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XIANDONG CHEN

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STABLY CLOSED FRAME HOMOMORPHISMS

by CHEN XIANDONG

RESUME. Dans cet article, on donne divers résultats sur les sommes fibrées d'homomorphismes de cadres ("frames"). Les homomorphismes stablement fermés et parfaits sont étudiés et caractérisés dans les catégories des cadres cohérents, des cadres continus et des cadres complètement réguliers.

As the counterparts of the classical closed continuous maps of topological spaces and dual to the open frame homomorphisms, closed frame homomorphisms have been defined naturally. The importance of this notion has been shown in Dowker-Papert [13], Pultr-Tozzi [21] and Chen [11], dealing with paracompactness, the pointfree Kuratowski-Mrówka theorem and local connectedness, respectively.

This paper arises from the desire to consider the frame homomorphisms whose pushouts in the category of frames are closed. It is known that coproducts and pushouts need not preserve closedness of homomorphisms in the category of frames. Thus perfect and stably closed homomorphisms are introduced naturally (Definition 2.2, 2.3), and then analyzed in section 2. Interesting characterizations of these homomorphisms in the categories of coherent frames, regular continuous frames and completely regular frames are presented in sections 3, 4, and 5, respectively. Finally, we apply the concept of perfect homomorphisms to the study of perfect-injectives and Gleason envelope in the category of completely regular frames.

For general background on frames, we refer to Johnstone [17].

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1 General Facts

Let L be a frame. The top (bottom) element of L will be denoted by e (0). For $a \in L$, its pseudocomplement is denoted by a^*, defined by $a^* = \bigvee\{x \in L | x \wedge a = 0\}$. For a frame homomorphism $h : L \longrightarrow M$, its right adjoint is denoted by $h_* : M \longrightarrow L$ and is given by $h_*(b) = \bigvee\{x \in L | h(x) \leq b\}$. A homomorphism $h : L \longrightarrow M$ is called dense if $h(x) = 0$ implies $x = 0$; it is called codense if $h(x) = e$ implies $x = e$.

For a frame L, we use $\mathfrak{C}L$ to denote its congruence frame. The correspondence $\mathfrak{C} : \mathbf{Frm} \longrightarrow \mathbf{Frm}$ is a faithful functor such that, for any $h : L \longrightarrow M$, $\mathfrak{C}h : \mathfrak{C}L \longrightarrow \mathfrak{C}M$ takes a congruence on L to the congruence generated by its image under $h \times h$. The right adjoint of $\mathfrak{C}h$ is simply $(h \times h)^{-1} : \mathfrak{C}M \longrightarrow \mathfrak{C}L$. The top and bottom of $\mathfrak{C}L$ are denoted by ∇ and Δ. For any $a \in L$, $\nabla_a = \{(x, y) | x \vee a = y \vee a\}$, called closed, is the least congruence containing $(0, a)$; $\Delta_a = \{(x, y) | x \wedge a = y \wedge a\}$, called open, is the least congruence containing (e, a). Each ∇_a is complemented in $\mathfrak{C}L$ with complement Δ_a. The map $\nabla^L : L \longrightarrow \mathfrak{C}L$ defined by $a \rightsquigarrow \nabla_a$ is a frame embedding which is also an epimorphism in $\mathbf{Frm}$, whereas, the map $a \rightsquigarrow \Delta_a$ is a dual poset embedding $L \longrightarrow \mathfrak{C}L$ taking finitary $\wedge$ to finitary$\vee$ and arbitrary $\bigvee$ to arbitrary $\bigwedge$.

Let B be a sub-join-semilattice of L. We use $\mathfrak{C}_B L$ to denote the subframe of $\mathfrak{C}L$ consisting of congruences of L expressible as joins of congruences of the form $\nabla_a \wedge \Delta_b$, where $a \in L$ and $b \in B$. We can easily obtain the next result, which has been presented in Jibladze and Johnstone [15] for the case of B as a subframe of L.

Proposition 1.1 *(1) The map $\nabla^L : L \longrightarrow \mathfrak{C}_B L$ defined by $x \rightsquigarrow \nabla_x$ is a frame embedding which is also an epimorphism in* $\mathbf{Frm}$.

(2) $\nabla^L : L \longrightarrow \mathfrak{C}_B L$ is universal among all homomorphisms $h : L \longrightarrow M$ such that $h[B]$ is contained in the Boolean part BM, the set of complemented elements of M.

