q^{-1}(S)$ of Y and $q|S: Y|S \to S$ the restriction of q to $Y|S$. If q is an $\mathcal{E}$-overspace, then $q|S$ is clearly also an $\mathcal{E}$-overspace.

The $\mathcal{E}$-overspace q will be said to be an *$\mathcal{E}$-fibration* if it satisfies the *$\mathcal{E}$-weak covering homotopy property* (= $\mathcal{E}WCHP$) [Bo2, p.136]. In the case where $\mathcal{E}$ consists of topological spaces without extra structure this is the $WCHP$ of [Do, section 5]; fibrations satisfying the $WCHP$ will be called *Dold fibrations.*

Let q be an $\mathcal{E}$-overspace such that, for each U in a numerable cover $\mathcal{U}$ of C, there is an $\mathcal{E}$-space $Q = Q(U)$ such that the restriction $q|U$ is *$\mathcal{E}$-fibre homotopy equivalent* (= $\mathcal{E}FHE$) [Bo2, p.130] to the projection $\mathcal{E}$-overspace $Q \times U \to U$. Then q will be said to be *$\mathcal{E}$-locally homotopy trivial* or $\mathcal{E}LHT$ [Bo2, p.142].

Let $\mathcal{W}$ denote the class of all spaces having the homotopy type of a CW-complex. If $C \in \mathcal{W}$, then an $\mathcal{E}$-overspace q is an $\mathcal{E}$-fibration if and only if it is $\mathcal{E}LHT$ [Bo2, thm.6.3].

We will always assume - unless we specify otherwise - that F is a given $\mathcal{E}$-space and use $\mathcal{F}$ to denote the ***category of fibres*** **containing F, i.e. the category of all $\mathcal{E}$-spaces that are $\mathcal{E}$-homotopy equivalent to F and all $\mathcal{E}$-homotopy equivalences between them.**

Let $\mathcal{F}(F)$ denote the monoid under composition of self-$\mathcal{F}$-homotopy equivalences of F, topologized as described below. Then $\mathcal{F}(F)'$ will denote $\mathcal{F}(F)$ with a whisker grown at its identity 1_F. We notice that $\mathcal{F}(F)'$ also carries the structure of a topological monoid. It can be used with the bar construction to construct an associated $\mathcal{F}$-overspace, i.e. a *universal $\mathcal{F}$-quasi-fibration* $q_{\mathcal{F}}: Y_{\mathcal{F}} \to C_{\mathcal{F}}$ (see §3 for more details).

May used either a Γ-*completeness* [Ma2, def.5.1] or a Γ'-*completeness* [Ma2, def.5.4] assumption to convert his $q_{\mathcal{F}}$ into a *universal* $\mathcal{F}$*-fibration* $\Gamma q_{\mathcal{F}}$ or $\Gamma' q_{\mathcal{F}}$ [Ma2, thm.9.2 (a) and (b)]. This theory can be very useful in the area of applications (see, for example, [Ma3]). The Γ- and Γ'-completeness assumptions also, however, have the drawback that their use detracts from the simplicity and generality of the classification result obtained.

The Brown Representability Theorem approach, given in [Bo3] and [Bo4], allows the construction of an alternative universal $\mathcal{F}$-fibration $p_{\mathcal{F}}: X_{\mathcal{F}} \to B_{\mathcal{F}}$ (see §3 for more details). Further, this procedure avoids the above problems (compare [Bo4, thm.5.3] with [Ma2, thm.9.2, (a) and (b)]). On the other hand, $p_{\mathcal{F}}$ cannot be applied in the same direct and flexible fashion that is possible with the former approach. These and other issues, concerning the merits of differing approaches to the classification problem, will be reviewed in [Bo5].

We use a fibred mapping space argument to equate $p_{\mathcal{F}}$ and $q_{\mathcal{F}}$. In particular, we equate the classifying spaces $B_{\mathcal{F}}$ and $C_{\mathcal{F}}$. Neither a Γ-completeness nor a Γ'-completeness condition is assumed. *Then properties of the bar construction can be used for applications, in the simplified and more general Brown's theorem context.*

If $p: X \to B$ and $q: Y \to C$ are maps, and $h: X \to Y$ and $g: B \to C$ are maps such that $qh = gp$, then we write that $<h, g>$ is a *pairwise map from p to q*. If p and q are $\mathcal{E}$-overspaces and $h|(X|b): X|b \to Y|g(b)$ is an $\mathcal{E}$-map for each $b \in B$, then we will write that $<h, g>$ is an $\mathcal{E}$*-pairwise map from p to q*. In section 2 we consider some relevant properties of *fibred mapping spaces*, leading to theorem 2.7 which gives sufficient conditions for the existence of an $\mathcal{F}$-pairwise map from a given $\mathcal{F}$-fibration to a given $\mathcal{F}$-overspace. This result, together with the properties of the geometric bar construction and of the Brown's theorem universal fibration that are reviewed in section 3, enables us to show in section 4 (theorem 4.2) that there is an $\mathcal{F}$-pairwise map $<h, g>$ from $p_{\mathcal{F}}$ to $q_{\mathcal{F}}$. Further $h: X_{\mathcal{F}} \to Y_{\mathcal{F}}$ and $g: B_{\mathcal{F}} \to C_{\mathcal{F}}$ are weak homotopy equivalences. Thus we have achieved our objective.

