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EXAMPLES OF CATEGORIFICATION

by Louis CRANE & David N. YETTER

RESUME. Les auteurs considèrent, dans les cas particuliers des algèbres de groupes finis et de leurs doubles quantiques, le problème de la classification des catégories semi-simples avec tenseur et bi-tenseur ayant un "rig" de Grothendieck (anneau sans négation) donné. Les résultats obtenus dans les deux cas ont de fortes résonance.

1 Introduction

The suggestion was made in work of Crane and Frenkel [4, 3] that the inverse relation to the Grothendieck rig (fusion rig) construction should shed light on the relation between topological quantum field theories (TQFT's) in various dimensions, and, as well, should provide constructions for TQFT's in dimension 4. In particular, the construction gives rise to categorical analogues of bialgebras suggested in [4] which provide the initial data for the construction proposed in [4] for a 4-dimensional state-sum TQFT. On the other hand, monoidal bicategories of semi-simple categories equipped with an action of such a category are candidates for the initial data for fully bicategorical state-sums of Crane/Yetter type [6, 8] which are not immediately reducible to signature as are invariants of Crane/Yetter type constructed from 1-object monoidal bicategories (that is from braided monoidal categories).

It is the purpose of this paper to consider several simple cases of the inverse relation to the Grothendieck rig construction as constructions of tensor and bitensor categories.

Definition 1.1 A rig R is a set equipped with a monoid structure, $(R, +, 0)$ and a semi-group structure (R, \cdot) satisfying, moreover,

$$\begin{array}{ll} \text{commutativity} & a + b = b + a \\ \text{right-distributivity} & (a + b) \cdot c = a \cdot c + b \cdot c \\ \text{left-distributivity} & a \cdot (b + c) = a \cdot b + a \cdot c. \end{array}$$

A rig is **unital** if the multiplicative semi-group is a monoid with unit 1. A rig is **of finite rank** if the additive monoid $(R, +, 0)$ is finitely generated.

Observe that in analogy to the coproduct of rings—tensor product of rings over \mathbb{Z} —rigs admit a coproduct, “tensor product over \mathbb{N} ”.

The one other general property of rigs (of finite rank) which we will need in the sequel is

Proposition 1.2 *If R is a rig of finite rank, then R has a unique minimal set of additive generators.*

proof: Suppose S and T are minimal sets of generators. Then there exist expressions

$$s = \sum_{t \in T} n_t^s t$$

and

$$t = \sum_{s \in S} m_s^t s$$

Observe then that $[n_t^s]$ and $[m_s^t]$ are mutually inverse matrices of natural numbers. The only such pairs are pairs of inverse permutation matrices. \square

Definition 1.3 *If \mathcal{C} is an (essentially small) tensor category, that is an abelian category equipped with a (bi-)exact monoidal product (not necessarily with unit object), then the set of isomorphism classes of objects in \mathcal{C} equipped with the operations induced by direct sum and tensor product is called the **Grothendieck rig** of \mathcal{C} , and denoted $Groth(\mathcal{C})$.*

Notice that if \mathcal{C} has a unit object, then $Groth(\mathcal{C})$ is unital. Similarly if \mathcal{C} is Artinian semi-simple, then $Groth(\mathcal{C})$ is of finite rank.

Observe that the universal property of direct sum imposes some constraints on the structure of a Grothendieck rig, the most salient of which is the condition that all 1-generator sub-additive-monoids are free.

Definition 1.4 *An abstract fusion-rule algebra is a rig, in which all 1-generator sub-additive-monoids are free. An abstract fusion-rule bialgebra is an abstract fusion rule algebra A equipped with a co-operation $\Delta : A \rightarrow A \otimes_{\mathbb{N}} A$ which is a rig homomorphism (or, equivalently, which satisfies the usual compatibility relations for bialgebras). It is counital if it is equipped with a co-operation $\epsilon : A \rightarrow \mathbb{N}$ which is a rig homomorphism. In the case where A is unital, we require Δ and ϵ to preserve the unit.*

This latter, the notion of an abstract fusion-rule bialgebra, is important because of the construction proposed by Crane and Frenkel [4] of state-sum invariants of 4-manifolds using as initial data categorical analogues of Hopf-algebras, and the result of Crane and Yetter [5] showing that any 4D TQFT with factorization at corners has as part of its structure a formal “bialgebra category.”

In order to explain the structure of these analogues to Hopf-algebras, and to fix the context in which our reversal of the Grothendieck rig construction will take place, first fix an algebraically closed field K . We will briefly mention the difficulties which non-algebraically closed fields present later. Let $VECT_K$ or simply $VECT$ denote the category of finite-dimensional vector-spaces over K with its usual monoidal structure.

Definition 1.5 *Let $VECT_K - mod$ denote the 2-category of all (small) Artinian semi-simple K -linear categories, that is K -linear categories equivalent to finite powers of $VECT_K$ with exact functors and natural transformations as 1- and 2-arrows. We refer to the objects of this 2-category as $VECT$ -modules. Let \boxtimes denote a chosen bifunctor from $VECT_K - mod^2$ to $VECT_K - mod$ which selects an object $C \boxtimes D$ equipped with a functor from $C \times D$ exact in each variable separately and universal among such.*

The existence of \boxtimes has been shown in a more general setting in [9] and by explicit construction applicable in the present case in [18]. It can be readily verified for $VECT_K$ -modules by mimicking the construction of the tensor product of vector-spaces, except that instead of identifying objects as one identified elements, one must adjoin an isomorphism. It

follows readily from the universality properties that \boxtimes makes $VECT_K - mod$ into a monoidal bicategory (cf. Gordon/Power/Street [13] and Kapranov/Voevodsky [12]), whose underlying bicategory is a 2-category.