(3) For a homomorphism $h : L \longrightarrow M$, the restriction of $\mathfrak{C}h$ on $\mathfrak{C}_B L$ determines a homomorphism $\mathfrak{C}_B h : \mathfrak{C}_B L \longrightarrow \mathfrak{C}_{h[B]} M$. Moreover the following diagram is a pushout:

$$\begin{array}{ccc} L & \xrightarrow{h} & M \\ \downarrow & & \downarrow \\ \mathfrak{C}_B L & \xrightarrow[\mathfrak{C}_B h]{} & \mathfrak{C}_{h[B]} M \end{array}$$

Now, let us recall some results on binary coproducts in the category of frames. Consider two frames L_1 and L_2, and let $D(L_1 \times L_2)$ be the frame of all downsets of $L_1 \times L_2$. Then the coproduct $L_1 \oplus L_2$ is represented (cf. [6], [11]) by the frame $\mathrm{Fix}(\pi)$, where π is the nucleus on $D(L_1 \times L_2)$ such that, for any $U \in D(L_1 \times L_2)$, $U = \pi(U)$ if and only if

$$X \times \{y\} \subseteq U \text{ implies } (\bigvee X, y) \in U, \text{ and } \{x\} \times Y \subseteq U \text{ implies } (x, \bigvee Y) \in U.$$

For $a \in L_1$ and $b \in L_2$, let $a \oplus b$ denote $\downarrow (a,b) \cup \downarrow (0,e) \cup \downarrow (e,0)$, the smallest downset containing (a,b) and fixed by π. Then, the coproduct maps $q_i : L_i \longrightarrow L_1 \oplus L_2$ $(i = 1,2)$ are given by $q_1(x) = x \oplus e$ and $q_2(y) = e \oplus y$.

Before we recall Proposition 1.2, we introduce the following nuclei (for detail, see [10], [12]): $\pi_1, \hat{\pi}_1, \pi_2, \hat{\pi}_2 : D(L_1 \times L_2) \longrightarrow D(L_1 \times L_2)$ are defined respectively by

$$\pi_1(U) = \{(\bigvee X, y) | X \times \{y\} \subseteq U\},$$

$$\hat{\pi}_1(U) = \{(\bigvee X, y) | X \text{ is finite and } X \times \{y\} \subseteq U\},$$

$$\pi_2(U) = \{(x, \bigvee Y) | \{x\} \times Y \subseteq U\},$$

$$\hat{\pi}_2(U) = \{(x, \bigvee Y) | Y \text{ is finite and } \{x\} \times Y \subseteq U\}.$$

Proposition 1.2 **([2], [10], [12], [22])** *For any $U \in D(L_1 \times L_2)$, if $a \in L_1$ is compact and $(a,b) \in \pi(U)$, then $(a,b) \in \pi_2 \circ \hat{\pi}_1(U)$.*

Concerning the frame version of Hausdorff spaces, the following results are known. For any frame L, the codiagonal map $\nabla : L \oplus L \longrightarrow L$, given by $x \oplus y \rightsquigarrow x \wedge y$, is the coequalizer of the copoduct maps: $q_1, q_2 : L \longrightarrow L \oplus L$. As usual, ∇ has a dense factorization: $\nabla : L \oplus L \xrightarrow{(.)\vee s} \uparrow s \longrightarrow L$, where $s = \bigvee\{a \oplus b | a, b \in L, a \wedge b = 0\}$, called the separator of L.

We shall call a frame L separated if the codiagonal map ∇ is closed, that is, $\nabla \cong (.) \vee s$ for $s = \bigvee\{a \oplus b | a, b \in L, a \wedge b = 0\}$ (such L is also called strongly Hausdorff by Isbell [14]).

Proposition 1.3 **([1], [12])** *The following are equivalent for any frame L:*

(1) *L is separated.*

(2) *In $L \oplus L$, $(e \oplus a) \vee s = (a \oplus e) \vee s$ for all $a \in L$, where s is the separator of L.*

(3) *For any $h_1, h_2 : L \longrightarrow M$, $(.) \vee t : M \longrightarrow \uparrow t$ is the coequalizer, where $t = \bigvee\{h_1(a) \wedge h_2(b) | a, b \in L, a \wedge b = 0\}$.*

(4) *For any $h_1, h_2 : L \longrightarrow M$, $h_1(a) \vee t = h_2(a) \vee t$ for all $a \in L$.*