It follows (corollary 4.3) that the bar construction space $C_{\mathcal{F}}$ acts as a classifying space for $\mathcal{F}$-fibrations. It does so without the requirement of a topological condition, such as compactness or belonging to $\mathcal{W}$, on the underlying spaces of the fibres involved.

Our discussion should be taken to be in the context of the category $\mathcal{T}$ of *compactly generated* **or** *cg-spaces* ([Bo2, p.128-129] and [V, sec.5, ex.(ii)]). Thus if X and Y are cg-spaces, then $\mathcal{T}(X,Y)$ will denote the space of all maps from X to Y equipped with the *cg-ified* modification [Bo2, p.129] of the compact-open topology. $\mathcal{F}(F)$ will be topologized in the same fashion.

We recall that a space B is *weak Hausdorff* [Bo2, p.129] if the *diagonal subset* $\Delta_B = \{(b,b) \mid b \in B\}$ is closed in the (cg-ified) product space $B \times B$.

Lemma 1.1 *If B is a weak Hausdorff space and S is a subspace of B, then S is weak Hausdorff.*

Proof. The diagonal subset $\Delta_S = (S \times S) \cap \Delta_B$ is closed in $S \times S$.

2 Fibred Mapping Spaces and Structure Preserving Maps

We will assume, throughout this section, that B is a weak Hausdorff space and that $p: X \to B$ and $q: Y \to C$ are $\mathcal{E}$-overspaces.

We recall the concept of the $\mathcal{E}$-overspace $q_g : Y \sqcap D \to D$, *induced* by *pulling q back over the map $g: D \to C$* [Bo2, p.130]. Also that there is a fibred mapping space $X \square Y$ with underlying set $\bigcup_{b \in B, c \in C} \mathcal{E}(X|b, Y|c)$ and a map $p\square q : X \square Y \to B \times C$, with $(p \square q)(f) = (b,c)$, where $f \in \mathcal{E}(X|b, Y|c)$, $b \in B$ and $c \in C$ [Bo2, p.131-132]. We use $p\square_1 q : X\square Y \to B$ and $p \square_2 q : X \square Y \to C$ to denote the maps that are the composites of $p \square q$ with the projections $\pi_B : B \times C \to B$ and $\pi_C : B \times C \to C$, respectively. So if $f \in \mathcal{E}(X|b, Y|c)$ where $b \in B$ and $c \in C$, then $(p\square_1 q)(f) = b$ and $(p \square_2 q)(f) = c$.

In this section we develop some basic properties of $p \square_1 q$. Versions of some of these (theorem 2.1, corollaries 2.2 and 2.4 and a weaker form of proposition 2.6) have been given elsewhere (see [BHP1], [BHP2], [BHMP] and [Bo1]). However this material is not all in one place, the details of the proofs are sometimes rather skimpy and the results are often based on data somewhat different from ours.

Theorem 2.1 : Fibred exponential law. *Let $f: D \to B$ be a map. Then there is a bijective correspondence between:*

(i) the set of $\mathcal{E}$-pairwise maps $<h,g>$ from p_f to q and

(ii) the set of maps $h^0: D \to X\Box Y$ over B, i.e. relative to f and $p\Box_1 q$,

determined by $h(x,d) = h^0(d)(x)$ and $g = (p\Box_2 q)h^0$, where $p(x) = f(d)$.

Proof. This is just a slight modification of the proof of another fibred exponential law, i.e. theorem 2.2 of [Bo2]; we notice that B is required to be weak Hausdorff in that result.

If $s: D \to B$ and $t: D \to C$ are maps then the map $D \to B\times C$, with $d \to (s(d), t(d))$ where $d \in D$, will be denoted by (s,t). Let $<h,g>$ be an $\mathcal{E}$-pairwise map from p_f to q. Then there is a map $h^0: D \to X\Box Y$, defined by the equality specified above, and such that $(p\Box q)h^0 = (f,g)$ (see [Bo2, thm.2.2]). So $(p\Box_1 q)h^0 = \pi_B(p\Box q)h^0 = \pi_B(f,g) = f$, and h^0 is a map over B as required.

Conversely, let $h^0: D \to X\Box Y$ be a map satisfying $(p\Box_1 q)h^0 = f$. Then $(p\Box q)h^0 = (p\Box_1 q, p\Box_2 q)h^0 = ((p\Box_1 q)h^0, (p\Box_2 q)h^0) = (f,g)$, where $g = (p\Box_2 q)h^0: D \to C$. So h^0 satifies the condition specified in [Bo2, thm.2.2], the associated h is well defined and $<h,g>$ is as required.

Corollary 2.2 *There is a bijective correspondence between :*

(i) the set of $\mathcal{E}$-pairwise maps $<h,g>$ from p to q, and

(ii) the set of sections h^0 to $p\Box_1 q$,

determined by $h|(X|b) = h^0(b)$ and $g = (p\Box_2 q)h^0$, where $b \in B$.

