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FIBRATIONS AND CLASSIFYING SPACES: AN AXIOMATIC APPROACH I

by Peter I. BOOTH

Résumé
On développe une théorie unifée des fibrations et des espaces de classification. De telles théories des fibrations sont caractérisées par la categorie $\mathcal{E}$ où leurs fibres doivent se trouver; on demontre ici qu'il existe une théorie intéressante des fibrations-$\mathcal{E}$ et leurs espaces de classification, si $\mathcal{E}$ a des sous-espaces, des cylindres et des cylindres de transformation qui possédent certaines bonnes propriétés, et aussi si $\mathcal{E}$ satisfait à une condition élémentaire de la théorie des ensembles.

1 Introduction

There are numerous published accounts of portions of the theory of fibrations and classifying spaces, and a unifying approach [M] that brings many such ideas together. *This is one of a family of papers that will describe a "streamlined" unified theory. Taken together they will improve the computability, generality, simplicity and unity of what is known in the area.* An account of our overall objectives and results will be given in [B3]; other papers in the series are [B1], [B2] and [B4].

We use terminology derived from that of [M]: $\mathcal{E}$ will denote a *category of enriched topological spaces* or *$\mathcal{E}$-spaces* and $\mathcal{F}$ a *category of fibres* in $\mathcal{E}$, i.e. consisting of all $\mathcal{E}$-spaces of a given $\mathcal{E}$-homotopy type and all $\mathcal{E}$-homotopy equivalences between them. If the fibres of a theory of fibrations are required to lie in such a category $\mathcal{F}$, then we use $\mathcal{F}$ to characterize that theory of enriched or *$\mathcal{F}$-fibrations.* The objectives and results of this paper can be summarized as follows.

Problem *Find conditions on $\mathcal{E}$ that are sufficient to ensure that, for all $\mathcal{F}$ in $\mathcal{E}$, there is a "good" theory of classifying spaces for $\mathcal{F}$-fibrations.*

Solution *It is enough for $\mathcal{E}$ to possess an underlying space functor, subspaces, cylinders and mapping cylinders that are well behaved (definition 2.3), and to satisfy a set-theoretical condition (definition 2.1).*

The argument used *has a modular structure*; to explain this we refer to two potential properties of $\mathcal{F}$-fibrations.

(i) If $f: Y \to Z$ is a homotopy equivalence between spaces and $MC(f)$ denotes the mapping cylinder of f, then both Y and Z can be viewed as strong deformation retracts of $MC(f)$. If an analagous property holds whenever Y and Z are the total spaces of $\mathcal{F}$-fibrations over the same CW-complex and f is a fibre homotopy equivalence in the appropriate sense ($= \mathcal{F}FHE$), then the *double retraction property* will be said to hold for $\mathcal{F}$-fibrations (see definition 3.2).

(ii) Let $q: Y \to C$ be an $\mathcal{F}$-fibration over a CW-complex C, and B be a subcomplex of C. The restriction of q over B will be denoted by $q|B$. If, in such situations, $q|B$ can be replaced by any $\mathcal{F}$-fibration over B that is within an $\mathcal{F}$FHE h of $q|B$, and this procedure does not change the $\mathcal{F}$FHE type of the resulting $\mathcal{F}$-fibration $q \lozenge h$ over C, then the *subfibration replacement property* will be said to hold for $\mathcal{F}$-fibrations (see definition 4.2).

Our main line of argument breaks down into four modules. In *Module I* ($=$ section 3) it is shown that if $\mathcal{E}$ is a well behaved category of enriched spaces and $\mathcal{F}$ is a category of fibres in $\mathcal{E}$, then the double retraction property must be valid for $\mathcal{F}$-fibrations. We show in *Module II* ($=$ section 4), that the last mentioned condition is sufficient to justify the subfibration replacement property for $\mathcal{F}$-fibrations. In *module III* ($=$ sections 5 and 6) the latter property is used, along with our set-theoretical condition, to verify that *Mayer-Vietoris* and *Wedge conditions* hold for $\mathcal{F}$-fibrations. We prove a particular form of the *Brown Representability Theorem* in *Module IV* ($=$ section 7), and use it with the two last mentioned conditions to obtain the required classification of $\mathcal{F}$-fibrations. A condensed version of our main result and the full version of that theorem are given in theorem 2.5 and section 8, respectively. There is some discussion of examples of our theory in section 6 of [B2]; a detailed account will appear in [B3].

Application of Brown's theorem in this area has two apparent disadvantages: the set-theoretical condition needs to be verified, and the relationship between the classifying spaces so produced and those obtained by bar construction procedures seems unclear [M, p.(vi) and (vii)]. The first difficulty is resolved in [B2] and the second in [B4].

We adopt the conventions, notation and terminology of [B1] ; in particular we work in the context of the category $\mathcal{T}$ of *cg- (= compactly generated) spaces* and maps [B1, p.128-129]. The symbols $\simeq$ and $\simeq^0$ will denote homotopy in the free and pointed senses, respectively. Given spaces X and Y, $[X, Y]$ will denote the set of free homotopy classes of free maps from X to Y; for pointed spaces $(X, *)$ and $(Y, *)$, $[X, Y]^0$ will denote the set of pointed homotopy classes of pointed maps from $(X, *)$ to $(Y, *)$. We will use $\mathcal{W}$ to denote the class of spaces that have the (free) homotopy types of CW-complexes, and $\mathcal{W}^0$ the class of pointed spaces that have the pointed homotopy types of pointed CW-complexes.

The following properties of cg-spaces will be useful later.

Lemma 1.1 *Let X be a space and $f: Y \to Z$ be an identification map. Then:*

(i) $(1_X)\times f: X\times Y \to X\times Z$ is also an identification map,

(ii) if V is an open or closed subset of Z, then the subspace topology on V coincides with the identification topology derived from the subspace $Y|V = f^{-1}(V)$ and the surjection $f|V: Y|V \to V$, and

(iii) if X has a finite closed cover, W is a space and $f: X \to W$ is a function, then f is continuous if and only if the restrictions of f to all members of the cover are continuous.

Proof. Result (i) is standard [V1, cor.3.8 and thm.5.1(a)]. In the ordinary topological category, (ii) is known [B(R), 4.3.1, cor 1, (c) and (d)] and (iii) is well known. The cg-versions of (ii) and (iii) can be derived because quotient spaces [V1, cor.2.2 and thm.5.1(a)] and open and closed subspaces [V1, axioms 1* and 1, and thm.5.1(a)] of cg-spaces are themselves cg-spaces.

The author thanks the referee for numerous helpful comments, including the suggestion that a modular structure is appropriate for this work.

2 Terminology, Notation and Classification Theorem

A *category of enriched spaces* $(\mathcal{E}, U)$ consists of a category $\mathcal{E}$ and a faithful functor $U : \mathcal{E} \to \mathcal{T}$. The "underlying space" functor U is such that if Y is an $\mathcal{E}$-object, S is a space and $f : S \to UY$ is a homeomorphism onto the space UY, then there exists a unique $\mathcal{E}$-object X and a unique $\mathcal{E}$-isomorphism $g : X \to Y$, such that $UX = S$ and $Ug = f$. The objects, morphisms and isomorphisms of $\mathcal{E}$ will henceforth be called *$\mathcal{E}$-spaces, $\mathcal{E}$-maps* and *$\mathcal{E}$-homeomorphisms*, respectively.

The definition as written in [B1, p.129] contained a misprint: "into" should have been replaced by "onto". In practise we will often omit U, e.g. many of the following "$\mathcal{E}$-definitions" should really have been phrased as "$(\mathcal{E}, U)$ - definitions".

Let X and Y be $\mathcal{E}$-spaces and $G : X \times I \to Y$ be a map. Then G will be said to be an *$\mathcal{E}$-homotopy* if the composite maps: $X = X \times \{t\} \subset X \times I \overset{G}{\to} Y$ are $\mathcal{E}$-maps, for all $t \in I$. The map $H : X \times I \times I \to Y$ will be said to be an *$\mathcal{E}$-homotopy of homotopies* if the composite maps: $X = X \times \{(t, u)\} \subset X \times I \times I \overset{H}{\to} Y$ are $\mathcal{E}$-maps, for all t and $u \in I$.

If X and Y are $\mathcal{E}$-spaces and there is an *$\mathcal{E}$-inclusion* $i : X \to Y$, i.e. an $\mathcal{E}$-map $i : X \to Y$ such that $Ui : UX \to UY$ is a homeomorphism into, then the $\mathcal{E}$-space X will be said an *$\mathcal{E}$-subspace* of the $\mathcal{E}$-space Y.

An *$\mathcal{E}$-overspace* is a map $q : Y \to C$ together with, for each $c \in C$, an associated structure of an $\mathcal{E}$-space on the *fibre* $Y|c = q^{-1}(c)$.

If $q_0 : Y_0 \to C_0$ and $q_1 : Y_1 \to C_1$ are $\mathcal{E}$-overspaces then an *$\mathcal{E}$-pairwise map* $<f, g>$ from q_0 to q_1 consists of maps $f : Y_0 \to Y_1$ and $g : C_0 \to C_1$ such that $q_1 f = g q_0$, and with the property that, for each $c \in C_0$, $f|(Y_0|c) : Y_0|c \to Y_1|g(c)$ is an $\mathcal{E}$-map.

Taking $C_0 = C_1 = C$ and fixing g to be 1_C, the $\mathcal{E}$-pairwise map concept reduces to *$\mathcal{E}$-map over C*. Taking $C_0 = C \times I$, $C_1 = C$ and fixing g as the projection $C \times I \to C$, there is an obvious associated idea of *$\mathcal{E}$-homotopy over C*, and hence of *$\mathcal{E}$-fibre homotopy equivalence (or $\mathcal{E}$FHE) over C*. An *$\mathcal{E}$-homeomorphism over C* is an isomorphism amongst $\mathcal{E}$-maps over C, i.e. a homeomorphism and $\mathcal{E}$-map over C that is an $\mathcal{E}$-homeomorphism on all individual fibres.

An $\mathcal{E}$-*fibration* is an $\mathcal{E}$-overspace that satisfies the $\mathcal{E}$-*weak covering homotopy property* or $\mathcal{E}WCHP$ [B1, p.136]. We recall that if $q: Y \to C$ is an $\mathcal{E}$-overspace and $C \in \mathcal{W}$ then q is an $\mathcal{E}$-fibration if and only if q is $\mathcal{E}$-*locally homotopy trivial* or $\mathcal{E}LHT$ (see [B1, p.142 and theorem 6.3, also p.141 for numerably contractible]).

Let F be a given $\mathcal{E}$-space. We define $\mathcal{F}$, *the category of fibres* containing F, to be the subcategory of $\mathcal{E}$ consisting of all $\mathcal{E}$-spaces that are $\mathcal{E}$-homotopy equivalent to F and all $\mathcal{E}$-homotopy equivalences between them. Taking $U|\mathcal{F}$ to denote the restriction of U to $\mathcal{F}$, the category of enriched spaces $(\mathcal{F}, U|\mathcal{F})$ will be called *the category of enriched fibres* containing F.

Definitions 2.1 *Let $\mathcal{F}FHE(C)$ denote the class of all $\mathcal{F}FHE$ classes of $\mathcal{F}$-fibrations over C. The category $\mathcal{E}$ will be said to be $\mathcal{E}FHE$ set-valued if, for all categories of fibres $\mathcal{F}$ in $\mathcal{E}$ and all CW-complexes C, the class $\mathcal{F}FHE(C)$ is a set. Hence $\mathcal{F}$ is $\mathcal{F}FHE$ set-valued if, for all CW-complexes C, the class $\mathcal{F}FHE(C)$ is a set.*

If $q: Y \to C$ is an $\mathcal{F}$-overspace and $f: D \to C$ is a map, then there is an *induced $\mathcal{F}$-overspace* $q_f: Y \sqcap D \to D$ [B1, p.130]. If q is an $\mathcal{F}$-fibration, then it is easily seen that q_f is also an $\mathcal{F}$-fibration. If $g: D \to C$ is also a map and $f \simeq g$, then q_f is $\mathcal{F}$FHE to q_g [B1, prop.6.2]. So, if $f: D \to C$ is a homotopy equivalence, the rule $[q] \to [q_f]$ determines a bijection $\mathcal{F}FHE(C) \to \mathcal{F}FHE(D)$. Hence, if $\mathcal{E}$ is $\mathcal{E}FHE$ set-valued, $\mathcal{F}$ is a category of fibres in $\mathcal{E}$ and $C \in \mathcal{W}$, then $\mathcal{F}FHE(C)$ is a set.

Let $\mathcal{F}$ be $\mathcal{F}$FHE set-valued, $q: Y \to C$ be an $\mathcal{F}$-fibration and $f: D \to C$ be a map. Then the rule $([q], [f]) \to [q_f]$ determines a function $\mathcal{F}FHE(C) \times [D, C] \to \mathcal{F}FHE(D)$. Hence the rules $C \to \mathcal{F}FHE(C)$ and $[f] \to ([q] \to [q_f])$ define a contravariant functor $\mathcal{F}FHE(-)$, from the homotopy category of spaces in $\mathcal{W}$ to the category of sets and functions. If we fix q, the rule $[f] \to [q_f]$ defines a natural function $\theta = \theta(q): [D, C] \to \mathcal{F}FHE(D)$, relative to all spaces D in $\mathcal{W}$.

Definitions 2.2 *Let $\mathcal{F}$ be an $\mathcal{F}FHE$ set-valued category of fibres and $p_{\mathcal{F}}: X_{\mathcal{F}} \to B_{\mathcal{F}}$ be a given $\mathcal{F}$-fibration. If, for every $D \in \mathcal{W}$, the cor-*

responding function $\theta(p_{\mathcal{F}}) : [D, B_{\mathcal{F}}] \rightarrow \mathcal{F}FHE(D)$ *is a bijection, then* $p_{\mathcal{F}}$ *will be said to be free universal amongst* $\mathcal{F}$*-fibrations. In that case* $B_{\mathcal{F}}$ *will be called a classifying space for* $\mathcal{F}$*-fibrations.*

In preparation for stating sufficient conditions for the existence of such a $p_{\mathcal{F}}$, we will embark on a discussion of spaces under a fixed space A. The reader may recall that in the main classification theorem 9.2 of [M] there are separate cases (a) and (b) covering the situations where the fibres are "unpointed" and "pointed", respectively; thus the fibres of Hurewicz and principal fibrations are under the empty space $\emptyset$, and those of sectioned fibrations are under a one-point space $*$. In this paper we will develop a single classification theorem that will cover both cases. Thus our A will be either $\emptyset$ or $*$, when our theory is applied to these classical theories. It turns out that this complication is only a "temporary" feature of our argument. Once we have completed module I, by establishing the double retraction property of theorem 3.7, these "under A" ideas will no longer be needed.