In this setting if a $VECT$ -module \mathcal{C} is equipped with a monoidal structure (with unit object), exact in each variable, we may regard this as being given by (exact) functors $\otimes : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$ (and $I : VECT \rightarrow \mathcal{C}$, sending K to the monoidal identity object), equipped with natural transformation(s) α (and ρ and λ) satisfying the usual pentagon (and triangle) coherence condition(s). Similarly the structural transformations of exact monoidal functors between $VECT$ -modules \mathcal{C} and \mathcal{D} may be understood as natural transformations between functors from $\mathcal{C} \boxtimes \mathcal{C}$ to \mathcal{D} (and $VECT$ to \mathcal{D}). We will refer to monoidal categories of this sort as (*Artinian semi-simple*) *tensor categories*.¹ We call exact monoidal functors *tensor functors*.

It is now sensible to consider in this setting duals to the notions of tensor category and tensor functor:

Definition 1.6 *An Artinian semi-simple (counital) cotensor category over K is a $VECT$ -module \mathcal{C} equipped with functors $\Delta : \mathcal{C} \rightarrow \mathcal{C} \boxtimes \mathcal{C}$ (and $\epsilon : \mathcal{C} \rightarrow VECT$), together with natural transformations β (and r and l) satisfying the obvious pentagon (and triangle) relation(s). A strong cotensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor equipped with natural isomorphisms $F_\sim : (F \boxtimes F)(\Delta) \rightarrow \Delta(F)$ (and $F^0 : \epsilon \rightarrow \epsilon(F)$) satisfying coherence conditions formally dual to those for strong monoidal functors.*

We can now succinctly define the categorical analogue of a bialgebra as given by Crane and Frenkel [4] including the conditions on the unit and counit functors which were omitted in [4]:

Definition 1.7 *An Artinian semi-simple bitensor category over K is an Artinian semi-simple category equipped with both a tensor category structure and a cotensor category structure for which the structure functors for the cotensor category structure are (strong) tensor functors, the structure functors for the tensor structure are cotensor functors, and the*

¹More generally, we advocate the use of “tensor category” to refer to a monoidal abelian category with \otimes exact in each variable.

structural transformations for the tensor functor and cotensor functor structures coincide whenever their sources and targets coincide. It is biunital if the tensor structure is unital and the cotensor structure is counital.

In particular, we fix notation for these “compatibility transformations” as follows:

$$\Phi = \tilde{\Delta} = \otimes_{\sim}^{-1}$$

$$\eta = \epsilon^0 = I_0^{-1}$$

$$\tau = \tilde{\epsilon} = \otimes_0^{-1}$$

$$\delta = \Delta^0 = I_{\sim}^{-1}$$

The other two “coherence cubes” of [4] (besides the pentagon and dual pentagon) are simply the coherence condition for a tensor functor and its dual.

Thus the complete structure of a biunital bitensor category \mathcal{C} is given by four functors $\otimes, I, \Delta,$ and ϵ and ten natural isomorphisms $\alpha, \rho, \lambda, \beta, r, l, \Phi, \eta, \tau,$ and δ satisfying coherence conditions which can be read off from the definitions of tensor and cotensor categories and tensor and cotensor functors.

The appropriate notion of structure preserving functors between bitensor categories is given by

Definition 1.8 *A (strong) bitensor functor is a 5-tuple*

$$(F, \tilde{F}, F_{\sim}, F^0, F_0),$$

where $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor between bitensor categories, and such that (F, \tilde{F}, F_0) (resp. (F, F_{\sim}, F^0)) is a tensor (resp. cotensor) functor and which moreover satisfies the following, in which primes (') indicate structural functors or natural transformations belonging to \mathcal{D} :

$$F \boxtimes F(\Phi_{A,B})][\oplus \tilde{F}_{A(1),A(2)} \boxtimes \tilde{F}_{B(1),B(2)}][F_{\sim A} \boxtimes F_{\sim B}] = \\ F_{\sim A \otimes B} \Delta'(\tilde{F}_{A,B}) \Phi'_{F(A),F(B)}$$

$$F_{A \otimes B}^0 \epsilon'(\tilde{F}_{A,B}) \tau'_{F(A),F(B)} = \tau_{A,B}(F_A^0 \otimes F_B^0)$$

$$F_{\sim I} \Delta'(F_0) \delta' = (F \boxtimes F)(\delta)(F_0 \boxtimes F_0)$$

and

$$F_n^0 F_0 \eta' = \eta$$

If either category is not unital or counital, the appropriate functor and attendant natural isomorphisms are omitted.

Likewise, we can define cotensor and bitensor natural transformations: a natural transformation is a cotensor transformation if it satisfies the dual condition to monoidal naturality, and is a bitensor transformation if it is both a monoidal natural transformation and a cotensor transformation.

Now, observe that if \mathcal{C} is a bitensor category, $\text{Groth}(\mathcal{C})$ has the structure of an abstract fusion rule bialgebra with the co-operations induced by the cotensor and counit functors on the category. We will refer to this as the **Grothendieck birig** of the bitensor category.