Proof. This follows from theorem 2.1 if we take $D = B, f = 1_B$ and identify p_f with p.

Corollary 2.3 *If S is a subspace of B, then $(X\Box Y)|S = (p\Box_1 q)^{-1}(S)$ and $(X|S)\Box Y$ are identical, and $(p\Box_1 q)|S = (p|S)\Box_1 q$.*

Proof. Clearly the underlying sets of the two spaces coincide and the functions $(p\Box_1 q)|S$ and $(p|S)\Box_1 q$ are identical. Let us consider any (of course cg-) space D, map $f: D \to S$ and function $k: D \to (X\Box Y)|S$ that is over S. We will prove that k is continuous into $(X\Box Y)|S$ if and only if it is continuous into $(X|S)\Box Y$. The result will then follow.

Let $i: S \to B$ and $j: (X\Box Y)|S \to X\Box Y$ denote the inclusions. If k is continuous into $(X\Box Y)|S$, it follows that $jk: D \to X\Box Y$ is a map over B, relative to the maps $if: D \to B$ and $p\Box_1 q: X\Box Y \to B$. Applying our fibred exponential law, there is a corresponding $\mathcal{E}$-pairwise map $<h, g>$ from p_{if} to q, determined by $h(x,d) = k(d)(x)$, where $p(x) = if(d) = f(d)$ and $g = (p\Box_2 q)jk$. The $\mathcal{E}$-overspaces $p_{if}: X \sqcap D \to D$ and $(p|S)_f: (X|S) \sqcap D \to D$ are identical, so $<h, g>$ is an $\mathcal{E}$-pairwise map from $(p|S)_f$ to q. Now S is weak Hausdorff (lemma 1.1), and so it follows from theorem 2.1 that there is a map $l: D \to (X|S)\Box Y$ over S, i.e. with $((p|S)\Box_1 q)l = f$. Further, $l(d)(x) = h(x,d) = k(d)(x)$ where $f(d) = p(x)$, so $l = k$ and $k: D \to (X|S)\Box Y$ is continuous.

Now each step in the above paragraph is clearly reversible, so the continuity of k in the two senses is equivalent and the proof is complete.

The subset $\bigcup_{c\in C} \mathcal{E}(F, Y|c)$ of the set $\mathcal{T}(F,Y)$ determines a subspace $Prin_F Y$ of the space $\mathcal{T}(F,Y)$. Then the composite of the inclusion map $Prin_F Y \to \mathcal{T}(F,Y)$, the map $\mathcal{T}(F,Y) \to Y$ that evaluates at an arbitrarily chosen point of F and $q: Y \to C$ is the obvious projection $prin_F q: Prin_F Y \to C$. Hence this projection is continuous.

Corollary 2.4 *If* $b \in B$, *then the fibre of* $p\Box_1 q$ *over* b *is* $Prin_{X|b} Y$.

Proof. It follows from corollary 2.3 that the fibre of $p\Box_1 q$ over $b \in B$, i.e. $(X\Box Y)|b$, is $(X|b)\Box Y$. Hence we have to prove that $(X|b)\Box Y = Prin_{X|b} Y$. Clearly the underlying sets are identical, so we just have to prove that the topologies agree. Let D be a space and $h^0: D \to (X|b)\Box Y$ be a function. We will show that h^0 is continuous into $(X|b)\Box Y$ if and only if it is continuous into $Prin_{X|b} Y$. The result then follows. Let us apply the fibred exponential law to $h^0: D \to (X|b)\Box Y$, considered as a map over a point, and the "ordinary" exponential law [Bo2, (0.1)] to

$h^0 : D \to Prin_{X|b}Y \subset \mathcal{T}(X|b, Y)$. We see that in each case the continuity of h^0 is equivalent to the continuity of $h : (X|b) \times D \to Y$, where $h(x, d) = h^0(d)(x)$, with $x \in X$ and $d \in D$. Hence the proof is complete.

If $b \in B$, then $\phi_b : F \to F \times \{b\}$ will denote the homeomorphism determined by $\phi_b(z) = (z, b)$, where $z \in F$. It follows - from the definition of category of enriched spaces - that there is a unique $\mathcal{E}$-space structure on $F \times \{b\}$, making ϕ_b into an $\mathcal{E}$-homeomorphism for each $b \in B$. We recall that this allows us to view the projection $\pi_B : F \times B \to B$ as an $\mathcal{E}$-overspace.

Corollary 2.5 *Let $\pi : (Prin_F Y) \times B \to B$ denote the projection. There is a homeomorphism $\Phi : (F \times B) \square Y \cong (Prin_F Y) \times B$, that is over B in the sense that $\pi\Phi = \pi_B \square_1 q$.*

Proof. If $b \in B$, $c \in C$ and $g \in \mathcal{E}(F \times \{b\}, Y|c)$, then $g\phi_b \in \mathcal{E}(F, Y|c) \subset Prin_F Y$. We define the function $\Phi : (F \times B) \square Y \to (Prin_F Y) \times B$ by the rule $\Phi(g) = (g\phi_b, (\pi_B \square_1 q)(g))$. Then $\pi\Phi(g) = \pi(g\phi_b, (\pi_B \square_1 q)(g)) = (\pi_B \square_1 q)(g)$, so $\pi\Phi = \pi_B \square_1 q$ and the function Φ is over B.