Let A be a given topological space. Then (X, i) is a *space under* A if X is a space and $i : A \rightarrow X$ is a map. If (Y, j) is another space under A then a map $f : X \rightarrow Y$ such that $fi = j$ will be called a *map under* A. The category of spaces and maps under A will be denoted by $\mathcal{A}$. There is a functor $U_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{T}$ that forgets both A and maps out of A, and $(\mathcal{A}, U_{\mathcal{A}})$ is clearly a category of enriched spaces. Hence we immediately have associated concepts of $\mathcal{A}$*-homotopy* (denoted by $\simeq^A$) and $\mathcal{A}$*-homotopy of homotopies.*

Let X be a space under A. The $\mathcal{A}$*-cylinder* $X \times_{\mathcal{A}} I$ is defined to be the quotient space of $X \times I$ via the relation $(i(a), t) \sim (i(a), u)$, for all $a \in A$ and t and $u \in I$. There is a map $i^{\perp} : A \rightarrow X \times_{\mathcal{A}} I$ given by $i^{\perp}(a) = [(i(a), 0)]$, where $a \in A$; so $(X \times_{\mathcal{A}} I, i^{\perp})$ is a space under A.

Let $f : X \rightarrow Y$ be a map under A and $j_1 : X \rightarrow X \times_{\mathcal{A}} I$ denote the $\mathcal{A}$-inclusion $j_1 : X \rightarrow X \times_{\mathcal{A}} I$, where $j_1(x) = [(x, 1)]$ for $x \in X$. Then the adjunction space $(X \times_{\mathcal{A}} I) \cup_f Y$, obtained using f and j_1, will be denoted by $\mathcal{A}MC(f)$; it will be called the $\mathcal{A}$*-mapping cylinder for* f. This involves associated maps $f^- : X \times_{\mathcal{A}} I \rightarrow \mathcal{A}MC(f)$ with $f^-[(x, t)] = [(x, t)]$ for $x \in X$ and $t \in I$, and $j_1^- : Y \rightarrow \mathcal{A}MC(f)$ with $j_1^-(y) = [y]$

and $y \in Y$. There is a map $j_1^- j : A \to \mathcal{A}MC(f)$, and so $(\mathcal{A}MC(f), j_1^- j)$ is a space under A. Clearly f^- and j_1^- are maps under A.

$$\begin{array}{ccccc} X & \xrightarrow{\quad f \quad} & Y & \xleftarrow{\quad j \quad} & A \\ \Big\downarrow{\scriptstyle j_1} & & \Big\downarrow{\scriptstyle j_1^-} & & \\ X \times_{\mathcal{A}} I & \xrightarrow[\quad f^- \quad]{} & \mathcal{A}MC(f) = (X \times_{\mathcal{A}} I) \cup_f Y & & \end{array}$$

In the case where $A = \emptyset$, we have $\mathcal{A} = \mathcal{T}$ and $X . \times_{\mathcal{A}} I = X \times I$. The "basic" *mapping cylinder, $\mathcal{T}MC(f)$*, will simply be written as $MC(f)$.

The following definition refers to categories of $\mathcal{E}$-spaces that are under A, in the sense that their underlying spaces and morphisms are in $\mathcal{A}$. Our conditions (iii) and (vii) ensure that the $\mathcal{E}$-spaces $X \times_{\mathcal{E}} I$ and $\mathcal{E}MC(g)$ are under A.

Definition 2.3 *A category of well enriched spaces under a given space A is a triple $(\mathcal{E}, U_{\mathcal{E}}, \{X \times_{\mathcal{E}} I\})$ consisting of a category $\mathcal{E}$ of $\mathcal{E}$-spaces and $\mathcal{E}$-maps, a faithful "underlying $\mathcal{A}$-space" functor $U_{\mathcal{E}}$ from $\mathcal{E}$ to $\mathcal{A}$ and, for each $\mathcal{E}$-space X, an associated $\mathcal{E}$-space $X \times_{\mathcal{E}} I$, to be known as the $\mathcal{E}$-cylinder of X. Given that both X and Y represent arbitrary $\mathcal{E}$-spaces, these data must satisfy the following axioms:*

(i) if S is a space under A and $f : S \to U_{\mathcal{E}}(Y)$ is an $\mathcal{A}$-homeomorphism into, then S can carry at most one corresponding $\mathcal{E}$-subspace structure, i.e. there can be at most one $\mathcal{E}$-space X with $U_{\mathcal{E}}(X) = S$ and such that there exists an $\mathcal{E}$-map $g : X \to Y$ with $U_{\mathcal{E}}(g) = f$,

(ii) in particular, if $f : S \to U_{\mathcal{E}}(Y)$ is an $\mathcal{A}$-homeomorphism onto then there exists such an $\mathcal{E}$-space X and $\mathcal{E}$-homeomorphism $g : X \to Y$ with $U_{\mathcal{E}}(g) = f$,

(iii) $U_{\mathcal{E}}(X \times_{\mathcal{E}} I) = U_{\mathcal{E}}(X) \times_{\mathcal{A}} I$,

(iv) *given* $t \in I$, *the rule* $(U_{\mathcal{E}}(j_t))(x) = [(x,t)]$ *defines an* $\mathcal{E}$*-map* $j_t : X \to X \times_{\mathcal{E}} I$,

(v) *if* $f : U_{\mathcal{E}}(X) \times_{\mathcal{A}} I \to U_{\mathcal{E}}(Y)$ *is a map under* A, *then there is an* $\mathcal{E}$*-map* $g : X \times_{\mathcal{E}} I \to Y$, *with* $U_{\mathcal{E}}(g) = f$, *if and only if there are* $\mathcal{E}$*-maps* $g_t : X \to Y$, *with* $U_{\mathcal{E}}(g_t) = f(U_{\mathcal{E}}(j_t))$, *for all* $t \in I$,

(vi) *given an* $\mathcal{E}$*-map* $g : X \to Y$ *and the map* $j_1 : X \to X \times_{\mathcal{E}} I$, *there is an* $\mathcal{E}$*-space* $\mathcal{E}MC(g)$, *the* $\mathcal{E}$*-mapping cylinder for* g, *and* $\mathcal{E}$*-maps* $j_1^- : Y \to \mathcal{E}MC(g)$ *and* $g^- : X \times_{\mathcal{E}} I \to \mathcal{E}MC(g)$, *making* $\mathcal{E}MC(g)$ *a pushout in* $\mathcal{E}$, *and*

(vii) *if* g *and* j_1 *are as in (vi), then* $U_{\mathcal{E}}(\mathcal{E}MC(g)) = \mathcal{A}MC(U_{\mathcal{E}}(g))$, *and* $U_{\mathcal{E}}$ *takes the pushout square described in (vi) to the adjunction space square that defines* $\mathcal{A}MC(U_{\mathcal{E}}(g))$.

Lemma 2.4 *If* $(\mathcal{E}, U_{\mathcal{E}}, \{X \times_{\mathcal{E}} I\})$ *is a category of well enriched spaces under a space* A, *then there is an associated category of enriched spaces* $(\mathcal{E}, U_{\mathcal{A}}U_{\mathcal{E}})$.

Proof. Clearly $U_{\mathcal{A}}U_{\mathcal{E}} : \mathcal{E} \to \mathcal{T}$ is a faithful functor. Let X be a space, Y be an $\mathcal{E}$-space and $f : X \to U_{\mathcal{A}}U_{\mathcal{E}}(Y)$ a homeomorphism onto. Now $U_{\mathcal{E}}(Y)$ is a space under A via a map $j : A \to U_{\mathcal{A}}U_{\mathcal{E}}(Y)$, so $(X, f^{-1}j)$ is a space under A and f is a $\mathcal{A}$-homeomorphism onto. It follows from condition (ii) of definition 2.3 that X carries an $\mathcal{E}$-structure and there is an $\mathcal{E}$-homeomorphism $g : X \to Y$ with $U_{\mathcal{E}}(g) = f$. The $\mathcal{A}$-structure on X is the only one making f into a $\mathcal{A}$-homeomorphism and hence, using (i) of definition 2.3, the $\mathcal{E}$-structure on X is the only one for which g is an $\mathcal{E}$-homeomorphism and $U_{\mathcal{A}}U_{\mathcal{E}}(g) = f$.

The following classification theorem for $\mathcal{F}$-fibrations is the (a) $\Rightarrow$ (e) portion of our main result, theorem 8.1.

Theorem 2.5 *Let* $(\mathcal{E}, U_{\mathcal{E}}, \{X \times_{\mathcal{E}} I\})$ *be a category of well enriched spaces under a given space* A, *where* $\mathcal{E}$ *is* $\mathcal{E}FHE$ *set-valued. Then, for each category of fibres* $\mathcal{F}$ *in* $\mathcal{E}$, *there is a free universal* $\mathcal{F}$*-fibration* $p_{\mathcal{F}} : X_{\mathcal{F}} \to B_{\mathcal{F}}$ *over a path-connected CW-complex* $B_{\mathcal{F}}$.

The functors $U_{\mathcal{E}}$ and $U_{\mathcal{A}}U_{\mathcal{E}}$ will often be omitted in what follows, e.g. we may use X to denote an $\mathcal{E}$-space, the $\mathcal{A}$-space $U_{\mathcal{E}}(X)$ and the $\mathcal{T}$-space $U_{\mathcal{A}}U_{\mathcal{E}}(X)$.

3 MODULE I: On Retractions between $\mathcal{F}$-fibrations

Definitions 3.1 *Let $(\mathcal{E}, U)$ be a category of enriched spaces, X be an $\mathcal{E}$-subspace of the $\mathcal{E}$-space Y via the $\mathcal{E}$-inclusion $i : X \to Y$, and $r : Y \to X$ be an $\mathcal{E}$-map such that $ri = 1_X$. If, further, there is an $\mathcal{E}$-homotopy under X, from ir to 1_Y, then r will be said to be an $\mathcal{E}$-strong deformation retraction of Y onto X, and X an $\mathcal{E}$-strong deformation retract of Y.*

Let $(\mathcal{E}, U)$ be such that, for every $\mathcal{E}$-homotopy equivalence $h : X \to Y$,
(i) there is an associated $\mathcal{E}$-space $Z(h)$, such that both X and Y are $\mathcal{E}$-strong deformation retracts of $Z(h)$ via $\mathcal{E}$-inclusions $\imath : X \to Z(h)$ and $\jmath : Y \to Z(h)$ and retractions $r : Z(h) \to X$ and $s : Z(h) \to Y$, and
(ii) $s\imath = f$, for all such h.
Then the $\mathcal{E}$-double retraction property will be said to be satisfied.

Let $\mathcal{F}$ be a category of fibres in $\mathcal{E}$. The *category of $\mathcal{F}$-fibrations and $\mathcal{F}$-maps over* the space C will be denoted by $\mathcal{F}C$. If $q : Y \to C$ is an $\mathcal{F}$-fibration, then we will define $U^{\vdash}(q)$ to be the topological space Y. Taking $q_0 : Y_0 \to C$ and $q_1 : Y_1 \to C$ to be $\mathcal{F}$-fibrations and $f : Y_0 \to Y_1$ to be an $\mathcal{F}$-map over C, we define $U^{\vdash}(f)$ to be the map $f : Y_0 \to Y_1$. This determines a functor $U^{\vdash} : \mathcal{F}C \to \mathcal{T}$, and $(\mathcal{F}C,\ U^{\vdash})$ is clearly a category of enriched spaces.

Definition 3.2 *If the $\mathcal{F}C$-double retraction property holds, for all CW-complexes C, then the double retraction property will be said to hold for $\mathcal{F}$-fibrations.*

We assume, for the rest of this module, that $(\mathcal{E}, U_{\mathcal{E}}, \{X \times_{\mathcal{E}} I\})$ is a category of well enriched spaces under A and $\mathcal{F}$ is the category of fibres containing the $\mathcal{E}$-space F. *Our objective here is to show that the double retraction property holds for $\mathcal{F}$-fibrations.*

Lemma 3.3 *(i) Let X be an $\mathcal{E}$-space. Then $k_X : X \times_{\mathcal{E}} I \to X$, where $k_X([(x,t)]) = x$ for $x \in X$ and $t \in I$, is an $\mathcal{E}$-homotopy equivalence. For each $t \in I$, the $\mathcal{E}$-map $j_t : X \to X \times_{\mathcal{E}} I$, $j_t(x) = [(x,t)]$, where $x \in X$, is an $\mathcal{E}$-homotopy inverse to k_X.*

(ii) Let Y and Z also be $\mathcal{E}$-spaces and $f \in \mathcal{E}(X,Y)$. We will assume that $g : \mathcal{E}MC(f) \to Z$ is an $\mathcal{A}$-map. Then g is an $\mathcal{E}$-map if and only if gj_1^- and gf^-j_t are $\mathcal{E}$-maps, for all $t \in I$.

(iii) Let $i_u : \mathcal{E}MC(f) \to \mathcal{E}MC(f) \times I$ be the map defined by $i_u(w) = (w,u)$, where $u \in I$ and $w \in \mathcal{E}MC(f)$. If $G : \mathcal{E}MC(f) \times I \to Z$ is an $\mathcal{A}$-homotopy, then G is an $\mathcal{E}$-homotopy if and only if $Gi_uj_1^-$ and $Gi_uf^-j_t$ are $\mathcal{E}$-maps, for all t and $u \in I$.