Proposition 1.4 *For any frame homomorphism $h : L \longrightarrow M$, there exists a unique onto homomorphism $G(h) : L \oplus M \longrightarrow M$ given by $x \oplus y \rightsquigarrow h(x) \wedge y$, which is the coequalizer of*

$$(id_L \oplus h) \circ q_1,\ (id_L \oplus h) \circ q_2 : L \rightrightarrows L \oplus L \longrightarrow L \oplus M.$$

Moreover, the following square is a pushout:

$$\begin{array}{ccc} L \oplus L & \xrightarrow{\nabla} & L \\ {\scriptstyle id_L \oplus h}\downarrow & & \downarrow{\scriptstyle h} \\ L \oplus M & \xrightarrow[G(h)]{} & M \end{array}$$

2 Closed Homomorphisms

Recall that a homomorphism $h : L \longrightarrow M$ is called closed if

$$h_*(h(x) \vee y) = x \vee h_*(y) \text{ for any } x \in L,\ y \in M.$$

An interesting characterization is that a homomorphism $h : L \longrightarrow M$ is closed if and only if

$$(h \times h)^{-1}(\nabla_u) = \nabla_{h_*(u)} \text{ for each } u \in M.$$

The next result easily follows from the definition.

Proposition 2.1 *Consider* $L \xrightarrow{f} M \xrightarrow{g} N$.

(1) *If* f *and* g *are closed, so is* $g \circ f$.

(2) *If* $g \circ f$ *is closed and* g *is one-one , then* f *is closed.*

(3) *If* $g \circ f$ *is closed and* f *is onto, then* g *is closed.*

Definition 2.1 *For any homomorphism* $h : L \longrightarrow M$, *define a set*

$$C_h(L) = \{x | h_*(h(x) \vee y) = x \vee h_*(y) \text{ for all } y \in M\}.$$

Lemma 2.1 $C_h(L)$ *is a sublattice of* L.

Lemma 2.2 *If* $x \in L$ *is complemented, then* $x \in C_h(L)$ *and hence* $C_h(L)$ *contains the Boolean part* BL.

PROOF. For any $y \in M$, let $r = h_*(h(x) \vee y)$. Then $r = (r \wedge x) \vee (r \wedge x^*)$ with $h(r \wedge x^*) = h(r) \wedge h(x^*) \leq (h(x) \vee y) \wedge h(x^*) = y \wedge h(x^*) \leq y$, hence $r \leq x \vee h_*(y)$. This shows $h_*(h(x) \vee y) \leq x \vee h_*(y)$, that is, $x \in C_h(L)$. ∎

Lemma 2.3 *Let* $h : L \longrightarrow M$ *be dense. If* $x \in C_h(L)$ *and* $h(x) \in BM$, *then* $x \in BL$.

PROOF. On the one hand, $e = h_*(h(x) \vee h(x)^*) = x \vee h_*(h(x)^*)$. On the other hand, $h(x) \wedge hh_*(h(x)^*) \leq h(x) \wedge h(x)^* = 0$, implying $x \wedge h_*(h(x)^*) = 0$. Hence $h_*(h(x)^*)$ is the complement of x. ∎

Proposition 2.2 *A frame* L *is Boolean if and only if any homomorphism* $h : L \longrightarrow M$ *is closed.*

PROOF. The " only if " part follows Lemma 2.2. For the " if " part, apply Lemma 2.3 to the homomorphism $\nabla : L \longrightarrow \mathfrak{C}L$. ∎

Given a homomorphism $h : L \longrightarrow M$ and a congruence $\theta \in \mathfrak{C}L$, there exists an induced homomorphism $h_\theta : L/\theta \longrightarrow M/\mathfrak{C}h(\theta)$. Actually, we get a pushout square:

$$\begin{array}{ccc} L & \xrightarrow{h} & M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ L/\theta & \xrightarrow[h_\theta]{} & M/\mathfrak{C}h(\theta) \end{array}$$

Question: When is h_θ closed?