Let $\psi_b : F \times \{b\} \to F$ denote the homeomorphism $\psi_b(z, b) = z$, where $z \in F$ and $b \in B$. Then ψ_b is the $\mathcal{E}$-homeomorphism that is inverse to ϕ_b.

If $f \in Prin_F Y$ and $b \in B$, then $f\psi_b \in \mathcal{E}(F \times \{b\}, Y|c) \subset (F \times B) \square Y$, where $c = (prin_F q)(f)$. Let $\Psi : (Prin_F Y) \times B \to (F \times B) \square Y$ be the function with $\Psi(f, b) = f\psi_b$. It is easily seen that $\Psi\Phi$ and $\Phi\Psi$ are identity functions, so Φ and Ψ are inverse bijections.

On applying theorem 2.1 to the identity on $(F \times B) \square Y$, we obtain an $\mathcal{E}$-pairwise map $<e, \pi_B \square_2 q>$ from $(\pi_B)_{(\pi_B \square_1 q)}$ to q. The evaluation map $e : (F \times B) \sqcap ((F \times B) \square Y) \to Y$ is determined by $e(z, b, g) = g(z, b)$, with $z \in F$, $b \in B$ and $g \in \mathcal{E}(F \times \{b\}, Y|c)$ for some $c \in C$. Further, there is a homeomorphism $((F \times B) \square Y) \times F \cong (F \times B) \sqcap ((F \times B) \square Y)$ determined by the rule $(g, z) \to (z, b, g)$, where $b = (\pi_B \square_1 q)(g)$. Hence we see, by composition, that there is a map $((F \times B) \square Y) \times F \to Y, (g, z) \to g(z, b)$. Applying the exponential law of [Bo2, (0.1)] to this map, the rule $g \to (z \to g(z, b))$ determines a map $(F \times B) \square Y \to \mathcal{T}(F, Y)$. In this case $g \in (F \times B) \square Y$, $z \in F$ and $b = (\pi_B \square_1 q)(g)$. Now $g(-, b) : F \to Y|c$, which takes z to $g(z,b)$, is $g\phi_b \in \mathcal{E}(F,Y|c)$, where $c = (\pi_B \square_2 q)(g)$. So we have a map $(F \times B) \square Y \to Prin_F Y, g \to g\phi_b$ and it follows that Φ is continuous.

We now define maps e', λ and μ such that $<e'\lambda, \mu>$ is an $\mathcal{E}$-pairwise map corresponding to Ψ. If the ordinary exponential law is applied to the inclusion $Prin_F Y \subset \mathcal{T}(F,Y)$, we obtain an evaluation map:

$$e' : (Prin_F Y) \times F \to Y, \ e'(f,z) = f(z),$$

where $f \in Prin_F Y$ and $z \in F$. Let us define

$$\lambda : (F \times B) \sqcap ((Prin_F Y) \times B) \to (Prin_F Y) \times F$$

by $\lambda((z,b),(f,b)) = (f,z)$, where $z \in F$, $b \in B$ and $f \in Prin_F Y$, and

$$\mu : (Prin_F Y) \times B \to C \text{ by } \mu(f,b) = (prin_F q)(f),$$

where $f \in Prin_F Y$ and $b \in B$. So $qe'\lambda(z,b,f,b) = qf(z) = (prin_F q)(f) = \mu(f,b) = \mu((\pi_B)\pi(z,b,f,b))$, where $z \in F$, $b \in B$ and $f \in Prin_F Y$. Hence $<e'\lambda, \mu>$ is an $\mathcal{E}$-pairwise map from $(\pi_B)\pi$ to q. Applying theorem 2.1 there is a map $(Prin_F Y) \times B \to (F \times B) \square Y$, determined by the rule $(f,b) \to ((z,b) \to f(z))$, where $f \in Prin_F Y$, $b \in B$ and $z \in F$. Now this is Ψ, so Ψ is continuous. Hence Φ is a homeomorphism over B.

Proposition 2.6 *If B is a CW-complex, $p : X \to B$ is an $\mathcal{E}$-fibration and $q : Y \to C$ is an $\mathcal{E}$-overspace, then $p \square_1 q$ is a Dold fibration.*

Proof. We know that p is $\mathcal{E}LHT$. Thus there is a numerable cover $\mathcal{U}$ of B and, for each $U \in \mathcal{U}$, there is an object Q of $\mathcal{E}$ such that $p|U$ is $\mathcal{E}FHE$ to the $\mathcal{E}$-overspace and projection $\pi_U : Q \times U \to U$. It follows, via corollary 2.3, that $(p \square_1 q)|U = (p|U) \square_1 q$, which in turn is FHE to $(\pi_U) \square_1 q$. We see from lemma 1.1 that U is weak Hausdorff, and hence from corollary 2.5 that $(\pi_U) \square_1 q$ is homeomorphic over U to the projection $Prin_Q(X) \times U \to U$. Hence $p \square_1 q$ is locally homotopy trivial, i.e. $\mathcal{T}$LHT, and therefore a Dold fibration.