Definition 1.9 *A K -categorification of an abstract fusion-rule algebra (resp. bialgebra) A is a K -linear tensor category (resp. bitensor category) whose Grothendieck rig (resp. birig) is A . If A is of finite rank, we call a categorification **semi-simple** if it is semi-simple as a tensor category, and, moreover, the objects whose images under the Grothendieck rig construction are the additive generators of A are simple objects.*

Notice in general, in the non-finite rank case, one must specify a set of additive generators to make sense of the notion of semi-simple categorification. We will have no need of this more general notion in this paper.

In the remainder of the paper, we consider examples of categorifications and their relevance to the construction of TQFT's. We restrict our attention to the case of K algebraically closed because this covers the most interesting case of $K = \mathbb{C}$ and removes the possibility of having objects in semi-simple categories whose endomorphism algebras are division-algebra extensions of the ground field.

2 Categorifying $\mathbb{N}[G]$, Dijkgraaf-Witten Theory, and the Turaev-Viro Construction

Our first example sheds light on the relationship between two well-known constructions of (2+1)-dimensional TQFT's: Dijkgraaf-Witten theory [11], in particular its simplicial construction as in Wakui [16], (cf. also Yetter [17]), and the generalized Turaev-Viro construction, cf. Barrett and Westbury [1]. As examples of tensor categories, these are reasonably well-known (cf. [2], [10]) and are included here merely as the simplest examples of categorification.

Fix a finite group G . Observe that the group rig $\mathbb{N}[G]$ is an abstract fusion-rule bialgebra with the operations induced on the basis G by $g \cdot h = gh$ (the null-infix denoting the group-law) and $\Delta(g) = g \otimes g$. The following shows that the categorifications of $\mathbb{N}[G]$ are essentially classified by the 3-cocycles on G with coefficients in K^\times when K is algebraically closed.

In this and all subsequent proofs, it first should be observed that each equivalence class of K -linear semi-simple categories contains skeletal categories (i.e. categories with only one object in each isomorphism class). We begin by restricting our attention to the structure of these skeletal categories up to isomorphism, then consider the question of monoidal equivalence. Notice, we are taking an approach orthogonal to

that of the Mac Lane Coherence Theorem [14]: we will not be able to strictify the structure maps and retain skeletal-ness.

Theorem 2.1 *Let G be a finite group, and K an algebraically closed field, then the isomorphism classes of skeletal semi-simple K -categorifications of $\mathbb{N}[G]$ in the sense listed in the left-hand column of Table 1 are in canonical one-to-one correspondence with choices of the data given in the corresponding entry of the right-hand column of Table 1.*

abstract fusion-rule algebra	a K^\times -valued 3-cocycle on G
unital abstract fusion-rule algebra	a pair (α, ρ) , α a K^\times -valued 3-cocycle on G , $\rho \in K^\times$
abstract fusion-rule bialgebra	a K^\times -valued 2-cochain on G
biunital abstract fusion-rule bialgebra	a triple (ϕ, ρ, r) , ϕ a K^\times -valued 2-cochain on G , $\rho \in K^\times$, and r a K^\times -valued 1-cochain on G

Table 1:

proof: We begin by proving the first two statements, that skeletal categorifications as an algebra (resp. unital algebra) are given by 3-cocycles (resp. pairs of a 3-cocycle and a scalar).

Now, in any skeletal categorification of $\mathbb{N}[G]$, we may identify the object whose image under the Grothendieck rig construction is $g \in G$ with g . Since the tensor product (resp. identity object) must be carried to the multiplication (resp. unit) in the fusion ring, the functors of the monoidal structure are given by

$$g \otimes h = gh \quad I = e$$

To specify a categorification (as an algebra), it remains only to describe the rest of the monoidal structure: in this case the structure maps become families of maps $\alpha_{g,h,k} : ghk \rightarrow ghk$, $\rho_g : g \rightarrow g$ and $\lambda_g : g \rightarrow g$. Observe, moreover, that these “maps” are just elements of K , that the semi-simplicity condition implies that any such families of maps will

satisfy the required naturality conditions, and that invertibility consists in restricting the choices to elements of K^\times .

The pentagon condition for a 4-tuple g, h, k, l can then be written:

$$\alpha_{g,h,k}\alpha_{g,hk,l}\alpha_{h,k,l} = \alpha_{gh,k,l}\alpha_{g,h,kl}$$

which is precisely the condition that $\alpha_{-, -, -}$ be a K^\times -valued 3-cocycle on G .

Upon including a unit, the triangle condition relating ρ, λ , and α becomes

$$\rho_g = \alpha_{g,e,h}\lambda_h$$

from which it follows that all components of ρ and λ are completely determined by α and the choice of a number ρ such that $\rho_e = \rho \cdot Id_e$. The exercise of verifying that the choice of λ_h 's determined by ρ_e , and the choice of ρ_g 's determined by λ_e satisfy the triangle condition for all pairs g and h is left to the reader. (Hint: use the cocycle condition with two indices equal to e .)

To specify a categorification as a bialgebra, notice first that the dual pentagon condition on the coassociator reduces to

$$\beta_g^3 = \beta_g^2$$

for all $g \in G$, and thus since β_g must be invertible, $\beta_g = 1$.

Similarly the compatibility condition between the connecting transformation ϕ and the coassociator β gives no restriction on the components of ϕ .