(iv) We now refer to the $\mathcal{E}$-homotopy of homotopies concept defined early in §2. Given u and $v \in I$, let $i_{u,v} : \mathcal{E}MC(f) \to \mathcal{E}MC(f) \times I \times I$ be the map defined by $i_{u,v}(w) = (w,u,v)$, where $w \in \mathcal{E}MC(f)$. Given an $\mathcal{A}$-homotopy of homotopies $H : \mathcal{E}MC(f) \times I \times I \to Z$, then H is an $\mathcal{E}$-homotopy of homotopies if and only if $Hi_{u,v}j_1^-$ and $Hi_{u,v}f^-j_t$ are $\mathcal{E}$-maps for all t, u and $v \in I$.

Proof. (i) The $\mathcal{A}$-map k_X is an $\mathcal{E}$-map via (v) of definition 2.3. Now $k_Xj_t = 1_X$ and, again using (v) of definition 2.3, j_tk_X is $\mathcal{E}$-homotopic to the identity on $X \times_{\mathcal{E}} I$. Part (ii) depends on parts (v) and (vi) of definition 2.3; (iii) on the definition of $\mathcal{E}$-homotopy and part (ii) of this result. For (iv), we use the definition of $\mathcal{E}$-homotopy of homotopies and part (ii) of this result.

Proposition 3.4 *The enriched category $(\mathcal{E},\ U_{\mathcal{A}}U_{\mathcal{E}})$ satisfies the double retraction property.*

Proof. (i) Let X and Y be $\mathcal{E}$-spaces and $h : X \to Y$ be an $\mathcal{E}$-homotopy equivalence. We recall that there are $\mathcal{E}$-maps $j_0^- : Y \to \mathcal{E}MC(h)$ and $h^- : X \times_{\mathcal{E}} I \to \mathcal{E}MC(h)$. Then X and Y are $\mathcal{E}$-subspaces of $\mathcal{E}MC(h)$, via associated $\mathcal{E}$-inclusions $\imath = h^-j_0 : X \to \mathcal{E}MC(h)$ and $\jmath = j_1^- : Y \to \mathcal{E}MC(h)$ (see definition 2.3, (vi) and (vii)).

(ii) (compare with [F]). Let $g : Y \to X$ be an $\mathcal{E}$-homotopy inverse of h, and $K : X \times I \to X$ be an $\mathcal{E}$-homotopy from 1_X to gh. We define a map $r = r(h): \mathcal{E}MC(h) \to X$ by $r(y) = g(y)$, where $y \in Y$, and

$$r[(x,t)] = \begin{cases} x & \text{if } 0 \leq t \leq 1/3 \\ K(x, 3t-1) & \text{if } 1/3 \leq t \leq 2/3 \\ gh(x) & \text{if } 2/3 \leq t \leq 1, \end{cases}$$

where $[(x,t)] \in X \times_{\mathcal{E}} I$.

Clearly r is an $\mathcal{A}$-map. We see via lemma 3.3(ii) that it is also an $\mathcal{E}$-map. Further $r\imath = rh^- \jmath_0 = 1_X$. In the $\mathcal{E} = \mathcal{T}$ case, a $\mathcal{T}$-homotopy from $\imath r = h^- \jmath_0 r$ to the identity on $MC(h)$ is defined on p.290 of [F]. That argument generalizes directly, giving an $\mathcal{E}$-homotopy from $\imath r$ to the identity on $\mathcal{E}MC(h)$. The details of the proof involve noticing that the procedure produces an $\mathcal{A}$-homotopy, and checking that this $\mathcal{A}$-homotopy is indeed an $\mathcal{E}$-homotopy. In particular we have to use lemma 3.3(iii) to verify that the combination of $\mathcal{E}$-maps and $\mathcal{E}$-homotopies, by composition and track addition, gives rise to new $\mathcal{E}$-homotopies.

The reader will notice that the justification of a statement on p.289 of [F], i.e. that "$f^*[L] = f_*[K]$", is not given in detail. However what we need at the corresponding point in our argument, in the terminology of [F], is the following: given the existence of f, g and K, we require that there should be an $\mathcal{E}$-homotopy $L: Y \times I \to Y$ from fg to 1_Y such that there is an $\mathcal{E}$-homotopy of homotopies $fK \simeq L(f \times 1_I)$ relative to its endpoints. Now this is justified in the $\mathcal{T}$-case by an elementary argument given in lemmas 1 and 2 of [V2], which generalizes directly and easily using lemma 3.3(iv) to the category $\mathcal{E}$. So r is an $\mathcal{E}$-strong deformation retract of $\mathcal{E}MC(h)$ onto X.

(iii) There is a *squeeze* $\mathcal{A}$-map $s = s(h): \mathcal{A}MC(h) \to Y$ defined by $s(y) = y$, where $y \in Y$, and $s[(x,t)] = h(x)$, where $x \in X$ and $t \in I$. Then $s\jmath = s\jmath_1^- = 1_Y$ and $sh^- \jmath_t = h$, so it follows from lemma 3.3(ii) that s is an $\mathcal{E}$-map. Let us now define an $\mathcal{A}$-homotopy $H: \mathcal{A}MC(h) \times I \to \mathcal{A}MC(h)$ from the identity on $\mathcal{A}MC(h)$ to $\jmath s = \jmath_1^- s$ by $H(y,t) = y$, and

$$H([(x,t)], u) = \begin{cases} (x, 1) & if\ \ t+u \geq 1 \\ (x, t+u) & if\ \ t+u \leq 1\ , \end{cases}$$

where $[(x,t)] \in \mathcal{E}MC(h), y \in Y$ and $t \in I$.

Then $H:\mathcal{E}MC(h)\times I \to \mathcal{E}MC(h)$ is an $\mathcal{E}$-homotopy by lemma 3.3(iii), H is clearly under Y and it follows that s is an $\mathcal{E}$-strong deformation retraction of $\mathcal{E}MC(h)$ onto Y.

(iv) The proof is completed by noticing that $si = sh^{-}j_0 = h$.

Proposition 3.5 *If* $U_{\mathcal{F}} = U_{\mathcal{E}}|\mathcal{F}:\mathcal{F}\to\mathcal{A}$ *and, for each* $\mathcal{F}$*-space* X, $X\times_{\mathcal{F}}I$ *is the* $\mathcal{E}$*-space* $X\times_{\mathcal{E}}I$, *then* $(\mathcal{F}, U_{\mathcal{F}}, \{X\times_{\mathcal{F}}I\})$ *is a category of well enriched spaces under* A.

Proof. We have to check that, for any $\mathcal{F}$-spaces and $\mathcal{E}$-homotopy equivalences between those $\mathcal{F}$-spaces, the $\mathcal{E}$-spaces and $\mathcal{E}$-maps that must then exist according to definition 2.3 are in fact in $\mathcal{F}$.

We know that $j_t:X\to X\times_{\mathcal{E}}I$ is an $\mathcal{E}$-homotopy equivalence (lemma 3.3(i)); hence, if X is in $\mathcal{F}$, so also is $X\times_{\mathcal{F}}I$. For (v), $g_t = gj_t$ for all $t\in I$, so g is an $\mathcal{E}$-homotopy equivalence if and only if all g_t all have that same property.

With respect to (vi), j_1^{-} is an $\mathcal{E}$-homotopy equivalence by (iii) in the proof of proposition 3.4, so if g is a morphism of $\mathcal{F}$, then $\mathcal{E}MC(g)$ is in $\mathcal{F}$. We define $\mathcal{F}MC(g) = \mathcal{E}MC(g)$. Then $g^{-}j_1 = j_1^{-}g$, so $g^{-}j_1$ is an $\mathcal{E}$-homotopy equivalence. But j_1 is an $\mathcal{E}$-homotopy equivalence, hence so also is g^{-}.

We will now show that $\mathcal{F}$-fibrations over C and $\mathcal{F}$-maps over C can be viewed as belonging to *the category* $\mathcal{P}$, *of spaces and maps under the product space* $A\times C$.

Let $q:Y\to C$ an $\mathcal{F}$-fibration. There is an $\mathcal{A}$-space $(Y|c,\ i_c:A\to Y|c)$, for each $c\in C$. We define a *function* $i_Y: A\times C\to Y$ by the rule $i_Y(a,c) = i_c(a)\in Y|c\subset Y$, where $a\in A$ and $c\in C$.

If $i:A\to F$ is the map that makes F into an $\mathcal{A}$-space and $c\in C$, then we define $i_c:A\to F\times\{c\}$ by $i_c(a) = (i(a),c)$. Taking $\pi_C:F\times C\to C$ to be the projection, we have $i_{F\times C} = i\times 1_C: A\times C\to F\times C$. If $q:Y\to C$ is an $\mathcal{F}$-fibration that is $\mathcal{F}$FHE to π_C, then it follows by composition that the function i_Y is continuous. Now any $\mathcal{F}$-fibration $q:Y\to C$ is $\mathcal{F}$LHT relative to a numerable open cover $\mathcal{V}$ of C, so $A\times C$ has the weak topology relative to $\{A\times V\mid V\in\mathcal{V}\}$. It follows that $i_Y|(A\times V) = i_{Y|V}:A\times V\to Y|V$ is continuous, for all choices of $V\in\mathcal{V}$, so i_Y is continuous. Hence $(Y,\ i_Y)$ is a space under $A\times C$.

If $q_0 : Y_0 \to C$ and $q_1 : Y_1 \to C$ are $\mathcal{F}$-fibrations, and $f : Y_0 \to Y_1$ is an $\mathcal{F}$-map over C, then it is easily seen that f is a map under $A \times C$. Hence there is a functor $U_{\mathcal{F}C} : \mathcal{F}C \to \mathcal{P}$, defined by $U_{\mathcal{F}C}(q) = (Y, i_Y)$ and $U_{\mathcal{F}C}(f) = f$.

Let $q : Y \to C$ be an $\mathcal{F}$-overspace of the T_1-space C. Then there is a *cylinder projection map* $cyl(q) : Y \times_{\mathcal{P}} I \to C$, defined by the rule $cyl(q)([(y,t)]) = q(y)$ where $[(y,t)] \in Y \times_{\mathcal{F}C} I$. The T_1-property of C ensures that $Y|c$ is closed in Y, for all $c \in C$. Applying lemma 1.1(ii), with the "f" of that lemma being the identification $Y \times I \to Y \times_{\mathcal{P}} I$, and the closed subset "V" being $(Y|c) \times_{\mathcal{A}} I$, it follows that the quotient space $(Y|c) \times_{\mathcal{A}} I$ is a subspace of $Y \times_{\mathcal{P}} I$. Hence $(Y|c) \times_{\mathcal{A}} I$ is the fibre of $cyl(q)$ over $c \in C$. We give $\mathcal{F}$-space structures $(Y|c) \times_{\mathcal{F}} I$ to such fibres, and then $Y \times_{\mathcal{P}} I$ can be viewed as an $\mathcal{F}$-overspace $Y \times_{\mathcal{F}C} I$ of C. Thus we have an $\mathcal{F}$-overspace $cyl(q) : Y \times_{\mathcal{F}C} I \to C$.

There is an $\mathcal{F}$-map over C, $k_Y : Y \times_{\mathcal{F}C} I \to Y$, $k_Y([(y,t)]) = y$, where $y \in Y$ and $t \in I$. We see, via the line of argument of lemma 3.3(i), that k_Y is an $\mathcal{F}$FHE from $cyl(q)$ to q. Now q is an $\mathcal{F}$-fibration, so it follows that $cyl(q)$ is an $\mathcal{F}$-fibration.

Proposition 3.6 *Let C be a given T_1-space. The corresponding triple $(\mathcal{F}C,\ U_{\mathcal{F}C},\ \{cyl(q) : Y \times_{\mathcal{F}C} I \to C\})$ is then a category of well enriched spaces under $A \times C$. Further $U^{\vdash}$, as defined after definitions 3.1, is $U_{\mathcal{P}} U_{\mathcal{F}C}$ and the associated enriched category is $(\mathcal{F}C,\ U^{\vdash})$.*

Proof. The proof involves routine checking that the properties of definition 2.3 for $\mathcal{F}C$ follow from the analagous properties for $\mathcal{F}$ (see proposition 3.5). We will just check condition (v) - a typical condition - and construct $\mathcal{F}C$-mapping cylinders; the rest is left to the reader.

(a)*Proof of (v).* Let $q_0 : Y_0 \to C$ be an $\mathcal{F}$-fibration. Then there is an $\mathcal{F}C$-map $j_t : Y_0 \to Y_0 \times_{\mathcal{F}C} I$ and there are corresponding $\mathcal{F}$-maps $j_{c,t} : Y_0|c \to (Y_0|c) \times_{\mathcal{F}} I$, where $c \in C$ and $t \in I$ (see definition 2.3(iv)). Let $q_1 : Y_1 \to C$ be an $\mathcal{F}$-fibration, and $f : Y_0 \times_{\mathcal{F}C} I \to Y_1$ be an under $A \times C$ over C map. Then f is an $\mathcal{F}$-map over C if and only if the restriction $f_c = f|((Y_0|c) \times_{\mathcal{F}} I) : (Y_0|c) \times_{\mathcal{F}} I \to Y_1|c$ is an $\mathcal{F}$-map, for all $c \in C$. Now this holds when $f_c j_{c,t} : Y_0|c \to Y_1|c$ is an $\mathcal{F}$-map for all $c \in C$

and $t \in I$, which is equivalent to $fj_t : Y_0 \to Y_1$ being an $\mathcal{F}$-map over C for all $t \in I$.

(b) *$\mathcal{F}C$-mapping cylinders.* Let $q_0 : Y_0 \to C$ and $q_1 : Y_1 \to C$ be a pair of $\mathcal{F}$-fibrations and $f : Y_0 \to Y_1$ an $\mathcal{F}C$-map. Now $U_{\mathcal{F}C}(f)$ is a map under $A \times C$, and we define the *mapping cylinder projection* $mc(f) : \mathcal{P}MC(f) \to C$ by $mc(f)([(y,t)]) = q_0(y)$ where $y \in Y_0$ and $t \in I$, and $mc(f)(y) = q_1(y)$ where $y \in Y_1$. Then, by an argument similar to that used for $\mathcal{F}C$-cylinders, the fibre of $mc(f)$ over $c \in C$ has the same underlying $\mathcal{A}$-space $\mathcal{A}MC(f|(Y_0|c))$ as does the $\mathcal{F}$-space $\mathcal{F}MC(f|(Y_0|c))$. We give such fibres the corresponding $\mathcal{F}$-space structures, and hence obtain an $\mathcal{F}$-overspace $mc(f) : (\mathcal{F}C)MC(f) \to C$.