The last result, which was announced without proof in [Bo1, (7.2)], should not be confused with certain similar published results. Thus all of [Bo2, prop.4.5], [BHP1, prop.6], [BHP2, axiom A4] and [BHMP, prop.2.3] state that if p and q are (in some sense) $\mathcal{E}$-fibrations, then $p\Box q$ is (some sort of) a fibration. These results prove, by composition with $\pi_B : B\times C \to C$, that $p\Box_1 q$ is a fibration. However, they assume something that we cannot assume about q, i.e. that it is at least an $\mathcal{E}$-fibration. The fact that q need not possess such a property is crucial for our verification of theorems 2.7 and 4.2.

From this point on our argument will usually focus on a category of fibres $\mathcal{F}$, rather than on $\mathcal{E}$; the $\mathcal{E}$ of previous results can, of course, be taken to be such an $\mathcal{F}$. So $X\Box Y$ and $Prin_F Y$ will now consist entirely of $\mathcal{F}$-homotopy equivalences between spaces that are always $\mathcal{F}$-homotopy equivalent to F.

A space S will be said to be *weakly contractible* if $\pi_i(S) = 0$, for all non-negative integers i.

Theorem 2.7 *Let B be a CW-complex, $p : X \to B$ be an $\mathcal{F}$-fibration and $q : Y \to C$ be an $\mathcal{F}$-overspace. If $Prin_F Y$ is weakly contractible, then there is an $\mathcal{F}$-pairwise map from p to q.*

Proof. It follows from proposition 2.6 that $p\Box_1 q$ satifies the WCHP. The fibre of $p\Box_1 q$ over $b \in B$ is $Prin_{X|b} Y$ (corollary 2.4), which has the homotopy type of $Prin_F Y$ and is therefore a weakly contractible space. Considering the exact homotopy sequence of $p\Box_1 q$, we see that this map is a weak homotopy equivalence. Factoring $p\Box_1 q$ as the composite of a homotopy equivalence $X\Box Y \to R$ and a Hurewicz fibration $\rho : R \to B$, we see via [Do, thm.6.1] that ρ is FHE to $p\Box_1 q$. Hence ρ is a weak homotopy equivalence and, by [Sp, thm.7.6.23], ρ has a homotopy section, i.e. a map σ such that $\rho\sigma \simeq 1_B$. Applying the CHP we see that ρ has a section, hence so also does $p\Box_1 q$. The result follows from corollary 2.2.

3 Constructing Classifying Spaces

We will now quote some results from [Ma2]. The concept of compactly generated space, as used in that memoir, incorporates the weak Hausdorffness condition. Hence, from this point on, **our discussion should**

be taken to be in the context of the category of weak Hausdoff cg-spaces [Mc, section 2].

Let G be a topological monoid that is *grouplike*, i.e. $\pi_0(G)$ with the obvious induced operation is a group. Further, let us assume that the identity element ι of G is a *strongly non-degenerate base point*, in the sense that $(G, \{\iota\})$ is a strong NDR-pair (see [Ma1, definition A.1]). Let X and Y be left and right G-spaces, respectively. Then [Ma2, p.31] describes a two-sided geometric bar construction, that defines a space $B(Y, G, X)$. Further, if $*$ is a one-point space, there is a unique left action $G\times * \to *$ and the map $X \to *$ induces a quasi-fibration $p_{Y,X} : B(Y, G, X) \to B(Y, G, *)$ [Ma2, p.34]. Then $p_{Y,X}$ has fibres homeomorphic to X [Ma2, p.35].

If H is any (of course cg-sense) topological monoid, we will use H' to denote the space consisting of H with a whisker ($= I$) grown at its identity ι. Thus we simply identify ι with 0. Then H' carries the structure of a topological monoid, under the operation that extends both the operation on H and multiplication on I, and has $th = ht = h$ for all $t \in I$ and $h \in H$. The identity for H' is of course 1. If H is a grouplike topological monoid, then we notice that H' is also a grouplike topological monoid. The relevance of H' lies in the fact that it has a strongly non-degenerate base point (see [Ma1, definition A.8] for further details).

The evaluation map $e : \mathcal{F}(F)\times F \to F$, $e(f, z) = f(z)$ where $f \in \mathcal{F}(F)$ and $z \in F$, and the composition map $c : \mathcal{F}(F) \times \mathcal{F}(F) \to \mathcal{F}(F)$, $c(f, g) = fg$ where f and $g \in \mathcal{F}(F)$, are left actions of $\mathcal{F}(F)$ on F and $\mathcal{F}(F)$, respectively. Let us now define a morphism of topological monoids, i.e. $\beta : \mathcal{F}(F)' \to \mathcal{F}(F)$, by $\beta(t) = 1_F$ and $\beta(h) = h$, where $t \in I$ and $h \in \mathcal{F}(F)$. Then there are left actions of $e(\beta \times 1_F) : \mathcal{F}(F)' \times F \to F$ of $\mathcal{F}(F)'$ on F, and $c(\beta \times 1_{\mathcal{F}(F)}) : \mathcal{F}(F)' \times \mathcal{F}(F) \to \mathcal{F}(F)$ of $\mathcal{F}(F)'$ on $\mathcal{F}(F)$.