On the other hand, the compatibility between ϕ and α is given by

$$\phi_{g,h}\phi_{gh,k}\alpha_{g,h,k} = \alpha_{g,h,k}\alpha_{g,h,k}\phi_{h,k}\phi_{g,hk}$$

which reduces to

$$\phi_{g,h}\phi_{g,hk}^{-1}\phi_{gh,k}\phi_{h,k}^{-1} = \alpha_{g,h,k}$$

Thus, in this case the structure is completely determined by the connecting transformation ϕ , which is simply a 2-cochain on G .

Finally in the case of a biunital bialgebra categorification, the remaining structure maps are similarly completely determined by the component maps at simple objects, which are again given by scalars in K^\times , $r_g, l_g, \tau_{g,h}$; and since $e = I$ is simple, scalars δ and η . The coherence conditions then become:

triangle for tensor structure	$\rho_g = \alpha_{g,e,k} \lambda_k$
triangle for cotensor structure	$r_g = \beta_g l_g$
Δ respects right unit	$\phi_{g,e} \delta \rho_g^2 = \rho_g$
Δ respects left unit	$\phi_{g,e} \delta \lambda_g^2 = \lambda_g$
\otimes respects right counit	$\phi_{g,k} \tau_{g,k} r_g r_k = r_{gk}$
\otimes respects left counit	$\phi_{g,k} \tau_{g,k} l_g l_k = l_{gk}$
ϵ preserves \otimes	$\alpha_{g,k,m} \tau_{g,km} \tau_{k,m} = \tau_{g,k} \tau_{gk,m}$
I preserves Δ	$\beta_e \delta^2 = \delta^2$
ϵ respects right unit	$\tau_{g,e} \eta = \rho_g$
ϵ respects left unit	$\tau_{e,g} \eta = \lambda_g$
I respects right counit	$\delta \eta = r_e$
I respects left counit	$\delta \eta = l_e$

We can then analyse these equations to determine a minimal set of data and conditions for specifying the unital and counital structures on a categorification of $\mathbb{N}[G]$. We assume that α and ϕ have been chosen so as to specify a bitensor categorification without unit or counit.

First, observe that it follows from the two triangle conditions that ρ, λ, r and l are completely determined by the values of $\rho = \rho_e$ and r_g ($g \in G$) by the formulas

$$\rho_g = \alpha_{g,e,e} \rho$$

$$\lambda_k = \alpha_{e,e,k}^{-1} \rho$$

$$l_g = r_g$$

The only restriction on these needed to ensure that they unambiguously determine cochains satisfying the first two equations has already been imposed by the condition that α be a coboundary in the suitable sense.

In a similar way, \odot respecting both the right and left counit conditions is equivalent to

$$\tau(g, k) = d(r^{-1})(g, k)\phi_{g,k}^{-1}.$$

Similarly, Δ respecting both the right and left unit conditions is equivalent to

$$\delta = \rho^{-1}\phi_{e,e}^{-1}$$

Given these last two equations, the conditions that ϵ preserve \otimes and I preserve Δ follow from the cocycle conditions and the fact that d distributes over multiplication of cochains and (remember we write the operation on cochains multiplicatively since the coefficients are in K^\times).

The four remaining conditions are all equivalent to

$$\eta = \rho r_e \phi_{e,c}$$

and we are done. \square

Turning to the more interesting question of monoidal equivalence classes of categorifications, we have:

Definition 2.2 *We say two algebra (resp. bialgebra) categorifications are **equivalent** if there exists a monoidal equivalence (resp. bitensor equivalence) between them which induces the identity on Grothendieck rigs.*

Theorem 2.3 *The equivalence classes of categorifications of $\mathbb{N}[G]$ as an abstract fusion-rule algebra (whether unital or not) are in 1-1 correspondence with $H^3(G, K^\times)$. The bialgebra categorification of $\mathbb{N}[G]$ is unique up to equivalence.*

proof: Now, observe that the structure of a monoidal equivalence which induces the identity on the Grothendieck rig is given entirely by the structure transformations of the monoidal equivalence. In this case, these are given by a choice of units $\psi_{a,b}$ for pair of group elements, and satisfying

$$\alpha_{g,h,k}\psi_{h,k}\psi_{g,hk} = \psi_{g,h}\psi_{gh,k}\alpha'_{g,h,k}$$

and

$$\rho_g\psi_{g,e} = \rho'_g$$

where $\alpha_{g,h,k}$ and $\alpha'_{g,h,k}$ (resp. ρ_g and ρ'_g) are the components of the associativity (resp. unit) transformations on the two categorifications.

Solving gives the condition that α and a be cohomologous, while the condition on the unit transformations is just a normalization condition which can be trivially satisfied.

The statement for bialgebra categorifications follows immediately from this and the condition that the associativity constraint be the coboundary of the connecting constraint: ψ must be the ratio of the two connecting constraints.

The additional structure of unital-counital categorifications is effaced by equivalence.

A similar analysis to that in the proof of the previous theorem shows that the structural transformations for an equivalence of unital and counital categorifications are determined by the 2-cochain ψ which determines the non-unital non-counital equivalence, together with a scalar f_0 and a 1-cochain f^0 .

The conditions besides the cobounding conditions reduce to

$$f_0 = \frac{\rho}{\rho'\psi_{e,e}}$$

$$f^0(g) = \frac{r_g}{r'_g}$$

which can always be solved for any choice of ρ and ρ' (resp. r and r').
□

The algebra case of this result in fact shows that the Dijkgraaf-Witten invariants of 3-manifolds are examples of the generalized Turaev/Viro construction of Barrett and Westbury [1]. It is not hard to show that the categorifications of a finite group rig are spherical categories in the sense of [1]. The construction as given by Wakui [16] is then immediately seen to be a special case of their general construction.