Lemma 2.4 *For any $\theta \in \mathfrak{C}L$, the closed quotient maps of L/θ are exactly those expressed as $L/\theta \longrightarrow L/(\theta \vee \nabla_u)$ for some $u \in L$.*

PROOF. The following squares are pushouts:

$$\begin{array}{ccc} L & \longrightarrow & L/\theta \\ \downarrow & & \downarrow \\ L/\nabla_u & \longrightarrow & L/(\theta \vee \nabla_u) \end{array} \quad \text{and} \quad \begin{array}{ccc} L & \longrightarrow & L/\theta \\ \downarrow & & \downarrow \\ \uparrow u & \longrightarrow & \uparrow [u] \end{array}$$

where $[u]$ is the image of u under the quotient map $L \longrightarrow L/\theta$. Therefore $L/\theta \longrightarrow \uparrow [u]$ is same as $L/\theta \longrightarrow L/(\theta \vee \nabla_u)$. ∎

Proposition 2.3 *For any frame homomorphism $h : L \longrightarrow M$ and $\theta \in \mathfrak{C}L$, the induced homomorphism $h_\theta : L/\theta \longrightarrow M/\mathfrak{C}h(\theta)$ is closed if and only if, for each $u \in M$,*

$$(h \times h)^{-1}(\mathfrak{C}h(\theta) \vee \nabla_u) = \theta \vee \nabla_v \textit{ for some } v \in L.$$

PROOF. The homomorphism h_θ is closed if and only if, for every $u \in M$, there exists $v \in L$ such that the right square in the following diagram commutes and the homomorphism g is one-one.

$$\begin{array}{ccccc} L & \longrightarrow & L/\theta & \longrightarrow & L/(\theta \vee \nabla_v) \\ {\scriptstyle h}\downarrow & & {\scriptstyle h_\theta}\downarrow & & \downarrow{\scriptstyle g} \\ M & \longrightarrow & M/\mathfrak{C}h(\theta) & \longrightarrow & M/(\mathfrak{C}h(\theta) \vee \nabla_u) \end{array}$$

But the right square commutes if and only if the outer square commutes, and the latter means $(h \times h)^{-1}(\mathfrak{C}h(\theta) \vee \nabla_u) = \theta \vee \nabla_v$ when g is one-one. ∎

Proposition 2.4 *If $h : L \longrightarrow M$ is closed and θ is complemented in $\mathfrak{C}L$, then $h_\theta : L/\theta \longrightarrow M/\mathfrak{C}h(\theta)$ is closed.*

PROOF. By Lemma 2.2, we have

$$(h \times h)^{-1}(\mathfrak{C}h(\theta) \vee \nabla_u) = \theta \vee (h \times h)^{-1}(\nabla_u) \text{ for any } u \in M.$$

But $(h \times h)^{-1}(\nabla_u) = \nabla_{h_*(u)}$ since h is closed. Therefore

$$(h \times h)^{-1}(\mathfrak{C}h(\theta) \vee \nabla_u) = \theta \vee \nabla_{h_*(u)},$$

which means that h_θ is closed by Proposition 2.3. ∎

Remark: Recall that, in topological spaces, for a closed continuous map $f : X \longrightarrow Y$ and any subspace A of Y, the restriction map $f_A : f^{-1}(A) \longrightarrow A$ is a pullback of f and is closed. To consider the frame counterpart of this fact, it is natural to ask whether Proposition 2.4 also holds for non-complemented θ. We do not know the answer yet.

In general, closedness is not preserved under coproducts and pushouts, so we introduce two more concepts concerning closedness.

Definition 2.2 *A homomorphism $h : L \longrightarrow M$ is called perfect if $h \oplus id_N : L \oplus N \longrightarrow M \oplus N$ is closed for any frame N.*

Remark: Elsewhere, the term "*perfect*" has been introduced using some topos theoretical notions involving the corresponding sheaves (Johnstone [20]). We did not explore the precise relationship between these notions. The present definition is directly motivated by the usual topological definition, translated into the category of frames.

Definition 2.3 *A homomorphism $h : L \longrightarrow M$ is called stably closed if every pushout of h is closed.*

Since $h \oplus id_N : L \oplus N \longrightarrow M \oplus N$ is the pushout of h along $L \longrightarrow L \oplus N$, we know that

Proposition 2.5 *Any stably closed homomorphism is perfect.*

Conversely, we have

Proposition 2.6 *If $h : L \longrightarrow M$ is perfect and L is separated, then h is stably closed.*

PROOF. Any homomorphism $f : L \longrightarrow N$ can be factored as $L \xrightarrow{q_1} L \oplus N \xrightarrow{G(f)} N$. The following squares are pushout:

$$\begin{array}{ccccc} L & \xrightarrow{q_1} & L \oplus N & \xrightarrow{G(f)} & N \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle h \oplus id_N}\big\downarrow & & \big\downarrow{\scriptstyle \bar{h}} \\ M & \longrightarrow & M \oplus N & \longrightarrow & P \end{array}$$

The separatedness of L implies $G(f)$ is closed by Proposition 1.3 and 1.4. Then $\bar{h}$ is closed by Proposition 2.4. ∎

Remark: We do not know whether this result holds without assuming the separatedness of L.