The above left action $\mathcal{F}(F)'\times F \to F$ and the left and right actions of $\mathcal{F}(F)'$ on $*$ determine a map $p_{*,F} : B(*, \mathcal{F}(F)', F) \to B(*, \mathcal{F}(F)', *)$; we use $q_{\mathcal{F}} : Y_{\mathcal{F}} \to C_{\mathcal{F}}$ to denote this map.

Then $C_{\mathcal{F}} = B(*, \mathcal{F}(F)', *)$ is the *geometric bar construction classifying space for* $\mathcal{F}$.

Proposition 3.1 *The map* $q_{\mathcal{F}}$ *is a quasi-fibration and carries the struc-*

ture of an $\mathcal{F}$-overspace. The map $prin_F(q_{\mathcal{F}}): Prin_F(Y_{\mathcal{F}}) \to C_{\mathcal{F}}$ is a quasi-fibration with a contractible total space $Prin_F(Y_{\mathcal{F}})$.

Proof. The first two conditions are justified because, as explained previously, any $p_{Y,X}$ is a quasi-fibration with fibres homeomorphic to X; $prin_F(q_{\mathcal{F}})$ is a quasi-fibration by [Ma2, prop.7.10].

We just have to verify the contractibility condition. We see, via [Ma2, prop.7.10], that the space $Prin_F(Y_{\mathcal{F}}) = Prin_F(B(*, \mathcal{F}(F)', F))$ is homeomorphic to $B(*, \mathcal{F}(F)', \mathcal{F}(F))$. The latter bar construction space is defined using the above left action $\mathcal{F}(F)' \times \mathcal{F}(F) \to \mathcal{F}(F)$. It has the homotopy type of the space $B(*, \mathcal{F}(F)', \mathcal{F}(F)')$, that is obtained using the operation on $\mathcal{F}(F)'$ (see [Ma2, prop.7.3(ii)]). This, in turn, can be seen to be contractible (take $Y = *$ in [Ma2, prop.7.5]). Hence $Prin_F(Y_{\mathcal{F}})$ is contractible.

We now turn to a completely different construction of classifying spaces, i.e. the Brown Representability Theorem approach of [Bo3] and [Bo4]. The arguments of those papers are valid in the world of weak Hausdorff cg-spaces, if a suitable adjustment is made to the definition of *"space under a given space A"*, as stated in [Bo3, section 2].

We will assume that (X, i) is a space under A now means that $i: A \to X$ is a homeomorphism of A onto a closed subspace of X. This ensures that all adjunction spaces, that appear in this family of papers, are defined using a map out of a closed subspace of a given space. It follows that, in our new context, all of these adjunction spaces are weak Hausdorff (see [Mc, props.2.4 and 2.5]). The fibred mapping spaces that appear in this family of papers may, temporarily, take us outside the category of weak Hausdorff spaces. However, that does not prevent us from drawing conclusions that apply in that category.

An $\mathcal{F}$-fibration $p: X \to B$ will be said to be *universal* if it is universal in the weakly contractible sense, i.e. if $Prin_F(X)$ is weakly contractible. It is a shown in [Bo4, prop.3.7] that, if such a p exists and a space $D \in \mathcal{W}$, then the collection $\mathcal{F}FHE(D)$ of all $\mathcal{F}FHE$ classes of $\mathcal{F}$-fibrations over D is a set. Further, such a p is also *free universal amongst $\mathcal{F}$-fibrations.* Thus there is a bijection

$$\theta = \theta(p): [D, B] \to \mathcal{F}FHE(D), \quad [f] \to [p_f], \quad [f] \in [D, B],$$

that is natural, relative to D in $\mathcal{W}$ (see [Bo3, prop.7.4] and [Bo4, thm.3.4]).

We recall the concept of $\mathcal{E}$ carrying the structure of a *category of well enriched spaces under a given space* A [Bo3, def.2.3]. Thus $\mathcal{E}$, with its underlying space functor, is a category of enriched spaces that possesses well behaved subspaces, cylinders and mapping cylinders, all in the under A sense. Further, the category of enriched spaces structure on $\mathcal{E}$ is derived from the well enriched structure on that category (see [Bo3, lem.2.4]).

If, for every choice of a category of fibres $\mathcal{F}$ in $\mathcal{E}$ and of a space Z under A, the class of all associated $\mathcal{F}$-space structures on Z is a set, then $\mathcal{E}$ will be said to be *proper* [Bo4, def.5.1(ii)].