3 Categorifying $D(\mathbb{N}[G])$

We now turn to the question of categorifying the simplest really non-trivial Hopf algebra: the Drinfel'd double of a finite (non-commutative) group algebra (here taken with \mathbb{N} -coefficients to produce a birig).

The same techniques may be used to classify the semi-simple categorifications of arbitrary bicrossproducts of a finite group algebra with a dual finite group algebra (cf. Majid [15]).

Recall that the Drinfel'd double of a finite group algebra can be constructed by taking as a basis pairs (g, \hat{h}) , where g and h are elements of the group, and \hat{h} indicates the element in the dual basis corresponding to h as a basis element in $\mathbb{N}[G]$, with structure maps given by

$$\begin{aligned} \text{multiplication} & \quad (g, \hat{h}) \cdot (k, \hat{l}) = \delta_{k^{-1}hk, l}(gk, \hat{l}) \\ \text{unit} & \quad 1 = \sum_h (e, \hat{h}) \\ \text{comultiplication} & \quad \Delta(g, \hat{h}) = \sum_{kl=h} (g, \hat{k}) \otimes (g, \hat{l}) \\ \text{counit} & \quad \epsilon(g, \hat{h}) = \delta_{h, e} \end{aligned}$$

To categorify this as an abstract fusion-algebra (resp. -bialgebra) we must first consider what are the source and target data for a component of the associator (resp. components of the associator, coassociator and compatibility transformation).

The typical component of an associator is a map

$$\alpha_{(g, \hat{h}), (k, \hat{l}), (m, \hat{n})} : ((g, \hat{h}) \otimes (k, \hat{l})) \otimes (m, \hat{n}) \rightarrow (g, \hat{h}) \otimes ((k, \hat{l}) \otimes (m, \hat{n}))$$

Observe that the source and target objects are both 0 unless $l = k^{-1}hk$ and $n = m^{-1}lm$, in which case both are the object (gkm, \hat{n}) .

Thus the associator may be regarded as a family of non-zero scalars $\alpha(g, k, m; \hat{n})$ giving the non-zero components as multiples of the identity on (gkm, \hat{n}) . (Note: a choice of g, k, m , and \hat{n} contains enough information to recover the source and target data for a non-zero component of the associator.)

Similarly, the coassociator has components given by maps (in the tensor cube of the category)

$$\beta_{(g, \hat{h})} : \bigoplus_{pk=h} \bigoplus_{ij=p} (g, \hat{i}) \boxtimes (g, \hat{j}) \boxtimes (g, \hat{k}) \rightarrow \bigoplus_{iq=h} \bigoplus_{jk=q} (g, \hat{i}) \boxtimes (g, \hat{j}) \boxtimes (g, \hat{k})$$

Each such map is determined by its components on the various

$$(g, \hat{i}) \boxtimes (g, \hat{j}) \boxtimes (g, \hat{k})$$

and thus by a family of scalars $\beta(g; \hat{i}, \hat{j}, \hat{k})$ (As above, the given indices contain enough information to recover to which summand of which component of the coassociator this scalar belongs.)

Finally, the compatibility transformation or “coherer” has as components maps (in the tensor square of the category)

$$\phi_{(g, \hat{h}), (k, \hat{l})} : \oplus_{mn=l}(gk, \hat{m}) \boxtimes (gk, \hat{n}) \rightarrow \oplus_{ab=h}(gk, \widehat{k^{-1}ak}) \boxtimes (gk, \widehat{k^{-1}bk})$$

In this case, the sources and targets of components are given only in the case where the source and target are non-zero, which happens precisely when $k^{-1}hk = l$.

Thus the coherer is determined by a family of scalars $\phi(g, k; \hat{m}, \hat{n})$.

Simply writing down the pentagon coherence condition on the associator in terms of the scalars $\alpha(g, k, m; \hat{n})$ gives

$$\alpha(k, m, p; \hat{q})\alpha(g, km, p; \hat{q})\alpha(g, k, m; \widehat{pqp^{-1}}) = \alpha(gk, m, p; \hat{q})\alpha(g, k, mp; \hat{q})$$

Similarly, the dual pentagon in terms of the scalars becomes

$$\beta(g; \hat{j}, \hat{k}, \hat{l})\beta(g; \hat{i}, \hat{j}k, \hat{l})\beta(g; \hat{i}, \hat{j}, \hat{k}) = \beta(g; \hat{i}j, \hat{k}, \hat{l})\beta(g; \hat{i}, \hat{j}, \hat{k}l)$$

(A choice of 3-cocycle for each group element.)

The coherence condition provided by the compatibility transformation as the structure transformation for Δ as a monoidal functor becomes:

$$\alpha(g, k, m; \hat{p})\alpha(g, k, m; \hat{q})\phi(k, m; \hat{p}, \hat{q})\phi(g, km; \hat{p}, \hat{q}) = \phi(g, k; \widehat{mpm^{-1}}, \widehat{mqm^{-1}})\phi(gk, m; \hat{p}, \hat{q})\alpha(g, k, m; \widehat{p\hat{q}})$$

(The two occurrences of α on the left come from the associator for $\mathcal{C} \boxtimes \mathcal{C}$.)