There is an $\mathcal{F}$-map over C, $s : (\mathcal{F}C)MC(f) \to Y_1$, $s([(y,t)]) = f(y)$ where $y \in Y_0$ and $t \in I$, and $s(y) = y$ where $y \in Y_1$. Then, following the argument of the part (iii) portion of the proof of proposition 3.4, we see that s is an $\mathcal{F}FHE$ from $mc(f)$ to q_1. Now q_1 is an $\mathcal{F}$-fibration hence so also is $mc(f)$.

Theorem 3.7 *If $(\mathcal{E}, U_{\mathcal{E}}, \{X \times_{\mathcal{E}} I\})$ is a category of well enriched spaces under a space A and $\mathcal{F}$ a category of fibres in $\mathcal{E}$, then the double retraction property holds for $\mathcal{F}$-fibrations.*

Proof. This follows from propositions 3.4 and 3.6.

4 MODULE II: Pre–Pasting Procedures

We assume throughout this module that $(\mathcal{E}, U)$ is a category of enriched spaces and $\mathcal{F}$ is a category of fibres in $\mathcal{E}$.

To apply Brown's theorem to our problem we need to be able to paste together certain $\mathcal{F}$-fibrations, i.e. in cases where their restrictions over a common subcomplex B of their base CW-complexes agree to within $\mathcal{F}FHE$. This will be simplified if we can first replace such $\mathcal{F}$-fibrations by $\mathcal{F}$-fibrations that have identical restrictions over B. The essential features of this step are made precise in definition 4.2; we will establish conditions sufficient to justify such a property in this section.

Definitions 4.1 *Let B be a subspace of the space C and $p:X\to B$ and $q:Y\to C$ be $\mathcal{F}$-fibrations. If $h:X\to Y|B$ is an $\mathcal{F}FHE$ from p to $q|B$ then (q,h) will be said to be a p-grounded $\mathcal{F}$-fibration.*

Let $(q_\lambda:Y_\lambda\to C, h_\lambda:X\to Y_\lambda|B)$ be p-grounded $\mathcal{F}$-fibrations, for $\lambda=0$ and 1. Then (q_0,h_0) will be said to be p-grounded $\mathcal{F}FHE$ to (q_1,h_1) if there is an $\mathcal{F}FHE$ $g:Y_0\twoheadrightarrow Y_1$, such that $(g|(Y_0|B))h_0:X\to Y_1|B$ is $\mathcal{F}$-homotopic over B to h_1. In this case, we write $(q_0,h_0)\equiv_p(q_1,h_1)$. If $q|B=p$, then q will be said to be a p-extending $\mathcal{F}$-fibration.

Let $(q,\ h)$ be a p-grounded $\mathcal{F}$-fibration. If $h:X\to Y\,|\,B$ is an $\mathcal{F}$-homeomorphism over B, then we may identify X with $Y|B$ and view q as a p-extending $\mathcal{F}$-fibration.

Let B be a subspace of D, as well as of C, and $f:D\to C$ be a map that extends the identity on B. If $(q,\ h)$ is a p-grounded $\mathcal{E}$-fibration, then so also is $(q_f,\ h)$. If q is a p-extending $\mathcal{F}$-fibration, then so also is q_f.

Definition 4.2 *The subfibration replacement property will be said to be valid for $\mathcal{F}$-fibrations if, for all choices of a subcomplex B of a CW-complex C, an $\mathcal{F}$-fibration $p:X\to B$ and a p-grounded $\mathcal{F}$-fibration $(q:Y\to C,\ h)$, there exists a p-extending $\mathcal{F}$-fibration $q\lozenge h:Y\lozenge X\to C$ such that $(q\lozenge h,1_X)\equiv_p(q,h)$.*

We now give two technical lemmas that will be useful in this section and later.

Lemma 4.3 *Let $p:X\to B$ be an $\mathcal{F}$-fibration and $(q:Y\to C,\ h)$ be a p-grounded $\mathcal{F}$-fibration. Let B be a subspace of D, as well as of C, and $f:D\to C$ be a map that extends the identity on B.*

(i) If S is a space such that $B\subset S\subset D$ then $(q_f|S,h)=(q_{f|S},h)$.

(ii) If D is a CW-complex, $g:D\to C$ is also a map that extends the identity on B, and $f\simeq^B g$, then $(q_f,h)\equiv_p(q_g,h)$.

(iii) Let $(q_0,\ h_0)$ and $(q_1,\ h_1)$ be p-grounded $\mathcal{F}$-fibrations over C. If $k:X^\sharp\to X$ is an $\mathcal{F}FHE$ from the $\mathcal{F}$-fibration $p^\sharp:X^\sharp\to B$ to p, then $(q_0,\ h_0k)$ and $(q_1,\ h_1k)$ are $p^\sharp$-grounded $\mathcal{F}$-fibrations. If, also, $(q_0,\ h_0)$ is p-grounded $\mathcal{F}FHE$ to $(q_1,\ h_1)$, then $(q_0,\ h_0k)$ is $p^\sharp$-grounded $\mathcal{F}FHE$ to $(q_1,\ h_1k)$.

Proof. (i) This is obvious from the definitions involved.

(ii) Let $Y \sqcap D$ denote the space obtained by pulling Y back over f, f_q denote the projection $Y \sqcap D \to Y$ and $H : D \times I \to C$ be an under B homotopy, from f to g, that is stationary on $D \times [0, \frac{1}{2}]$ (see [B1, p.136]). Now $q(f_q) : (Y \sqcap D) \times \{0\} = Y \sqcap D \to C$ is $H(q_f \times 1_I)|((Y \sqcap D) \times \{0\})$ and q satisfies the $\mathcal{F}WCHP$, so there is a homotopy $K : (Y \sqcap D) \times I \to Y$, extending $f_q : (Y \sqcap D) \times \{0\} = Y \sqcap D \to Y$, for which $<K, H>$ is an $\mathcal{F}$-pairwise homotopy [B1, p.130] from $q_f \times 1_I$ to q. Then by the universal property of pullbacks $L = (K, (q_f \times 1_I)) : (Y \sqcap D) \times I \to Y \sqcap (D \times I)$ is an $\mathcal{F}$-map over $D \times I$. The restriction of L over each point $(d, 0)$ of $D \times I$ is the identity on $(Y \sqcap D) \times \{0\}$; it follows from [B1, thm.5.4] that L is an $\mathcal{F}$FHE. Restricting L over $D \times \{1\}$ and forgetting the $\{1\}$s, we obtain an $\mathcal{F}$FHE ℓ from q_f to q_g.

Now $((Y \sqcap D) \times I)|(B \times I)$ can be identified with $(Y|B) \times I$, so the restriction of L over $B \times I$ gives an $\mathcal{F}$-homotopy over B from $1_{Y|B}$ to $\ell|(Y|B)$. Hence ℓ is the required p-grounded $\mathcal{F}$FHE.

(iii) This is immediate.

Lemma 4.4 *(i) If $f : B \to C$ is a cellular map between CW-complexes, then $MC(f)$ is a CW-complex with $B = B \times \{0\}$ as a subcomplex.*

(ii) Let B be a subcomplex of the CW-complex C and $i : B \to C$ be the inclusion. Then $s = s(i) : MC(i) \to C$ – see the $\mathcal{T}$-version of (iii) in the proof of proposition 3.4 – is an under B homotopy equivalence. We will use $n = n(i) : C \to MC(i)$ to denote an under B homotopy inverse of s.

Proof. (i) follows from [FP, ex.2, p.63] and (ii) from [B(R), 7.2.8].

Let $p : X \to B$, $p_0 : X_0 \to B$, and $p_1 : X_1 \to B$ be $\mathcal{F}$-fibrations. We will assume that there are $\mathcal{F}B$-inclusions $X \to X_0$ and $X \to X_1$ (= "homeomorphisms into" that are also $\mathcal{F}$-maps over B). We then have the concept of $\mathcal{F}B$-maps (= $\mathcal{F}$-maps over B) from X_0 to X_1 that are under X, and $\simeq^X_{\mathcal{F}B}$ will denote homotopies of such maps.

Let $p : X \to B$ be an $\mathcal{F}$-fibration and $(q : Y \to C, h)$ be a p-grounded $\mathcal{F}$-fibration. Then $i : B \to C$ and $j : Y|B \to Y$ will denote the inclusion maps. Let us assume that the double retraction property holds for

$\mathcal{F}$-fibrations. Thus there must exist an $\mathcal{F}$-fibration $\mu: Z(h) \to B$, and $\mathcal{F}B$-maps $\imath: X \to Z(h), \jmath: Y|B \to Z(h), \rho: Z(h) \to X$ and $\sigma: Z(h) \to Y|B$, $\imath$ and $\jmath$ also being homeomorphisms into. These must satisfy:

$$\rho\imath = 1_X, \quad \imath\rho \simeq^X_{\mathcal{F}B} 1_{Z(h)}, \quad \sigma\jmath = 1_{Y|B}, \quad \jmath\sigma \simeq^{Y|B}_{\mathcal{F}B} 1_{Z(h)} \text{ and } \sigma\imath = h.$$

We define $Y \natural X$ to be the quotient set of

$$(X \times [0, \frac{1}{3}]) \cup (Z(h) \times [\frac{1}{3}, \frac{2}{3}]) \cup ((Y|B) \times [\frac{2}{3}, 1]) \cup Y,$$

obtained by identifying $(x, \frac{1}{3})$ with $(\imath(x), \frac{1}{3})$ where $x \in X$, $(y, \frac{2}{3})$ with $(\jmath(y), \frac{2}{3})$ where $y \in Y|B$, and $(y, 1)$ with y where $y \in Y|B$. We give $Y \natural X$ the obvious quotient topology. The function $q \natural h: Y \natural X \to MC(i)$ is defined by

$$\begin{aligned}
(q \natural h)(x,t) &= (p(x),t) && \text{if } x \in X,\ t \in [0, \tfrac{1}{3}],\\
(q \natural h)(z,t) &= (\mu(z),t) && \text{if } z \in Z(h),\ t \in [\tfrac{1}{3}, \tfrac{2}{3}],\\
(q \natural h)(y,t) &= (q(y),t) && \text{if } y \in Y|B,\ t \in [\tfrac{2}{3}, 1], \text{ and}\\
(q \natural h)(y) &= q(y) && \text{if } y \in Y.
\end{aligned}$$

Clearly $q \natural h$ is continuous.

If we take the "f" of lemma 1.1(ii) to be the identification map that takes the above union of spaces into $Y \natural X$, we see that the fibres of $q \natural h$ can be identified with the corresponding fibres of p, μ and q. Hence $q \natural h$ can be made into an $\mathcal{F}$-overspace in the evident fashion.

Proposition 4.5 *Let $(q: Y \to C,\ h)$ be a p-grounded $\mathcal{F}$-fibration. If $\mathcal{F}$ is such that the double retraction property holds for $\mathcal{F}$-fibrations, then $q \natural h$ is a p-extending $\mathcal{F}$-fibration.*

Proof. Let us define the function $\alpha: X \times [0, \frac{5}{9}) \to (Y \natural X)|(B \times [0, \frac{5}{9}))$, by $\alpha(x,t) = (x,t)$ for $x \in X$ and $t \in [0, \frac{1}{3})$, and $\alpha(x,t) = (\imath(x), t)$ for $x \in X$ and $t \in [\frac{1}{3}, \frac{5}{9})$. Then α is continuous by lemma 1.1(iii). Hence α is an $\mathcal{F}$-map over $B \times [0, \frac{5}{9})$. Now $\rho\imath = 1_X$, so we can define the $\mathcal{F}$-map $\beta: (Y \natural X)|(B \times [0, \frac{5}{9})) \to X \times [0, \frac{5}{9})$ over $B \times [0, \frac{5}{9})$ by $\beta(x,t) = (x,t)$ for $x \in X$ and $t \in [0, \frac{1}{3})$, and $\beta(z,t) = (\rho(z), t)$ for $z \in Z(h)$ and $t \in [\frac{1}{3}, \frac{5}{9})$. Again using $\rho\imath = 1_X$, we see that $\beta\alpha$ is the identity on $X \times [0, \frac{5}{9})$. Also

$\imath\rho \simeq^{X}_{\mathcal{F}B} 1_{Z(h)}$, so $\alpha\beta$ is $\mathcal{F}$-homotopic over $B\times[0,\frac{5}{9})$ to the identity function on $(Y \natural X)|(B\times[0,\frac{5}{9}))$. Thus α is an $\mathcal{F}$FHE and so $(q\natural h)|(B\times[0,\frac{5}{9}))$ is an $\mathcal{F}$-fibration.

If $f:P\to Q$ is a map and $t \in [0,1)$ then $MC(f,t)$ will denote the quotient space of $P\times(t,1]$ and Q that is obtained by identifying $(z,1)$ with $f(z)$, for all $z \in P$.

Now $q\times 1_I: Y\times I \to C\times I$ is an $\mathcal{F}$-fibration, $MC(i,\frac{4}{9}) \subset C\times(\frac{4}{9},1]$, and $(q\times 1_I)^{-1}(MC(i,\frac{4}{9})) = MC(\jmath,\frac{4}{9}) \subset Y\times(\frac{4}{9},1]$. It follows that the restriction $(q\times 1_I)|MC(i,\frac{4}{9})$, with underlying map $MC(\jmath,\frac{4}{9})\to MC(i,\frac{4}{9})$, is an $\mathcal{F}$-fibration.