Theorem 3.2 *(see [Bo4, thm.5.3]) Let $\mathcal{E}$ carry the structure of a proper category of well enriched spaces under a space A, F be an $\mathcal{E}$-space and $\mathcal{F}$ be the category of fibres determined by F. Then there exists a universal $\mathcal{F}$-fibration $p_{\mathcal{F}}: X_{\mathcal{F}} \to B_{\mathcal{F}}$ over a path connected CW-complex $B_{\mathcal{F}}$.*

The above is a consequence of Brown's theorem, so the space $B_{\mathcal{F}}$ will be said to be a *Brown Representability Theorem classifying space for* $\mathcal{F}$.

4 Main Result

Lemma 4.1 *If $q: Y \to C$ is an $\mathcal{F}$-fibration, then $prin_F q$ is a Dold fibration.*

Proof. Let us take W to be a given space, and $g: W \times \{0\} \to Prin_F Y$ and $G: W \times I \to C$ to be maps such that $(prin_F q)g = G|(W \times \{0\})$. It follows from the exponential law (0.1) of [Bo2] that there is a map $g': F \times W \times \{0\} \to Y$, determined by the rule $g'(z, w, 0) = g(w, 0)(z)$, where $w \in W$ and $z \in F$. If $\pi: F \times W \to W$ denotes the projection, then $qg'(z,w,0) = q(g(w,0)(z)) = (prin_F q)g(w,0) = G(w,0) = G(\pi \times 1_I)(z,w,0)$. Further, $g'(-,w,0) = g(w,0) \in \mathcal{F}(F, Y|G(w,0))$, and so $<g', G|(W \times \{0\})>$ is an $\mathcal{F}$-pairwise map from $\pi \times 1_{\{0\}}$ to q.

Now q has the $\mathcal{F}$WCHP, so there is a map $H': F \times W \times I \to Y$, such that $<H', G>$ is an $\mathcal{F}$-pairwise map from $\pi \times 1_I$ to q. We see, via the ordinary exponential law, that there is a map $H: W \times I \to \mathcal{T}(F, Y)$, where

$H(w,t)(z) = H'(z,w,t)$ for all $w \in W$, $t \in I$ and $z \in F$. Then the above $\mathcal{F}$-pairwise condition ensures that H factors through $Prin_F Y$, and we can take H to be a homotopy $W \times I \to Prin_F Y$. So $(prin_F q)H(w,t) = q(H(w,t)(z)) = qH'(z,w,t) = G(\pi \times 1_I)(z,w,t) = G(w,t)$, for all $w \in W$, $t \in I$ and $z \in F$. Hence $(prin_F q)H = G$. Further, $H(w,0)(z) = H'(z,w,0) = g'(z,w,0) = g(w,0)(z)$, for all $w \in W$ and $z \in F$. So $H(w,0) = g(w,0)$, for all $w \in W$.

Theorem 4.2 *Let $\mathcal{E}$ carry the structure of a proper category of well enriched spaces under a space A, F be a given $\mathcal{E}$-space and $\mathcal{F}$ be the category of fibres determined by F. Then:*

(i) there is an $\mathcal{F}$-pairwise map $<h, g>$ from $p_{\mathcal{F}}$ to $q_{\mathcal{F}}$ such that the maps $h: X_{\mathcal{F}} \to Y_{\mathcal{F}}$ and $g: B_{\mathcal{F}} \to C_{\mathcal{F}}$ are weak homotopy equivalences.

(ii) If $\mathcal{F}(F) \in \mathcal{W}$, then g is a homotopy equivalence; if also $F \in \mathcal{W}$, then h is a homotopy equivalence.

Proof. (i) It follows from theorem 2.7, proposition 3.1 and theorem 3.2 that there is an $\mathcal{F}$-pairwise map $<h, g>: p_{\mathcal{F}} \to q_{\mathcal{F}}$. The map $h: X_{\mathcal{F}} \to Y_{\mathcal{F}}$ induces a map $h_{\#}: Prin_F(X_{\mathcal{F}}) \to Prin_F(Y_{\mathcal{F}})$, $h_{\#}(f) = (h|(X_{\mathcal{F}}|b))f$, where $f \in Prin_F(X_{\mathcal{F}})$ and $b = (prin_F(p_{\mathcal{F}}))(f)$. Then $<h_{\#}, g>$ is a pairwise map from $prin_F(p_{\mathcal{F}})$ to $prin_F(q_{\mathcal{F}})$. The morphisms of $\mathcal{F}$ are $\mathcal{F}$-homotopy equivalences, so $h|(X_{\mathcal{F}}|b): X_{\mathcal{F}}|b \to Y_{\mathcal{F}}|b$ is an $\mathcal{F}$-homotopy equivalence, for all $b \in B$. Hence the induced map $h_{\#}|\mathcal{F}(F, X_{\mathcal{F}}|b): \mathcal{F}(F, X_{\mathcal{F}}|b) \to \mathcal{F}(F, Y_{\mathcal{F}}|b)$, where $f \to (h|(X_{\mathcal{F}}|b)f$ for all $b \in B_{\mathcal{F}}$, is a homotopy equivalence. Now $Prin_F(X_{\mathcal{F}})$ is weakly contractible (theorem 3.2) and $Prin_F(Y_{\mathcal{F}})$ is contractible (proposition 3.1), so $h_{\#}$ is a weak homotopy equivalence. Also $B_{\mathcal{F}}$ and $C_{\mathcal{F}}$ are path connected (theorem 3.2 and [Ma2, prop.7.1]). It follows from the "ladder" of exact homotopy sequences associated with the Dold fibration $prin_F(p_{\mathcal{F}})$ (lemma 4.1), the quasi-fibration $prin_F(q_{\mathcal{F}})$ (propopsition 3.1), and the pairwise map $<h_{\#}, g>$, that g is a weak homotopy equivalence.