Similarly, the coherence condition provided by the compatibility transformation as the structure transformation for \otimes as a cotensor functor becomes

$$\phi(g, k; \hat{p}; \hat{r})\phi(g, k; \widehat{p\hat{r}}, \hat{s})\beta(gk; \hat{p}, \hat{r}, \hat{s}) = \beta(g; \widehat{kpk^{-1}}, \widehat{krk^{-1}}, \widehat{ksk^{-1}})\beta(k; \hat{p}, \hat{r}, \hat{s})\phi(g, k; \hat{r}, \hat{s})\phi(g, k; \hat{p}, \widehat{r\hat{s}})$$

These conditions and those involving the other structural transformations become more intelligible if we introduce a general setting for such scalar valued functions:

For any finite group, G , let \hat{G} denote the set of characteristic functions of 1-element subsets of G . Let $C_{n,m}(G, K^\times)$ (or $C_{n,m}$ when G and K are clear from context) denote the abelian group of all functions from $G^n \times \hat{G}^m$ to K^\times . The groups $C_{n,m}$ then form a double complex (written multiplicatively) with differentials $d_2 : C_{n,m} \rightarrow C_{n,m+1}$, given by the Hochschild coboundary in the “hatted indices”, and $\tilde{d}_1 : C_{n,m} \rightarrow C_{n+1,m}$ given by the same formula as Hochschild coboundary in the “unhatted indices” except that when the last index is dropped, all hatted indices are left-conjugated by the dropped index.

In terms of these coboundary operations, the coherence conditions already interpreted for categorifications of $D(\mathbb{N}[G])$ become

$$\tilde{d}_1(\alpha) = 1$$

$$d_2(\beta) = 1$$

$$d_2(\alpha) = \tilde{d}_1(\phi)$$

$$d_2(\phi) = \tilde{d}_1(\beta)$$

that is, the triple (α, ϕ, β) forms a coboundary in the total complex of the double complex $(C_{n,m}, \tilde{d}_1, d_2, n, m \geq 1)$. We will index the cohomology of the total complex of $(C_{n,m}, \dots)$ by $n + m - 1$, and denote the groups by $\mathbb{H}(G, K^\times)$

By identical methods, one can verify that the structure transformations for the preservation of the tensor product and cotensor product

by a bitensor functor with the identity as underlying functor are determined by families of scalars $\tilde{f}(g, k; \hat{l})$ and $f_{\sim}(g; \hat{m}, \hat{n})$ such that the ratios of corresponding structure maps satisfy

$$\alpha\alpha'^{-1} = \tilde{d}_1(\tilde{f})$$

$$\phi\phi'^{-1} = d_2(\tilde{f})\tilde{d}_1(f_{\sim})$$

and

$$\beta\beta'^{-1} = d_2(f_{\sim}).$$

Thus we have

Theorem 3.1 *The skeletal semi-simple K -categorifications of $D(\mathbb{N}[G])$ as an abstract fusion-rule algebra (resp. abstract fusion-rule bialgebra) are in one-to-one correspondence with the 3-coboundaries in $(C_{n,1}, \tilde{d}_1)$ (resp. the 3-coboundaries in the total complex of the double complex $(C_{n,m}, \tilde{d}_1, d_2)$). Moreover, the equivalence classes of categorifications are in natural one-to-one correspondence with the elements of the cohomology group $H_{3,1}$ (resp. $\mathbb{H}_3(G, K^\times)$).*

As was done explicitly for the associator, coherer, and coassociator above, we can examine the components of each of the other structural transformations at a simple object. In this way we find that ρ (resp. λ, r, l) is determined by a 1, 1-cochain $\rho(g; \hat{h})$ (resp. $\lambda(g; \hat{h}), r(g; \hat{h}), l(g; \hat{h})$). For example, the typical component of ρ at a simple object is a map from $(g; \hat{h}) \otimes \oplus_k(e; \hat{k})$ to $(g; \hat{h})$, but the source is just $(g; \hat{h})$, so the map is a scalar multiple of $1_{(g; \hat{h})}$.

Likewise, δ is determined by 0, 2-cochain $\delta(\hat{k}, \hat{l})$, τ by a 2, 0-cochain, and η by a single element of K^\times . (Note: although the double complex actually used in defining categorifications does not have a 0-row or 0-column, it is helpful here and in what follows to consider the larger complex which does, since much of what is needed to handle the unital and counital structures is conveniently phrased in terms of cochains and coboundaries in the larger complex.)

Again, by way of example, δ is a map from $\Delta(I)$ to $I \boxtimes I$ (the latter being the unit object in $\mathcal{C} \boxtimes \mathcal{C}$), that is a map from $\bigoplus_h kl=h(e; \hat{k}) \boxtimes (e; \hat{l})$ to $\bigoplus_k l(e; \hat{k}) \boxtimes (e; \hat{l})$, and is thus determined by a choice of scalar for each summand (the simple summands of the two sides being the same, and each occurring with multiplicity one), that is a 0,2-cochain.

Of course, these functions satisfy conditions equivalent to the coherence conditions for the natural transformations they define. It is an easy exercise to write out each of the coherence conditions in turn, instantiate the objects with simple objects (or simple summands of I , as appropriate), and write out the corresponding equation on the cochains.