Let us define $\gamma: MC(\jmath,\frac{4}{9}) \to (Y\natural X)|MC(i,\frac{4}{9})$ by $(y,t)\to(\jmath(y),t)$, where $y \in Y|B$ and $t \in (\frac{4}{9},\frac{2}{3}]$, and as the identity function elsewhere. Then γ is continuous by lemma 1.1(iii). Hence γ is an $\mathcal{F}$-map over $MC(i,\frac{4}{9})$. We define $\delta:(Y\natural X)|MC(i,\frac{4}{9}) \to MC(\jmath,\frac{4}{9})$ by $(z,t)\to(\sigma(z),t)$, where $z \in Z(h)$ and $t \in (\frac{4}{9},\frac{2}{3}]$, and as the identity elsewhere. Now $\sigma\jmath = 1_{Y|B}$, so δ is well defined. Clearly δ is an $\mathcal{F}$-map over $MC(i,\frac{4}{9})$. It follows from $\sigma\jmath = 1_Y$ that $\delta\gamma$ is the identity on $MC(\jmath,\frac{4}{9})$. Also $\jmath\sigma \simeq^{Y|B}_{\mathcal{F}B} 1_{Z(h)}$ ensures that $\gamma\delta$ is $\mathcal{F}$-homotopic over $MC(i,\frac{4}{9})$ to the identity function on $(Y\natural X)|MC(i,\frac{4}{9})$. So γ is an $\mathcal{F}$FHE, and hence $(q\natural h)|MC(i,\frac{4}{9})$ is an $\mathcal{F}$-fibration.

Now $\{B\times[0,\frac{5}{9}), MC(i,\frac{4}{9})\}$ is a numerable cover of $MC(i)$, hence the $\mathcal{F}$-overspace $q\natural h$ is an $\mathcal{F}$-fibration [B1, thm.4.7]. Identifying the restriction $(q\natural h)|(B\times\{0\})$ with p, we see that $q\natural h$ is p-extending.

Corollary 4.6 *Let B be a subcomplex of the CW-complex C. If s denotes the under B retraction of lemma 4.4(ii), and $qs|(B\times\{0\})$ is identified with $q|B$, then we have $(q\natural h, 1_X) \equiv_p (qs, h)$.*

Proof. Let us define $g: Y\natural X\to Y$ by $g(x,t) = h(x)$ for $x \in X$ and $t \in [0,\frac{1}{3}]$, $g(z,t) = \sigma(z)$ for $z \in Z(h)$ and $t \in [\frac{1}{3},\frac{2}{3}]$, $g(y,t) = y$ for $y \in Y|B$ and $t \in [\frac{2}{3},1]$, and $g(y) = y$ for $y \in Y$. Now $\sigma\imath = h$ and $\sigma\jmath = 1_{Y|B}$ so g is a well defined map. Then $qg = s(q\natural h))$, $<g,\ s>$ is an $\mathcal{F}$-pairwise map from $q\natural h$ to q and $k = (g, q\natural h): Y\natural X\to Y\sqcap MC(i)$ an $\mathcal{F}$-map over $MC(i)$.

If $c\in C$ then $k\,|\,((Y\natural X)\,|\,c) = k\,|\,(Y\,|\,c)$ is the obvious canonical $\mathcal{F}$-homeomorphism $Y|c\to(Y|c)\times\{c\}$, and if $b \in B$ and $t \in I$ then (b,t)

is in the same path component of $MC(i)$ as $i(b) \in C$. Now $MC(i)$ is a CW-complex (lemma 4.4(i)), so it follows from [B1, thm.5.4] that k is an $\mathcal{F}$FHE. Identifying $X\times\{0\}$ with X and the space obtained by pulling Y back over $s|(B\times\{0\})$ with $Y|B$, we see that $k|X = h$. So k is the required p-grounded $\mathcal{F}$FHE.

Definition 4.7 *Let $(q: Y \to C,\ h)$ be a p-grounded $\mathcal{F}$-fibration. Taking $n = n(i)$ as in lemma 4.4(ii), we define $q \lozenge h : Y \lozenge X \to C$ to be the induced p-extending $\mathcal{F}$-fibration $(q \natural h)_n$.*

Theorem 4.8 *Let $(\mathcal{E}, U)$ be a category of enriched spaces and $\mathcal{F}$ be a category of fibres in $\mathcal{E}$. If the double retraction property is valid for $\mathcal{F}$-fibrations, then so also is the subfibration replacement property. If (q, h) is a p-grounded $\mathcal{F}$-fibration, then the p-extending $\mathcal{F}$-fibration $q \lozenge h$ is as specified in definition 4.7.*

Proof. We adopt the terminology of definition 4.7. Pulling the k of corollary 4.6 back over n we obtain a p-grounded $\mathcal{F}$FHE from $(q \lozenge h, 1_X)$ to $((q_s)_n, h)$. Now $((q_s)_n, h) = (q_{(sn)}, h)$, and $(q_{(sn)}, h) \equiv_p (q, h)$ by lemma 4.3(ii) since $sn \simeq^B 1_C$ by lemma 4.4(ii). Hence $(q \lozenge h, 1_X) \equiv_p (q, h)$.

5 MODULE IIIa: Combining Fibrations by Pasting

We assume, throughout this module, that $(\mathcal{E}, U)$ is a category of enriched spaces, F is a given $\mathcal{E}$-space and $\mathcal{F}$ is the category of fibres containing F.

Let us assume that $p: X \to B$ is a given $\mathcal{F}$-fibration and that $q: Y \to C$ is a p-extending $\mathcal{F}$-fibration. We use $i: B \to C$ and $j: X \to Y$ to denote the inclusion maps. Then we define $q^*: MC(j) \to MC(i)$ by $q^*(x,t) = (p(x), t)$, for $x \in X$ and $t \in I$, and $q^*(y) = y$, for $y \in Y$. Now $MC(j)$ and $MC(i)$ are subspaces of $Y\times I$ and $C\times I$, respectively, q^* is the restriction of the $\mathcal{F}$-fibration $q\times 1_I: Y\times I \to C\times I$ over $MC(i)$, and hence q^* is an $\mathcal{F}$-fibration. We will refer to q^* as the *p-cylindrical $\mathcal{F}$-fibration associated with q*. Identifying $p\times 1_{\{0\}}: X\times\{0\}\to B\times\{0\}$ with $p: X\to B$ we see that q^* becomes a p-extending $\mathcal{F}$-fibration, and $(q^*, 1_X)$ a p-grounded $\mathcal{F}$-fibration.

When p-extending $\mathcal{F}$-fibrations are pasted together by identifying their copies of X, we may not know if the resulting objects are themselves $\mathcal{F}$-fibrations. This difficulty does not, however, occur if we are able to use p-cylindrical $\mathcal{F}$-fibrations. In this section we investigate the relationship between p-extending $\mathcal{F}$-fibrations and their associated p-cylindrical $\mathcal{F}$-fibrations, and then describe a procedure for pasting such cylindrical $\mathcal{F}$-fibrations together.

Proposition 5.1 *Let B be a subcomplex of the CW- complex C, $p: X\to B$ be an $\mathcal{F}$-fibration and $q: Y\to C$ be a p-extending $\mathcal{F}$-fibration.*

(i) If $s: MC(i)\to C$ denotes the usual retraction, then there is an induced p-extending $\mathcal{F}$-fibration, $q_S: Y\sqcap MC(i)\to MC(i)$. Also there is an $\mathcal{F}$-homeomorphism $MC(j)\to Y\sqcap MC(i)$ over $MC(i)$, i.e. from q^* to q_S, that restricts over B to 1_X.

(ii) $((q^*)n, 1_X)\equiv_p (q, 1_X)$, where $n: C\to MC(i)$ is as in lemma 4.4(ii).

Proof. (i) Let $\pi: C\times I\to C$ denote the projection. Then the induced $\mathcal{F}$-fibrations $q_\pi: Y\sqcap(C\times I)\to C\times I$ and $q\times 1_I: Y\times I\to C\times I$ can be identified together, via an $\mathcal{F}$-homeomorphism $g: Y\times I\to Y\sqcap(C\times I)$ over $C\times I$, i.e. $g(y,t)=(y,(q(y),t))$ where $y\in Y$ and $t\in I$. Restricting these $\mathcal{F}$-fibrations over $MC(i)\subset C\times I$, we obtain the p-extending $\mathcal{F}$-fibrations q_S and q^*. Then $k=g|MC(j): MC(j)\to Y\sqcap MC(i)$ is an $\mathcal{F}$-homeomorphism over $MC(i)$, with $k|X=1_X$.

(ii) By part (i), $((q^*)n, 1_X)\equiv_p ((q_S)n, 1_X) = (q_{S}n, 1_X)$, and by lemma 4.3(ii) $(q_{S}n, 1_X)\equiv_p (q, 1_X)$.

We introduce a generalization of the wedge construction. Let B be a subspace of each of the family of spaces $\{C_\lambda\}_{\lambda\in\Lambda}$; we define $\nabla_{\lambda\in\Lambda}C_\lambda$ to be the quotient space obtained from $\bigsqcup_{\lambda\in\Lambda}C_\lambda$ by pasting (i.e. identifying) together the copies of B. If $f_\lambda: B\to C_\lambda$ are maps, for $\lambda\in\Lambda$, then we define the *multiple mapping cylinder* $MMC(f_\lambda:\Lambda)$ to be the space $\nabla_{\lambda\in\Lambda}MC(f_\lambda)$ that is obtained by identifying the copies of $B\times\{0\}$.

The identified copies of $B\times\{0\}$ will in turn be identified with B; hence B is a subspace of $MMC(f_\lambda:\Lambda)$. In particular if $\Lambda=\{0,1\}$ the corresponding *double mapping cylinder* will be denoted by $DMC(f_0,f_1)$. If also $B=C_0\cap C_1$ and f_0 and f_1 are inclusions then $\nabla_{\lambda\in\Lambda}C_\lambda$ can be identified with $C_0\cup C_1$.

Lemma 5.2 *(i) Let $J=\bigvee_{\lambda\in\Lambda}[0,\frac{1}{2})_\lambda$, the indexed intervals $[0,\frac{1}{2})_\lambda$ being attached by identifying 0s; then $B\times J$ is an open subspace of $MMC(f_\lambda:\Lambda)$.*

(ii) If $f_\lambda:B\to C_\lambda$ are cellular maps between CW-complexes, for all $\lambda\in\Lambda$, then $MMC(f_\lambda:\Lambda)$ is a CW-complex that contains all of the CW-complexes $MC(f_\lambda)$ as subcomplexes, for all $\lambda\in\Lambda$. In particular if $\Lambda=\{0,1\}$ then $DMC(f_0,f_1)$ is a CW-complex containing $MC(f_0)$ and $MC(f_1)$ as subcomplexes.

(iii) Let B be a subcomplex of each of the CW-complexes C_λ and $i_\lambda:B\to C_\lambda$ denote the inclusions, for all $\lambda\in\Lambda$. Then there is an under B homotopy equivalence $S:MMC(i_\lambda:\Lambda)\to\nabla_{\lambda\in\Lambda}C_\lambda$; N will denote an under B homotopy inverse of S.

(iv) If $B=C_0\cap C_1$ is a subcomplex of the CW-complexes C_0 and C_1, with inclusions $i_0:B\to C_0$ and $i_1:B\to C_1$, then there is an under B homotopy equivalence $S:DMC(i_0,i_1)\to C_0\cup C_1$. N will denote an under B homotopy inverse of S.

Proof. (i) The spaces $B\times J$ and $\nabla_{\lambda\in\Lambda}(B\times[0,\frac{1}{2})_\lambda)$ are the same quotient space of $B\times(\bigsqcup_{\lambda\in\Lambda}[0,\frac{1}{2})_\lambda)=\bigsqcup_{\lambda\in\Lambda}(B\times[0,\frac{1}{2})_\lambda)$ (use lemma 1.1(i)). Also $\bigsqcup_{\lambda\in\Lambda}(B\times[0,\frac{1}{2})_\lambda)$ is open in $\bigsqcup_{\lambda\in\Lambda}MC(f_\lambda)$, so applying lemma 1.1(ii) with the obvious identification $\bigsqcup_{\lambda\in\Lambda}MC(f_\lambda)\to MMC(f_\lambda:\Lambda)$, we see that $\nabla_{\lambda\in\Lambda}(B\times[0,\frac{1}{2})_\lambda)$ is an open subspace of $MMC(f_\lambda:\Lambda)$. Thus $B\times J$ is an open subspace of the latter space.

(ii) $MMC(f_\lambda:\Lambda)$ can be viewed as the adjunction space determined by the map $\bigsqcup_{\lambda\in\Lambda}B_\lambda\to B$ that identifies together a family of indexed copies B_λ of B, and the inclusion $\bigsqcup_{\lambda\in\Lambda}B_\lambda=\bigsqcup_{\lambda\in\Lambda}(B_\lambda\times\{0\})\subset\bigsqcup_{\lambda\in\Lambda}MC(f_\lambda)$. The result follows from lemma 4.4(i) and [FP, thm.2.3.1].

For (iii) we attach copies of $s_\lambda=s(i_\lambda):MC(i_\lambda)\to C_\lambda$ to form S, and of $n_\lambda=n(i_\lambda):C_\lambda\to MC(i_\lambda)$ to obtain N (see lemma 4.4(ii)).

If $\Lambda=\{0,1\}$ then (iii) reduces to (iv).

Let B be a subcomplex of the CW-complex C_λ, with inclusion $i_\lambda : B \to C_\lambda$, for all $\lambda \in \Lambda$. Also let $q_\lambda : Y_\lambda \to C_\lambda$ be a p-extending $\mathcal{F}$-fibration with inclusion $j_\lambda : X \to Y_\lambda$, for all $\lambda \in \Lambda$. Then we define

$$\Lambda^\bullet : MMC(j_\lambda : \Lambda) \to MMC(i_\lambda : \Lambda)$$

by $\Lambda^\bullet(y) = q_\lambda(y)$ if $y \in Y_\lambda$, and $\Lambda^\bullet(x,t) = (p(x), t)$, if $x \in X$ and t is in the indexed interval $I_\lambda = [0,1]_\lambda$. Clearly $\Lambda^\bullet$ is an $\mathcal{F}$-overspace.