Proposition 2.7 *Any closed onto homomorphism is stably closed.*

PROOF. Consider a closed onto homomorphism $h = (.) \vee u : L \longrightarrow \uparrow u$. For any homomorphism $g : L \longrightarrow M$, the pushout of h along g is $(.) \vee g(u) : M \longrightarrow \uparrow g(u)$. ∎

Proposition 2.8 *Consider $L \xrightarrow{f} M \xrightarrow{g} N$.*

1. *If f ang g are perfect (stably closed), so is $g \circ f$.*

2. If $g \circ f$ is perfect and f is onto, then g is perfect.

3. If $g \circ f$ is stably closed and f is epic, then g is stably closed.

4. If $g \circ f$ is perfect and M is separated, then g is stably closed.

PROOF. 1. Trivial.

2. For any frame Q, consider

$$L \oplus Q \xrightarrow{f \oplus id_Q} M \oplus Q \xrightarrow{g \oplus id_Q} N \oplus Q$$

The composite is closed and $f \oplus id_Q$ is again onto, and thus $g \oplus id_Q$ is closed by Proposition 2.1.

3. For any homomorphism $h : M \longrightarrow Q$, consider the diagram

$$\begin{array}{ccccc} L & \xrightarrow{f} & M & \xrightarrow{g} & N \\ & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \bar{h}} \\ & & Q & \xrightarrow{\bar{g}} & P \end{array}$$

where $\bar{g}$ is the pushout of g along h. Since f is epic, $\bar{g}$ is also the pushout of $g \circ f$ along $h \circ f$, hence $\bar{g}$ is closed.

4. Consider the commuting square:

$$\begin{array}{ccc} M \oplus L & \xrightarrow{G(f)} & M \\ {\scriptstyle id_M \oplus (g \circ f)}\downarrow & & \downarrow{\scriptstyle g} \\ M \oplus N & \xrightarrow[G(g)]{} & N \end{array}$$

Since M is separated, $G(g)$ is closed, therefore $G(g)$ is perfect since it is onto. Again $id_M \oplus (g \circ f)$ is perfect, hence $g \circ G(f) = G(g) \circ (id_M \oplus (g \circ f))$ is perfect, thus g is perfect since $G(f)$ is onto. Finally, applying Proposition 2.6, we know that g is stably closed. ∎

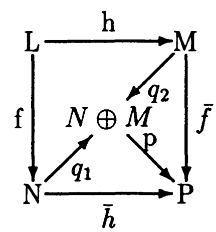
Now, we recall an important result, which is constructively valid and has been studied by [21], [22] and [10].

Proposition 2.9 *A frame M is compact if and only if $q_1 : L \longrightarrow L \oplus M$ is closed for any frame L.*

Corollary *If $h : L \longrightarrow M$ is perfect and L is compact, then M is also compact.*

Proposition 2.10 *If L is separated and M compact, then any homomorphism $h : L \longrightarrow M$ is stably closed.*

PROOF. Considering the standard construction of pushouts, we can get the pushout square as follows:



where q_1, q_2 are the coproduct maps and p is the coequalizer of $q_1 \circ f, q_2 \circ h$. Since L is separated, p is closed by Proposition 1.3. And since M is compact, q_1 is closed by Proposition 2.9. Therefore $\bar{h} = p \circ q_1$ is closed. ∎

3 Coherent Frames

Recall that a frame L is coherent if its compact elements generate L as a frame and compact elements of L form a sublattice KL of L, including $0, e \in L$. Coherent frames and coherent (=compactness preserving) homomorphisms constitute the category **CohFrm**.

The following fact is well known:

Lemma 3.1 *Any bounded distributive lattice A can be embedded into a Boolean algebra HA such that HA is generated, in Boolean terms, by A. Moreover, the correspondence $A \rightsquigarrow HA$ is functorial, providing the reflection of the category of bounded distributive lattices to the category of Boolean algebra.*

It is a familiar fact that a coherent frame L is isomorphic to the ideal lattice $\mathfrak{J}KL$ of KL. Thus, for a coherent frame L, the embedding $KL \longrightarrow HKL$ induces a coherent embedding $L \cong \mathfrak{J}KL \longrightarrow \mathfrak{J}HKL$. Moreover,

Proposition 3.1 **StFrm** *(the category of compact 0-dimensional frames) is reflective in* **CohFrm**, *with the reflection map $L \longrightarrow \mathfrak{J}HKL$.*