The table below summarizes the resulting equations:

triangle for tensor structure	$\rho(g; \widehat{klk^{-1}}) = \alpha(g, e, k; \hat{l})\lambda(k; \hat{l})$
triangle for cotensor structure	$r(g; \hat{k}) = \beta(g; \hat{k}, \hat{e}, \hat{l})l(g; \hat{l})$
Δ respects right unit	$\phi(g, c; \hat{k}, \hat{l})\delta(\hat{k}, \hat{l})\rho(g; \hat{k})\rho(g; \hat{l}) = \rho(g; \widehat{kl})$
Δ respects left unit	$\phi(e, g; \hat{k}, \hat{l})\delta(\hat{k}, \hat{l})\lambda(g; \hat{k})\lambda(g; \hat{l}) = \lambda(g; \widehat{kl})$
\otimes respects right counit	$\phi(g, k; \hat{l}, \hat{e})\tau(g, k)r(g; \widehat{klk^{-1}})r(g; \hat{l}) = r(gk; \hat{l})$
\otimes respects left counit	$\phi(g, k; \hat{l}, \hat{e})\tau(g, k)l(g; \widehat{klk^{-1}})l(g; \hat{l}) = l(gk; \hat{l})$
ϵ preserves \otimes	$\alpha(g, k, m; \hat{e})\tau(g, km)\tau(k, m) = \tau(g, k)\tau(gk, m)$
I preserves Δ	$\beta(e; \hat{k}, \hat{l}, \hat{m})\delta(\hat{k}, \widehat{lm})\delta(\hat{l}, \hat{m}) = \delta(\hat{k}, \hat{l})\delta(\widehat{kl}, \hat{m})$
ϵ respects right unit	$\tau(g, e)\eta = \rho(g; \hat{e})$
ϵ respects left unit	$\tau(e, g)\eta = \lambda(g; \hat{e})$
I respects right counit	$\delta(\hat{h}, \hat{e})\eta = r(e; \hat{h})$
I respects left counit	$\delta(\hat{e}, \hat{h})\eta = l(e; \hat{h})$

By way of example, the condition that Δ be a monoidal functor includes the condition that Δ respect the right unit transformation. Written out this become the equation

$$\Phi_{A,I}(Id \otimes_2 \delta)\rho_2 = \Delta(\rho)$$

where the subscripts $_2$ indicate the corresponding structure in $\mathcal{C} \boxtimes \mathcal{C}$. Now, $\rho_2 = \rho \boxtimes \rho$, and if $A = (g; \hat{h})$ then all of the sources and targets of the maps in the equation are $\oplus_{kl=h}(g; \hat{k}) \boxtimes (g; \hat{l})$, and each map is thus determined by a scalar for each triple $(g; \hat{k}, \hat{l})$. Writing out a component of the equation above gives

$$\phi(g, e; \hat{k}, \hat{l})\delta(\hat{k}, \hat{l})\rho(g; \hat{k})\rho(g; \hat{l}) = \rho(g; \widehat{k\hat{l}})$$

(Notice: on the right hand side, we use the fact that Δ preserves identities and scalar multiples.)

We can then analyse these equations to determine a minimal set of data and conditions for specifying the unital and counital structures on a categorification of $D(\mathbb{N}[G])$. We assume that α, ϕ and β have been chosen as in the previous theorem to specify a bitensor categorification without unit or counit.

First, observe that it follows from the two triangle conditions that ρ, λ, r and l are completely determined by the values of $\rho(e; \hat{m})$ ($m \in G$) and $r(g; \hat{e})$ ($g \in G$) by the formulas

$$\rho(g; \hat{m}) = \alpha(g, e, e; \hat{m})\rho(e; \hat{m})$$

$$\lambda(k; \hat{l}) = \alpha^{-1}(e, e, k; \hat{l})\rho(e; \widehat{k\hat{l}k^{-1}})$$

$$r(g; \hat{k}) = \beta(g; \hat{k}, \hat{e}, \hat{e})r(g; \hat{e})$$

$$l(g; \hat{k}) = \beta(g; \hat{e}, \hat{e}, \hat{k})r(g; \hat{e})$$

The only restriction on these needed to ensure that they unambiguously determine cochains satisfying the first two equations has already been imposed by the condition that α and β be coboundaries in the suitable sense.

In a similar way, \otimes respecting both the right and left counit conditions is equivalent to

$$\tau(g, k) = \tilde{d}_1(r^{-1})(g, k; \hat{e})\phi^{-1}(g, k; \hat{e}, \hat{e})$$

with no further conditions imposed on r or ϕ .

On the other hand, Δ respecting both the right and left unit conditions is equivalent to

$$\delta(\hat{k}, \hat{l}) = d_2(\rho^{-1})(e; \hat{k}, \hat{l})\phi^{-1}(e, e; \hat{k}, \hat{l})$$

together with the condition that δ be invariant under simultaneous conjugation of both indices by elements of G .

Given these last two equations, the conditions that ϵ preserve \otimes and I preserve Δ follow from the condition that (α, ϕ, β) be a cocycle and the fact that \tilde{d}_1 and d_2 distribute over multiplication of cochains and $\tilde{d}_1^2 = 1$ and $d_2^2 = 1$ (remember we write the operation on cochains multiplicatively since the coefficients are in K^\times).

The four remaining conditions are all equivalent to

$$\eta = \rho(e; \hat{e})r(e; \hat{e})\phi(e, e; \hat{e}, \hat{e})$$

A similar analysis shows that the structural transformations for an equivalence of unital and counital categorifications are determined by the 1,2-cochain and 2,1-cochain which determine the non-unital non-counital equivalence, together with a 0,1-cochain f_0 and a 1,0-cochain f^0 .