Proposition 5.3 *In the notation just given, the $\mathcal{F}$-overspace $\Lambda^\bullet$ is a p-extending $\mathcal{F}$-fibration. Further $\Lambda^\bullet | MC(i_\lambda) = (q_\lambda)^*$, for all $\lambda \in \Lambda$. Such a $\Lambda^\bullet$ will be said to be a p-multicylindrical $\mathcal{F}$-fibration.*

Proof. For all $\lambda \in \Lambda$, $MC(i_\lambda)$ and $MC(j_\lambda)$ are subspaces of the spaces $MMC(i_\lambda : \Lambda)$ and $MMC(j_\lambda : \Lambda)$, respectively. Hence each $\Lambda^\bullet | MC(i_\lambda)$ is the corresponding cylindrical $\mathcal{F}$-fibration $(q_\lambda)^*$, for all $\lambda \in \Lambda$.

Now $MC(i_\lambda, 0)$ is an open subspace of $MC(i_\lambda)$, so $\Lambda^\bullet | MC(i_\lambda, 0)$ coincides with $(q_\lambda)^* | MC(i_\lambda, 0)$ and is itself an $\mathcal{F}$-fibration, for all $\lambda \in \Lambda$. $B \times J$ and $X \times J$ are subspaces of $MMC(i_\lambda : \Lambda)$ and $MMC(j_\lambda : \Lambda)$, respectively (proposition 5.2(i)). So $\Lambda^\bullet | (B \times J)$ must be the $\mathcal{F}$-fibration $p \times 1_J : X \times J \to B \times J$. The CW-complex $MMC(i_\lambda : \Lambda)$ has a numerable open cover consisting of $B \times J$ and all open sets in $\{MC(i_\lambda, 0)\}_{\lambda \in \Lambda}$ [FP, thm.A.3.3]. It follows that $\Lambda^\bullet$ is an $\mathcal{F}$-fibration [B1, thm.4.7].

Clearly B and X are subspaces of $MMC(i_\lambda : \Lambda)$ and $MMC(j_\lambda : \Lambda)$, respectively, for all $\lambda \in \Lambda$. So $\Lambda^\bullet | B = p$ and $\Lambda^\bullet$ is p-extending.

We consider an *example* of the above construction. Given that $q : Y \to \nabla_{\lambda \in \Lambda} C_\lambda$ is a p-extending $\mathcal{F}$-fibration, we define q_λ to be the $\mathcal{F}$-fibration $q | C_\lambda : Y | C_\lambda \to C_\lambda$, for all $\lambda \in \Lambda$. The associated $\Lambda^\bullet$ will be denoted by $q^\bullet : Y^\bullet \to MMC(i_\lambda : \Lambda)$.

Proposition 5.4 *Let $i_\lambda : B \to C_\lambda$ denote the inclusion of the subcomplex B of the CW-complex C_λ, for all $\lambda \in \Lambda$. We take q, q_λ and $q^\bullet$ as in the preceding example and $j_\lambda : Y | C_\lambda \to Y$ as the inclusion.*

(i) If $S : MMC(i_\lambda : \Lambda) \to \nabla_{\lambda \in \Lambda} C_\lambda$ is the homotopy equivalence under B of lemma 5.2(iii), then $q_S : Y \sqcap MMC(i_\lambda : \Lambda) \to MMC(i_\lambda : \Lambda)$ is a p-extending $\mathcal{F}$-fibration. Furthermore, there is an $\mathcal{F}$-homeomorphism

$Y^\bullet \to Y \sqcap MMC(i_\lambda : \Lambda)$ *over the space* $MMC(i_\lambda : \Lambda)$, *i.e. from* $q^\bullet$ *to* q_S, *that restricts over* B *to* 1_X.

(ii) $((q^\bullet)_N, 1_X) \equiv_p (q, 1_X)$, *where* N *is as in lemma 5.2(iii).*

(iii) If $\Lambda^\bullet$ *is as in proposition 5.3, then there exists a p-extending* $\mathcal{F}$*-fibration* $q : Y \to \nabla_{\lambda\in\Lambda} C_\lambda$ *such that* $(q^\bullet, 1_X) \equiv_p (\Lambda^\bullet, 1_X)$.

Proof. (i) Let I_λ denote a copy of I that is indexed by $\lambda \in \Lambda$. We define the $\mathcal{F}$-map $\ell : Y^\bullet \to Y \sqcap MMC(i_\lambda : \Lambda)$ by $\ell(y) = (y, q(y))$ for $y \in Y_\lambda$, and $\ell(x,t) = (x, (p(x), t)$ for $(x,t) \in X \times I_\lambda$. Let $s(\lambda) = s(i_\lambda) : MC(i_\lambda) \to C_\lambda$ denote the usual retraction. We see, via lemma 5.1(i), that the restriction of ℓ over $MC(i_\lambda)$ is an $\mathcal{F}$-homeomorphism $MC(j_\lambda) \to (Y|C_\lambda) \sqcap MC(i_\lambda)$ over $MC(i_\lambda)$. It then follows, utilising lemma 1.1(iii), that ℓ is a homeomorphism over $MMC(i_\lambda : \Lambda)$.

(ii) This is a direct generalization of the proof of proposition 5.1(ii).

(iii) Let N and S be as in lemma 5.2(iii). We define q to be $(\Lambda^\bullet)_N$. Then $(q^\bullet, 1_X) = (((\Lambda^\bullet)_N)^\bullet, 1_X) \equiv_p (((\Lambda^\bullet)_N)_S, 1_X)$ by (i) of this proposition. Further $(((\Lambda^\bullet)_N)_S, 1_X) = ((\Lambda^\bullet)_{NS}, 1_X) \equiv_p (\Lambda^\bullet, 1_X)$ by lemma 4.3(ii). Hence we have $(q^\bullet, 1_X) \equiv_p (\Lambda^\bullet, 1_X)$.

6 MODULE IIIb: On the Mayer-Vietoris and Wedge Conditions

If B is a singleton space and base point $*$ for C, F a given $\mathcal{F}$-space and p the $\mathcal{F}$-fibration $F \to *$ then various p-concepts simplify.

Definitions 6.1 *An* F*-grounded* $\mathcal{F}$*-fibration* (q, h) *is an* $\mathcal{F}$*-fibration* $q : Y \to C$ *over a pointed space* C*, and an* $\mathcal{F}$*-homotopy equivalence* $h : F \to Y|*$*. For example, if* $\pi_C : F \times C \to C$ *denotes the projection, then* $(\pi_C, 1_F)$ *is the trivial* F*-grounded* $\mathcal{F}$*-fibration over* C.

If $(q_\lambda : Y_\lambda \to C,\ h_\lambda : F \to Y_\lambda|*)$*, for* $\lambda = 0$ *and* 1*, are* F*-grounded* $\mathcal{F}$*-fibrations then an* $\mathcal{F}FHE$ $g : Y_0 \to Y_1$ *will be said to be an* F*-grounded* $\mathcal{F}FHE$ *from* (q_0, h_0) *to* (q_1, h_1) *if* $h_1 \simeq_{\mathcal{F}} (g|(Y_0|*))h_0$*, where the range of this homotopy is restricted to* $Y_1|*$*. We then write* $(q_0, h_0) \equiv_F (q_1, h_1)$.

The class $\mathcal{F}FHE^F(C)$ *will consist of all* F*-grounded* $\mathcal{F}FHE$*-classes of* F*-grounded* $\mathcal{F}$*-fibrations over* C*. Then* $\mathcal{F}$ *will be said to be* $\mathcal{F}FHE^F$

set-valued if $\mathcal{F}FHE^F(C)$ is a set, for all choices of a pointed CW-complex C.

The set-theoretical difficulty referred to in §1 is the problem of ensuring that, for all spaces in $\mathcal{W}^0$, the classes $\mathcal{F}FHE^F(C)$ are sets. Otherwise $\mathcal{F}FHE^F$ is not a functor and Brown's theorem cannot be applied.

If (q, h) is an F-grounded $\mathcal{F}$-fibration over C and $f: D \to C$ is a pointed map then, using the identification $(Y|*)\times\{*\} = Y|*$, there is an *F-grounded $\mathcal{F}$-fibration induced from (q,h) by f*, i.e. (q_f, h). If D is a pointed CW-complex and f and g are pointed maps from D to C such that $f \simeq^0 g$, then $(q_f, h) \equiv_F (q_g, h)$ (lemma 4.3(ii)).

We argued after definition 2.1 that, if $\mathcal{F}$ is $\mathcal{F}$FHE set-valued and $C \in \mathcal{W}$, then $\mathcal{F}FHE(C)$ is a set. The parallel "pointed" argument uses lemma 4.3(ii) to show that, if $\mathcal{F}$ is $\mathcal{F}FHE^F$ set-valued and $C \in \mathcal{W}^0$, then $\mathcal{F}FHE^F(C)$ is a set.

Lemma 6.2 *If $\mathcal{F}$ is $\mathcal{F}FHE$ set-valued and the subfibration replacement property for $\mathcal{F}$-fibrations is satisfied, then $\mathcal{F}$ is $\mathcal{F}FHE^F$ set-valued.*

Proof. Let C be a pointed CW-complex. We see, from the subfibration replacement property for $\mathcal{F}$-fibrations, that each $\mathcal{F}$FHE class of $\mathcal{F}$-fibrations over C contains a member with distinguished fibre F. Let $R(C)$ be a set consisting of one representative from each $\mathcal{F}$FHE class of $\mathcal{F}$-fibrations over C with such fibres.

If $(q: Y \to C,\ \ h: F \to q^{-1}(*))$ is an F-grounded $\mathcal{F}$-fibration then there is a q' in $R(C)$ that is $\mathcal{F}FHE$ to q. Hence (q,h) is $\mathcal{F}FHE^F$ to (q', h'), for some choice of an $\mathcal{F}$-homotopy equivalence $h': F \to F$. So $[(q,h)] = [(q',h')]$, and $\mathcal{F}FHE^F(C)$ is a set. It follows that $\mathcal{F}$ is $\mathcal{F}FHE^F$ set-valued.

Let $\mathcal{F}$ be $\mathcal{F}FHE^F$ set-valued. Then $C \to (\mathcal{F}FHE^F(C),\ [(\pi_C, 1_F)])$, defines a function from $\mathcal{W}^0$ to the class of pointed sets, where $(\pi_C, 1_F))$ is as defined in definitions 6.1.

Also, any pointed map $f: D \to C$ between spaces in $\mathcal{W}^0$ determines a function $f^{\#}: \mathcal{F}FHE^F(C) \to \mathcal{F}FHE^F(D)$, with $([(q,h)]) \to [(q_f, h)])$, where (q, h) is an F-grounded $\mathcal{F}$-fibration over C. If g is also a pointed map from D to C and $f \simeq^0 g$, then $(q_f, h) \equiv_F (q_g, h)$, and so $f^\# = g^\#$. Further these functions $f^\#$ preserve the distinguished $\mathcal{F}FHE$-classes of F-grounded $\mathcal{F}$-fibrations, i.e. $f^\#([(\pi_C, 1_F)]) = [(\pi_D, 1_F)]$. Also, if $k: K \to D$ and $f: D \to C$ are pointed maps between pointed spaces, then $(fk)^\# = k^\# f^\#$ clearly holds.

Let us assume that $\mathcal{F}$ is $\mathcal{F}$FHE set-valued, and that the subfibration replacement property holds for $\mathcal{F}$-fibrations. Then *there is a contravariant functor $\mathcal{F}FHE^F(-)$, from the homotopy category of spaces in $\mathcal{W}^0$ to the category of pointed sets and pointed functions.* The rule for morphisms is given by $\mathcal{F}FHE^F([f]) = f^\#$, where f is a pointed map between spaces in $\mathcal{W}^0$.

Let H be a contravariant functor from the homotopy category of pointed CW-complexes to the category of pointed sets. If B is a pointed subcomplex of the pointed CW-complex C, i denotes the inclusion $B \to C$ and $u \in H(C)$, then $u|B$ will be used to denote $H([i])(u) \in H(B)$.

We consider the situation where C_0 and C_1 are pointed subcomplexes of the pointed CW-complex $C_0 \cup C_1$, with $B = C_0 \cap C_1$ being a pointed subcomplex of both C_0 and C_1. Then H will be said to satisfy the *Mayer-Vietoris condition* if, for all such C_λ and all $u_\lambda \in H(C_\lambda)$, for $\lambda = 0$ and 1, and such that $u_0|B = u_1|B$, then there exists a $u \in H(C_0 \cup C_1)$ with $u|C_\lambda = u_\lambda$, for $\lambda = 0$ and 1.

Theorem 6.3 *Let $(\mathcal{E}, U)$ be a category of enriched spaces, F be a given $\mathcal{E}$-space and $\mathcal{F}$ be the category of fibres containing F. If $\mathcal{F}$ is $\mathcal{F}FHE$ set-valued and the subfibration replacement property holds for $\mathcal{F}$-fibrations, then the functor $\mathcal{F}FHE^F(-)$ satisfies the Mayer-Vietoris condition.*

Proof. Let B, C_0 and C_1 be as specified in the statement of the Mayer-Vietoris condition. Also let $i(\lambda) : B \to C_\lambda$, $j(\lambda) : C_\lambda \to C_0 \cup C_1$ and $\iota(\lambda) : MC(i(\lambda)) \to DMC(i(0), i(1))$ denote the inclusions, for $\lambda \in \Lambda = \{0, 1\}$. If $*$ denotes the basepoint of B then $(*, 0)$ will be the basepoint of $MC(i(0)), MC(i(1))$, and $DMC(i(0), i(1))$. We then have based homotopy equivalences $s(0): MC(i(0)) \to C_0$, $s(1): MC(i(1)) \to C_1$, and $S: DMC(i(0), i(1)) \to C_0 \cup C_1$ (see lemmas 4.4(ii) and 5.2(iv)). We will use $n(0), n(1)$ and N to denote respective based homotopy inverses of these maps. We further assume that $(q_\lambda : Y_\lambda \to C_\lambda, h_\lambda)$ are F-grounded $\mathcal{F}$-fibrations, for $\lambda \in \{0,1\}$, such that $(q_0|B, h_0) \equiv_F (q_1|B, h_1)$ via an $\mathcal{F}$FHE $h : Y_0|B \to Y_1|B$. So in particular $(h|(Y_0|*))h_0$ is $\mathcal{F}$-homotopic to h_1.