A similar argument, applied to the ladder associated with the pairwise map $<h, g>$ from $p_{\mathcal{F}}$ to $q_{\mathcal{F}}$, now shows that h is also a weak homotopy equivalence.

(ii) If $\mathcal{F}(F) \in \mathcal{W}$ then, using the retraction $\beta: \mathcal{F}(F)' \to \mathcal{F}(F)$, we see that $\mathcal{F}(F)' \in \mathcal{W}$. So $C_{\mathcal{F}}$ is in $\mathcal{W}$ [Ma2, prop.7.2]. We already know that $B_{\mathcal{F}}$ is a CW-complex, so g is a weak homotopy equivalence between spaces in $\mathcal{W}$ and hence is a homotopy equivalence [Sp, 7.6.24].

If the base space and fibres of a Hurewicz fibration are both in $\mathcal{W}$, then so also is the total space [Sc, thm.2]. Any map factors as the composite of a Hurewicz fibration and a homotopy equivalence; if the map is a Dold fibration, then the homotopy equivalence is an FHE [Do, thm.6.1]. Hence the result, just stated for Hurewicz fibrations and spaces in $\mathcal{W}$, also applies when the fibrations are just assumed to be Dold fibrations. Now $p_{\mathcal{F}}: X_{\mathcal{F}} \to B_{\mathcal{F}}$ is an $\mathcal{F}$-fibration, so it is a Dold fibration [Bo2, prop.4.3]. Also $B_{\mathcal{F}}$ is a CW-complex, so if $F \in \mathcal{W}$ then $X_{\mathcal{F}} \in \mathcal{W}$. If also $\mathcal{F}(F) \in \mathcal{W}$, then $\mathcal{F}(F)' \in \mathcal{W}$ and $Y_{\mathcal{F}} = B(*, \mathcal{F}(F)', F) \in \mathcal{W}$ [Ma2, prop.7.2]. So h is a weak homotopy equivalence between spaces in $\mathcal{W}$ and hence a homotopy equivalence.

We have not shown that there is a universal $\mathcal{F}$-fibration over $C_{\mathcal{F}}$ in general, however we do have the following result.

Corollary 4.3 *(i) There is a bijection* $[D, C_{\mathcal{F}}] \to \mathcal{F}FHE(D)$ *that is natural in* D*, where* $D \in \mathcal{W}$.

(ii) If $\mathcal{F}(F) \in \mathcal{W}$*, then there is a universal* $\mathcal{F}$*-fibration over* $C_{\mathcal{F}}$.

Proof. (i) It follows from [Sp, cor.7.6.23] that the weak homotopy equivalence $g: B_{\mathcal{F}} \to C_{\mathcal{F}}$ determines a bijection $g_{\#}: [D, B_{\mathcal{F}}] \to [D, C_{\mathcal{F}}]$ by the rule $g_{\#}[f] = [gf]$, where $[f] \in [D, B_{\mathcal{F}}]$. The result follows by composing the inverse of $g_{\#}$ with the bijection $\theta(p_{\mathcal{F}}): [D, B_{\mathcal{F}}] \to \mathcal{F}FHE(D)$.

(ii) If $\mathcal{F}(F) \in \mathcal{W}$, then g is a homotopy equivalence. Taking k to be a homotopy inverse of g, then $(p_{\mathcal{F}})_k : X_{\mathcal{F}} \sqcap C_{\mathcal{F}} \to C_{\mathcal{F}}$ is an $\mathcal{F}$-fibration.

Let k_- denote the projection $X_{\mathcal{F}} \sqcap C_{\mathcal{F}} \to X_{\mathcal{F}}$. We then have an $\mathcal{E}$-pairwise map $<k_-, k>$ from $(p_{\mathcal{F}})_k$ to $p_{\mathcal{F}}$ which induces, by composition, a pairwise map $<k_{-\#}, k> : prin_F((p_{\mathcal{F}})_k) \to prin_F(p_{\mathcal{F}})$. Now $Prin_F(X_{\mathcal{F}})$ is weakly contractible, so it follows from the exact homotopy ladder associated with $<k_{-\#}, k>$ that $Prin_F(X_{\mathcal{F}} \sqcap C_{\mathcal{F}})$ is weakly contractible. Hence $(p_{\mathcal{F}})_k$ is universal.

[Ma2, thm.9.2 (a)(ii) and (b)(ii)] are similar to particular cases of (ii) of the last corollary, with $A = \emptyset$ in the former case and $A =$ *a one point space* in the latter case.

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