The conditions besides the cobounding conditions of Theorem 3.1 reduce to

$$f_0(\hat{k}) = \frac{\rho(e; \hat{k})}{\rho'(e; \hat{k})\tilde{f}(e, e; \hat{k})}$$

$$f^0(g) = \frac{r(g, \hat{e})}{r'(g, \hat{e})f_{\sim}(g; \hat{e}, \hat{e})}$$

Thus, it follows that within any equivalence class of categorifications, the *only* constraint upon the choice of the functions $\rho(e; \hat{h})$ and $r(g; \hat{e})$ is the condition that $\delta(\hat{k}, \hat{l})$ be invariant under simultaneous conjugation of both indices. We have thus almost shown

Theorem 3.2 *Every skeletal biunital semi-simple bitensor K -categorification of $D(\mathbb{N}[G])$ is determined by a choice of a 3-cocycle (α, ϕ, β) in the total complex of the double complex $C_{i,j}$, together with a choice of functions $\rho(e; \hat{k}) : \hat{G} \rightarrow K^\times$ and $r(g; \hat{e}) : G \rightarrow K^\times$ subject to the condition that $\phi(e, e; \hat{k}, \hat{l})d_2(\rho)(e, \hat{k}, \hat{l})$ be invariant under simultaneous conjugation of the hatted indices. Conversely, every such choice determines such a categorification up to isomorphism. The equivalence classes of unital counital bitensor categorifications of $D(\mathbb{N}[G])$ are in natural one-to-one correspondence with $\mathbb{H}_3(G, K^\times)$.*

proof: For the first two statements, it remains only to observe that the condition on δ is equivalent to the given condition on ϕ and ρ .

The final statement requires a little more work. By the preceding remark, it suffices to show that every cohomology class admits a representative for which the $\rho(e; \hat{k})$ can be chosen so the invariance condition holds. Let (α, ϕ, β) be an arbitrary 3-cocycle. Now, consider the 2-cochain $(1, f)$, where 1 is the constant 2,1-cochain, and $f(g; \hat{k}, \hat{l}) = \phi^{-1}(e, e; \hat{k}, \hat{l})$. Multiplying (α, ϕ, β) by the (total) coboundary of $(1, f)$ give a cohomologous 3-cocycle such that $\phi(e, e; \hat{k}, \hat{l}) = 1$. Thus any constant $\rho(e; \hat{k})$ suffices. \square

Now observe that for any group G there is at least one solution to the required equations: if we choose *all* of the families of scalars to be identically 1, we obtain a solution. We will refer to this and any equivalent bitensor categorifications as **trivial categorifications** of $D(\mathbb{N}[G])$.

Of course, it behooves us to exhibit a non-trivial bitensor categorification, since we as yet have no examples. The simplest family of such may be described as follows: let all of the families of scalars be identically 1 except $\beta(g; \hat{i}, \hat{j}, \hat{k})$. Observe that all conditions not involving β are trivially satisfied, and that the conditions involving β then reduce to those defining other quantities in terms of β and the other scalars, and the conditions

$$\beta(g; \widehat{h\hat{i}h^{-1}}, \widehat{h\hat{j}h^{-1}}, \widehat{h\hat{k}h^{-1}})\beta(h; \hat{i}, \hat{j}, \hat{k}) = \beta(gh; \hat{i}, \hat{j}, \hat{k})$$

and

$$\beta(g; \hat{i}, \hat{j}, \hat{k})\beta(g; \hat{i}, \hat{j}\hat{k}, \hat{l})\beta(g; \hat{j}, \hat{k}, \hat{l}) = \beta(g; \hat{i}\hat{j}, \hat{k}, \hat{l})\beta(g; \hat{i}, \hat{j}, \hat{k}\hat{l})$$

Thus, we may regard $\beta(g; -, -, -)$ as a function from G to 3-cocycles on G (written with hatted indices), satisfying the first equation. In particular, any group homomorphism from G to the (abelian) group of 3-cocycles invariant under simultaneous conjugation gives such a function.

For a specific example, let G be any group with C_2 , the cyclic group of order 2, as a quotient (e.g. $G = \mathfrak{S}_n$). Call an element of G even when its image in C_2 is the identity; odd otherwise. In this case β given by

$$\begin{aligned} \beta(g; \hat{i}, \hat{j}, \hat{k}) &= -1 \text{ if } g, i, j, k \text{ are all odd} \\ &= 1 \text{ otherwise} \end{aligned}$$

has all of the desired properties. In particular, $(1, 1, \beta)$ represents a non-trivial element of $\mathbb{H}_3(G, K^\times)$ (provided $\text{char}(K) \neq 2$).

4 Conclusions

The cohomological setting which provided a natural setting for these constructions and classification theorems suggests that the process of categorification should, at least in the semi-simple case, be viewed as a deformation process for tensor or bitensor categories.

The authors, in work in preparation [7] have constructed a general framework for the infinitesimal deformation of general (semi-simple) tensor (resp. bitensor) categories in terms of a similar complex (resp. double complex), and have isolated the obstructions to the existence of formal power-series deformations as cochains. At present it is unclear whether the obstruction cochains are closed or whether the obstructions have a non-cohomological aspect.

It still remains to use the examples presented herein to provide explicit examples of 4-manifold invariants of Crane/Frenkel type [4] and to construct the monoidal bicategory of representations of the bitensor categories constructed herein, thereby giving initial data for a fully bicategorical version of the Crane-Yetter construction (cf. [6], [8]).

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