It follows from the subfibration replacement property for $\mathcal{F}$-fibrations that there must be an $\mathcal{F}$-fibration $q_1 \lozenge h : Y_1 \lozenge (Y_0|B) \to C_1$ such that $(q_1 \lozenge h)|B = q_0|B$. Then, taking 1 to denote the identity on $Y_0|*$, $(q_1 \lozenge h,\ 1)$ is $(Y_0|*)$-grounded $\mathcal{F}$FHE to $(q_1,\ h|(Y_0|*))$. Hence by lemma 4.3(iii) $(q_1 \lozenge h,\ h_0) \equiv_F (q_1,\ (h|(Y_0|*))h_0) \equiv_F (q_1,\ h_1)$. Now q_0 and $q_1 \lozenge h$ are both $(q_0|B)$-extending $\mathcal{F}$-fibrations, and so proposition 5.3 allows us to define a $(q_0|B)$-extending $\mathcal{F}$-fibration $\Lambda^\bullet$ over $DMC(i(0), i(1))$. Then:

$((\Lambda^\bullet)_N|C_0, h_0) = ((\Lambda^\bullet)_{Nj(0)}, h_0)$ by lemma 4.3(i)
$\quad = ((\Lambda^\bullet)_{\iota(0)n(0)}, h_0)$ since $Nj(0) = \iota(0)n(0)$,
$\quad = (((\Lambda^\bullet)_{\iota(0)})_{n(0)}, h_0)$
$\quad \equiv_F (((q_0)^*)_{n(0)}, h_0)$ by proposition 5.3
$\quad \equiv_F (((q_0)_{s(0)})_{n(0)}, h_0)$ by proposition 5.1(i)
$\quad = ((q_0)_{s(0)n(0)}, h_0)$
$\quad \equiv_F (q_0, h_0)$ by lemma 4.3(ii)

Hence $((\Lambda^\bullet)_N|C_0, h_0) \equiv_F (q_0, h_0)$.

If we repeat this series of equivalences, but starting instead with $((\Lambda^\bullet)_N|C_1, h_0)$ and replacing $C_0, j(0), \iota(0), n(0), s(0), q_0$ and $(q_0)^*$ by C_1, $j(1), \iota(1), n(1), s(1), q_1 \lozenge h$ and $(q_1 \lozenge h)^*$ respectively, we will then obtain $((\Lambda^\bullet)_N|C_1, h_0) \equiv_F (q_1 \lozenge h, h_0)$. Now $(q_1 \lozenge h, h_0) \equiv_F (q_1, h_1)$, so $((\Lambda^\bullet)_N|C_1, h_0) \equiv_F (q_1, h_1)$. Hence $[((\Lambda^\bullet)_N, h_0)]$ is the required "u".

If C is a pointed space then the associated *whiskered space* C' will be C with a *whisker* grown at its distinguished point, i.e. $C' = MC(i)$ where $i : * \to C$ takes $*$ to the basepoint of C. Then $(*, 0)$ at the isolated end of the whisker will be the basepoint of C'. If C is a CW-complex then so is C', (lemma 4.4(i)) and $(*, 0)$ can be taken to be a 0-cell (and $\{(*, 0)\}$ a subcomplex) of C' [FP, lemma 2.3.7].

We will again take p to be the map $F \to *$; further p-concepts then simplify. Thus $q : Y \to C$ is an *F-extending $\mathcal{F}$-fibration* means that q is an $\mathcal{F}$-fibration with $Y|* = F$. We then have associated concepts of an $\mathcal{F}$-fibration over C' being *F-cylindrical* and of an $\mathcal{F}$-fibration over $\bigvee_{\lambda\in\Lambda} C'_\lambda$ being *F-multicylindrical.*

Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be a family of pointed CW-complexes and H be a contravariant functor from the homotopy category of pointed CW-complexes to the category of pointed sets. Then there is a function $w: H(\bigvee_{\lambda\in\Lambda} C_\lambda) \to \prod_{\lambda\in\Lambda} H(C_\lambda)$ defined by $w(u) \doteq \{u|C_\lambda\}_{\lambda\in\Lambda}$, where $u \in H(\bigvee_{\lambda\in\Lambda} C_\lambda)$. If w is a bijection for all choices of $\{C_\lambda\}_{\lambda\in\Lambda}$, then H will be said to satisfy the *wedge condition.*

Taking H to be $\mathcal{F}FHE^F(-)$, the wedge condition function is then $\omega: \mathcal{F}FHE^F(\bigvee_{\lambda\in\Lambda} C_\lambda) \to \prod_{\lambda\in\Lambda} \mathcal{F}FHE^F(C_\lambda)$, defined by the rule $\omega[(q,h)] = \{[(q|C_\lambda, h)]\}_{\lambda\in\Lambda}$, where (q,h) is an F-grounded $\mathcal{F}$-fibration over $\bigvee_{\lambda\in\Lambda} C_\lambda$. We now define an analagous function ω' for cylindrical and multicylindrical fibrations, and reduce the problem of proving the bijectivity of ω to the corresponding problem for ω'.

If C is a pointed CW-complex, then $\mathcal{F}FHE^F_{cyl}(C')$ will denote the set of all F-grounded $\mathcal{F}$FHE classes of pairs $(q^*, 1_F)$, where q is an F-extending $\mathcal{F}$-fibration over C. Further $\mathcal{F}FHE^F_{mcyl}(\bigvee_{\lambda\in\Lambda} C'_\lambda)$ will denote the set of F-grounded $\mathcal{F}$FHE classes of pairs $(q^\bullet, 1_F)$, where q is an F-extending $\mathcal{F}$-fibration over $\bigvee_{\lambda\in\Lambda} C_\lambda$. There is a function $\omega': \mathcal{F}FHE^F_{mcyl}(\bigvee_{\lambda\in\Lambda} C'_\lambda) \to \prod_{\lambda\in\Lambda} \mathcal{F}FHE^F_{cyl}(C'_\lambda)$, defined by the rule $\omega'[(q^\bullet, 1_F)] = \{[(q^\bullet|C'_\lambda, 1_F)]\}_{\lambda\in\Lambda}$, where q is an F-extending $\mathcal{F}$-fibration over $\bigvee_{\lambda\in\Lambda} C_\lambda$.

Lemma 6.4 *If $\mathcal{F}$ is $\mathcal{F}FHE$ set-valued and the subfibration replacement property holds for $\mathcal{F}$-fibrations, then ω is bijective if and only if ω' is bijective.*

Proof. We state and prove three conditions: the result then follows.

(i) *$\xi: \mathcal{F}FHE^F_{cyl}(C') \to \mathcal{F}FHE^F(C)$, $\xi([(q^*, 1_F)]) = [((q^*)n, 1_F)]$ is a bijection, where C is a pointed CW-complex, $n: C \to C'$ the based homotopy equivalence of lemma 4.4(ii), and q an F-extending $\mathcal{F}$-fibration over C.*

If (q,h) is an F-grounded $\mathcal{F}$-fibration over C then we know, by the subfibration replacement property for $\mathcal{F}$-fibrations, that there exists an

F-extending $\mathcal{F}$-fibration $q \diamondsuit h$ over C such that $(q \diamondsuit h, 1_F) \equiv_F (q, h)$. Then $\xi([((q \diamondsuit h)^*, 1_F)]) = [(((q \diamondsuit h)^*)n, 1_F)] = [(q \diamondsuit h, 1_F)]$ by proposition 5.1(ii). Now $[(q \diamondsuit h, 1_F)] = [(q, h)]$ and so ξ is surjective.

Let q_0 and q_1 be F-extending $\mathcal{F}$-fibrations over C and $\xi[(q_0^*, 1_F)] = \xi[(q_1^*, 1_F)]$, i.e. $((q_0^*)n, 1_F) \equiv_F ((q_1^*)n, 1_F)$. It follows from proposition 5.1(ii) that $(q_0, 1_F) \equiv_F ((q_0^*)n, 1_F) \equiv_F ((q_1^*)n, 1_F) \equiv_F (q_1, 1_F)$. So $(q_0^*, 1_F) \equiv_F (q_1^*, 1_F)$, and ξ is injective.

(ii) *If $\{C_\lambda\}_{\lambda \in \Lambda}$ is a family of pointed CW-complexes, then there is a bijection $\Xi : \mathcal{F}FHE^F_{mcyl}(\bigvee_{\lambda\in\Lambda} C'_\lambda) \to \mathcal{F}FHE^F(\bigvee_{\lambda\in\Lambda} C_\lambda)$, defined by $\Xi([(q^\bullet, 1_F)]) = [((q^\bullet)_N, 1_F)]$, where q is an F-extending $\mathcal{F}$-fibration over $\bigvee_{\lambda\in\Lambda} C_\lambda$.*

The proof of (i) applies verbatim once we replace ξ, C, C', $n: C \to C'$, q^*, $(q \diamondsuit h)^*$, q_0^*, q_1^*, and proposition 5.1(ii) by Ξ, $\bigvee_{\lambda\in\Lambda} C_\lambda$, $\bigvee_{\lambda\in\Lambda} C'_\lambda$ $N : \bigvee_{\lambda\in\Lambda} C_\lambda \to \bigvee_{\lambda\in\Lambda} C'_\lambda$, $q^\bullet$, $(q \diamondsuit h)^\bullet$, $q_0^\bullet$, $q_1^\bullet$, and proposition 5.4(ii), respectively.

(iii) *There is a bijection $\xi_\lambda : \mathcal{F}FHE^F_{cyl}(C'_\lambda) \to \mathcal{F}FHE^F(C_\lambda)$, for each $\lambda \in \Lambda$. If $\prod_{\lambda\in\Lambda} \xi_\lambda : \prod_{\lambda\in\Lambda} \mathcal{F}FHE^F_{cyl}(C'_\lambda) \to \prod_{\lambda\in\Lambda} \mathcal{F}FHE^F(C_\lambda)$ denotes the obvious associated bijection, then $\omega\,\Xi = (\prod_{\lambda\in\Lambda} \xi_\lambda)\omega'$.*

Let $j(\lambda) : C_\lambda \to \bigvee_{\lambda\in\Lambda} C_\lambda$ and $\iota(\lambda) : C'_\lambda \to \bigvee_{\lambda\in\Lambda} C'_\lambda$ denote the inclusions, and $n(\lambda)$ be a based homotopy equivalence $C_\lambda \to C'_\lambda$ (see lemma 4.4(ii)). Given an F-extending $\mathcal{F}$-fibration q over $\bigvee_{\lambda\in\Lambda} C_\lambda$, we have:

$$
\begin{aligned}
&(\textstyle\prod_{\lambda\in\Lambda} \xi_\lambda)\omega'[(q^\bullet, 1_F)] \\
&= (\textstyle\prod_{\lambda\in\Lambda} \xi_\lambda)\{[((q^\bullet)_{\iota(\lambda)}, 1_F)]\}_{\lambda\in\Lambda} \\
&= \{[(((q^\bullet)_{\iota(\lambda)})_{n(\lambda)}, 1_F)]\}_{\lambda\in\Lambda} \\
&= \{[((q^\bullet)_{\iota(\lambda)n(\lambda)}, 1_F)]\}_{\lambda\in\Lambda} \\
&= \{[((q^\bullet)_{Nj(\lambda)}, 1_F)]\}_{\lambda\in\Lambda} \quad \text{since } Nj(\lambda) = \iota(\lambda)n(\lambda) \\
&= \{[(((q^\bullet)_N)_{j(\lambda)}, 1_F)]\}_{\lambda\in\Lambda} \\
&= \omega([((q^\bullet)_N, 1_F)]) \\
&= \omega\Xi([(q^\bullet, 1_F)]).
\end{aligned}
$$

Theorem 6.5 *Let $(\mathcal{E}, U)$ be a category of enriched spaces, F be a given $\mathcal{E}$-space and $\mathcal{F}$ be the category of fibres containing F. If $\mathcal{F}$ is $\mathcal{F}FHE$ set-valued and the subfibration replacement property holds for $\mathcal{F}$-fibrations, then the functor $\mathcal{F}FHE^F(-)$ satisfies the wedge condition.*

Proof. We will verify the bijectivity of ω'; the result then follows from lemma 6.4. Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be a family of pointed CW-complexes.

(i) *The surjectivity of* ω'. A typical member of $\prod_{\lambda\in\Lambda}\mathcal{F}FHE^F_{cyl}(C'_\lambda)$ has the form $\{[((q_\lambda)^*, 1_F)]\}_{\lambda\in\Lambda}$, where $q_\lambda : Y_\lambda \to C_\lambda$ is an F-extending $\mathcal{F}$-fibration for each $\lambda\in\Lambda$. We know via proposition 5.3 that we can construct an F-multicylindrical $\mathcal{F}$-fibration $\Lambda^\bullet$ over $\bigvee_{\lambda\in\Lambda}C'_\lambda$, such that $\Lambda^\bullet|C'_\lambda = (q_\lambda)^*$, for all $\lambda\in\Lambda$. According to proposition 5.4(iii), there exists an F-grounded $\mathcal{F}$-fibration $q : Y \to \bigvee_{\lambda\in\Lambda}C_\lambda$ such that $(q^\bullet, 1_F) \equiv_F (\Lambda^\bullet, 1_F)$. Hence $(q^\bullet|C'_\lambda, 1_F) \equiv_F (\Lambda^\bullet|C'_\lambda, 1_F) = ((q_\lambda)^*, 1_F)$, for all $\lambda\in\Lambda$. Then $\omega'[(q^\bullet, 1_F)] = \{[(q^\bullet|C'_\lambda, 1_F)]\}_{\lambda\in\Lambda} = \{[((q_\lambda)^*, 1_F)]\}_{\lambda\in\Lambda}$, for all $\lambda\in\Lambda$. It follows that ω' is surjective.

(ii) *The injectivity of* ω'. Typical members of $\mathcal{F}FHE^F_{mcyl}(\bigvee_{\lambda\in\Lambda}C'_\lambda)$ take the forms $[(q_0^\bullet, 1_F)]$ and $[(q_1^\bullet, 1_F)]$, where $q_0 : Y_0 \to \bigvee_{\lambda\in\Lambda}C_\lambda$ and $q_1 : Y_1 \to \bigvee_{\lambda\in\Lambda}C_\lambda$ are F-extending $\mathcal{F}$-fibrations. If $\omega'[(q_0^\bullet, 1_F)] = \omega'[(q_1^\bullet, 1_F)]$, then there exist F-grounded $\mathcal{F}$FHEs $f_\lambda : Y_0^\bullet|C'_\lambda \to Y_1^\bullet|C'_\lambda$ from $(q_0^\bullet|C'_\lambda, 1_F)$ to $(q_1^\bullet|C'_\lambda, 1_F)$, for all $\lambda\in\Lambda$. Thus we have F-grounded $\mathcal{F}$FHEs $f_\lambda : (Y_0|C_\lambda)^* \to (Y_1|C_\lambda)^*$ from $((q_0|C_\lambda)^*, 1_F)$ to $((q_1|C_\lambda)^*, 1_F)$, for all $\lambda\in\Lambda$ (see proposition 5.3). Restricting over the basepoint $(*, 0)$, we obtain maps $f_\lambda|(*,0) : F\times\{0\} \to F\times\{0\}$ which satisfy $1_{F\times\{0\}} \simeq_{\mathcal{F}} f_\lambda|(*,0)$ via an $\mathcal{F}$-homotopy $F\times\{0\}\times I \to F\times\{0\}$. Identifying $F\times\{0\}$ with F, we obtain a family of $\mathcal{F}$-homotopies $H_\lambda : F\times I \to F$, with $H_\lambda(z, 0) = z$, and $(H_\lambda(z,1), 0) = f_\lambda(z, 0)$, where $z \in F$.

Let π_F denote the projection $F\times I \to F$. We define $f : Y_0^\bullet \to Y_1^\bullet$ by $f(y) = f_\lambda(y)$, if $y \in Y_0|C_\lambda$, and

$$f(z,t) = \begin{cases} (H_\lambda(z, 2t), t) & \text{if } z \in F \text{ and } t \in [0, \frac{1}{2}]_\lambda \\ (\pi_F f_\lambda(z, 2t-1), t) & \text{if } z \in F \text{ and } t \in [\frac{1}{2}, 1]_\lambda \end{cases}$$

Then f is [B1, thm.5.4] an F-grounded $\mathcal{F}FHE$ from $(q_0^\bullet, 1_F)$ to $(q_1^\bullet, 1_F)$.

7 MODULE IV: Brown's theorem applied

We assume, throughout this module, that $(\mathcal{E}, U)$ is a category of enriched spaces, F is a given $\mathcal{E}$-space and $\mathcal{F}$ is the category of fibres containing F.

The following version of the Brown Representability Theorem will be justified at the end of this section.

Theorem 7.1 *Let H be a contravariant functor from the homotopy category of pointed CW-complexes to the category of pointed sets and pointed functions, satisfying the Mayer-Vietoris and Wedge conditions and such that $H(S^0)$ contains only one element. Then there exist a path connected pointed CW-complex C_∞ and a universal element $u \in H(C_\infty)$, i.e. for each pointed CW-complex D the rule $\phi([f]) = H([f])(u)$ specifies a natural bijection $\phi: [D, C_\infty]^0 \to H(D)$, where $[f] \in [D, C_\infty]^0$.*

Let $\mathcal{F}$ be $\mathcal{F}FHE^F$ set-valued and $(q: Y \to C,\ h)$ be an F-grounded $\mathcal{F}$-fibration. It follows via lemma 4.3(ii) that $[f] \to [(q_f, h)]$ determines a natural transformation $\phi = \phi(q, h): [D,\ C]^0 \to \mathcal{F}FHE^F(D)$, relative to all $D \in \mathcal{W}^0$.

Definition 7.2 *Let us assume that $\mathcal{F}$ is $\mathcal{F}FHE^F$ set-valued and that there is an F-grounded $\mathcal{F}$-fibration $(p_\mathcal{F}: X_\mathcal{F} \to B_\mathcal{F},\ h_\mathcal{F}: F \to X_\mathcal{F}|*)$. Then $(p_\mathcal{F},\ h_\mathcal{F})$ will be said to be grounded universal amongst F-grounded $\mathcal{F}$-fibrations if, for all choices of spaces $D \in \mathcal{W}^0$, the associated function $\phi = \phi(p_\mathcal{F}, h_\mathcal{F}): [D,\ B_\mathcal{F}]^0 \to \mathcal{F}FHE^F(D)$ is a natural bijection.*

Proposition 7.3 *If $\mathcal{F}$ is $\mathcal{F}$FHE set-valued and $\mathcal{F}FHE^F(-)$ satisfies the Mayer-Vietoris and wedge conditions, then there is a path connected pointed CW-complex $B_\mathcal{F}$ and a grounded universal F-grounded $\mathcal{F}$-fibration $(p_\mathcal{F}: X_\mathcal{F} \to B_\mathcal{F},\ h_\mathcal{F})$.*

Proof. We first restrict the domain of our functor $\mathcal{F}FHE^F(-)$ to the homotopy category of pointed CW-complexes. Now $\mathcal{F}$ is a category of fibres, so clearly $\mathcal{F}FHE^F(S^0)$ is a singleton set. Taking H to be this restriction of $\mathcal{F}FHE^F(-)$, the existence of $(p_\mathcal{F}, h_\mathcal{F})$ follows immediately from our assumptions and theorem 7.1. Thus ϕ is a bijection for pointed CW-complexes D.

If C and D are in $\mathcal{W}^0$ and there is a based homotopy equivalence $C \to D$, then there are induced bijections $\mathcal{F}FHE^F(D) \to \mathcal{F}FHE^F(C)$ (see lemma 4.3(ii)) and $[D,\ B_\mathcal{F}]^0 \to [C,\ B_\mathcal{F}]^0$. Our result, i.e. for domain $\mathcal{W}^0$, follows via the naturality of ϕ.

Proposition 7.4 *If $(p_{\mathcal{F}}, h_{\mathcal{F}})$ is grounded universal amongst F-grounded $\mathcal{F}$-fibrations, then $p_{\mathcal{F}}$ is free universal amongst $\mathcal{F}$-fibrations.*

Proof. If $(p_{\mathcal{F}}: X_{\mathcal{F}} \to B_{\mathcal{F}}, h_{\mathcal{F}})$ is a grounded universal F-grounded $\mathcal{F}$-fibration and $D \in \mathcal{W}$ then $D \sqcup \{*\} \in \mathcal{W}^0$ and $\theta(p_{\mathcal{F}})$ is the composite bijection :

$$[D, B_{\mathcal{F}}] \approx [D \sqcup \{*\}, B_{\mathcal{F}}]^0 \approx \mathcal{F}FHE^F(D \sqcup \{*\}) \approx \mathcal{F}FHE(D).$$

Theorem 7.5 *Let $(\mathcal{E}, U)$ be a category of enriched spaces, F be a given $\mathcal{E}$-space and $\mathcal{F}$ be the category of fibres containing F. If $\mathcal{F}$ is $\mathcal{F}FHE$ set-valued and $\mathcal{F}FHE^F(-)$ satifies the Mayer-Vietoris and wedge conditions, there exists a free universal $\mathcal{F}$-fibration $p_{\mathcal{F}}: X_{\mathcal{F}} \to B_{\mathcal{F}}$ over a path connected CW-complex $B_{\mathcal{F}}$.*

Proof. The result is immediate from propositions 7.3 and 7.4.

Let us suppose that $f_\lambda : C \to D_\lambda$ and $g_\lambda : D_\lambda \to E$ are pointed maps between pointed CW-complexes, for $\lambda = 0$ and 1, such that $g_0 f_0 \simeq^0 g_1 f_1$. Then $\{f_0, f_1, g_0, g_1\}$ will be said to be a *weak pushout* if, for all choices of a pointed CW-complex K and pointed maps $g_0' : D_0 \to K$ and $g_1' : D_1 \to K$ such that $g_0' f_0 \simeq^0 g_1' f_1$, there is a pointed map $k : E \to K$ such that $g_0' \simeq^0 k g_0$ and $g_1' \simeq^0 k g_1$.

Let H be a contravariant functor from the homotopy category of pointed CW-complexes to the category of pointed sets and pointed functions. Then H will be said to satisfy the *weak pushout condition* if, for all weak pushouts $\{f_0, f_1, g_0, g_1\}$, $u_0 \in H(D_0)$ and $u_1 \in H(D_1)$ such that $H(f_0)(u_0) = H(f_1)(u_1)$, there exists a $u \in H(E)$ such that $H(g_0)(u) = u_0$ and $H(g_1)(u) = u_1$.

Proof of theorem 7.1. According to [B(E), thm.2.8] and the remark that follows it, we just need to verify the wedge and weak pushout conditions; hence it is sufficient for us to use the Mayer-Vietoris condition to prove the weak pushout condition.

Let us assume that $\{f_0, f_1, g_0, g_1\}$ is a weak pushout; the cellular approximation theorem allows us to take these four maps to be pointed cellular maps. If $*$ denotes the base point of C, then the points $(*,0),(*,0)$ and $[(*,0)]$ will be taken as base points for the CW-complexes $MC(f_0), MC(f_1)$ and $DMC(f_0,f_1)$ respectively (see lemmas 4.4(i) and 5.2(ii)). For each $\lambda \in \{0,1\}$, we will write $i_\lambda : C \to MC(f_\lambda)$ for the inclusion determined by $i_\lambda(c) = (c,0)$ with $c \in C$. The inclusion $MC(f_\lambda) \to DMC(f_0,f_1)$ will be denoted by ι_λ. Also there is a pointed homotopy equivalence $s_\lambda = s(f_\lambda) : MC(f_\lambda) \to D_\lambda$, with a pointed homotopy inverse $n_\lambda : D_\lambda \to MC(f_\lambda)$, for $\lambda \in \{0,1\}$ (see lemma 4.4(ii)).

We know that $f_\lambda = s_\lambda i_\lambda$ and that $n_\lambda s_\lambda$ is based homotopic to the identity on $MC(f_\lambda))$, so $n_\lambda f_\lambda = n_\lambda s_\lambda i_\lambda \simeq^0 i_\lambda$, for $\lambda \in \{0,1\}$. Hence $\iota_0 n_0 f_0 \simeq^0 \iota_0 i_0 = \iota_1 i_1 \simeq^0 \iota_1 n_1 f_1$. It follows from the weak pushout property that there is a pointed map $k : E \to DMC(f_0,f_1)$ such that $kg_\lambda \simeq^0 \iota_\lambda n_\lambda$, for $\lambda \in \{0,1\}$.

We will assume that $u_\lambda \in H(D_\lambda)$, for $\lambda \in \{0,1\}$, are such that $H(f_0)(u_0) = H(f_1)(u_1)$. Each n_λ is a pointed homotopy equivalence, so for each λ there is a unique $v_\lambda \in H(MC(f_\lambda))$ such that $H(n_\lambda)(v_\lambda) = u_\lambda$. Then $v_0|C = H(n_0 f_0)(v_0) = H(f_0)H(n_0)(v_0) = H(f_0)(u_0) = H(f_1)(u_1) = H(f_1)H(n_1)(v_1) = H(n_1 f_1)(v_1) = v_1|C$. We will now view I_0 and I_1 as two distinct copies of I, and $MC(f_\lambda) = (C \times I_\lambda) \cup_{f_\lambda} D$, for $\lambda = 0$ and 1. Identifying C with $C \times \{0_0\}$ and $C \times \{0_1\}$, we see that $C = MC(f_0) \cap MC(f_1)$ and $DMC(f_0,f_1) \cong MC(f_0) \cup MC(f_1)$. Hence the Mayer-Vietoris condition applies. So there is a $v \in H(DMC(f_0,f_1))$ such that $v|MC(f_\lambda) = v_\lambda$, for $\lambda = 0$ and 1. Let us define $u = H(k)(v) \in H(E)$. Then $H(g_\lambda)(u) = H(g_\lambda)H(k)(v) = H(kg_\lambda)(v) = H(\iota_\lambda n_\lambda)(v) = H(n_\lambda)H(\iota_\lambda)(v) = H(n_\lambda)(v_\lambda) = u_\lambda$, for $\lambda \in \{0,1\}$. So $H(g_\lambda)(u) = u_\lambda$ for each $\lambda \in \Lambda$, and the weak pushout condition is proved.

8 MAIN RESULT: Combining the Modules

Theorem 8.1 *Let $(\mathcal{E}, U)$ be a category of enriched spaces that is $\mathcal{E}FHE$ set-valued, F be an $\mathcal{E}$-space and $\mathcal{F}$ denote the category of fibres determined by F. Then the following conditions are related as indicated:*

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e).$$

(a) $(\mathcal{E}, U_{\mathcal{E}}, \{X \times_{\mathcal{E}} I\})$ *is a category of well enriched spaces under a space A, with associated category of enriched spaces* $(\mathcal{E}, U)$ *(see lemma 2.4).*

(b) The double retraction property holds for $\mathcal{F}$*-fibrations.*

(c) The subfibration replacement property holds for $\mathcal{F}$*-fibrations.*

(d) $\mathcal{F}$ *is* $\mathcal{F}FHE$ *set-valued, and the functor* $\mathcal{F}FHE^F(-)$ *satisfies the Mayer - Vietoris and wedge conditions.*

(e) There exists a free universal $\mathcal{F}$*-fibration* $p_{\mathcal{F}}: X_{\mathcal{F}} \to B_{\mathcal{F}}$*, over a path - connected CW-complex* $B_{\mathcal{F}}$*.*

We notice that this result includes the particular case where $\mathcal{E}$ is itself a category of enriched fibres, i.e. where $\mathcal{E} = \mathcal{F}$.

Proof. $(a) \Rightarrow (b), (b) \Rightarrow (c), (c) \Rightarrow (d)$, and $(d) \Rightarrow (e)$ follow from the main results of modules I, II, III, and IV respectively, i.e. theorems 3.7, 4.8, 6.3 and 6.5, and 7.5, respectively.

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