Now consider $\mathfrak{C}_{KL}L$, which is the subframe of $\mathfrak{C}L$ consisting of congruences expressible as joins of congruences of the form $\nabla_a \wedge \Delta_b$ with $a \in L, b \in KL$. Since the universal property of $\nabla^L : L \longrightarrow \mathfrak{C}_{KL}L$ by Proposition 1.1, there is a homomorphism $\bar{\imath} : \mathfrak{C}_{KL}L \longrightarrow \mathfrak{J}HKL$ making the diagram commuting:

$$\begin{array}{ccc} L & \xrightarrow{\nabla^L} & \mathfrak{C}_{KL}L \\ & \searrow^{i} & \downarrow \bar{\imath} \\ & & \mathfrak{J}HKL \end{array}$$

Notice that $\mathfrak{C}_{KL}L$ is 0-dimensional, thus $\bar{\imath}$ is closed. It is easy to check that $\bar{\imath}$ is dense, therefore is one-one according to the fact that *any dense closed homomorphism is one-one.* Thus $\mathfrak{C}_{KL}L$ is compact as a subframe of $\mathfrak{J}HKL$. For each $x \in KL$, ∇_x is also compact in $\mathfrak{C}_{KL}L$, that is, $\nabla^L : L \longrightarrow \mathfrak{C}_{KL}L$

is coherent. Now we can easily obtain the next fact, which could also be derived from results of Banaschewski and Brümmer [8].

Proposition 3.2 *The map $\nabla^L : L \longrightarrow \mathfrak{C}_{KL}L$ provides the reflection map for* **StFrm** *to be reflective in* **CohFrm**.

Definition 3.1 *Let L be a coherent frame. A congruence $\theta \in \mathfrak{C}L$ is called coherent if the quotient map $L \longrightarrow L/\theta$ is coherent.*

We have the following observations:

(1) Any ∇_a is coherent.

(2) For compact $a \in L$, Δ_a is coherent.

(3) A congruence θ is coherent if and only if $L/(\Delta_x \vee \theta)$ is compact for any compact $x \in L$.

(4) L/ϕ is compact if and only if, for any $X \subseteq L$, $\Delta_{\bigvee X} \leq \phi$ implies $\Delta_{\bigvee S} \leq \phi$ for some finite $S \leq X$.

(5) If θ_1, θ_2 are coherent, then $\theta_1 \wedge \theta_2$ is coherent by applying (3) and (4).

(6) For a set Θ of coherent congruences, $\bigvee\{\theta | \theta \in \Theta\}$ in $\mathfrak{C}L$ is determined by the multiple pushout, in **Frm**, of $\{L \longrightarrow L/\theta | \theta \in \Theta\}$. Since coherent frames are precisely the free frames generated by distributive lattices, the class of coherent frames is closed under colimits in **Frm**. Hence $\bigvee \Theta$ is coherent.

Proposition 3.3 *For a coherent frame L, $\mathfrak{C}_{KL}L$ consists of all coherent congruences of L.*

PROOF. From the above observation, we know that any element of $\mathfrak{C}_{KL}L$ is coherent.

Now we show that any coherent congruence θ belongs to $\mathfrak{C}_{KL}L$. Take

$$\phi = \bigvee\{\Delta_d \wedge \nabla_c | \Delta_d \wedge \nabla_c \leq \theta \text{ with compact } d, c\},$$

then $\phi \leq \theta$ and $\phi \in \mathfrak{C}_{KL}L$. We need to show $\theta \leq \phi$. Consider arbitrary $\Delta_x \wedge \nabla_y \leq \theta$. For every compact $c \leq y$, $\Delta_x \wedge \nabla_c \leq \theta$ implies $\Delta_x \leq \theta \vee \Delta_c$. By (4) and (6) there is a compact $d_c \leq x$ such that $\Delta_{d_c} \leq \theta \vee \Delta_c$, then $\Delta_x \wedge \nabla_c \leq \phi$ since $\Delta_{d_c} \wedge \nabla_c \leq (\theta \vee \Delta_c) \wedge \nabla_c \leq \theta$. Therefore $\Delta_x \wedge \nabla_y = \bigvee\{\Delta_x \wedge \nabla_c |$ compact $c \leq y\} \leq \phi$. This proves $\theta = \phi \in \mathfrak{C}_{KL}L$. ∎

Lemma 3.2 *Let $h : L \longrightarrow M$ be a homomorphism such that $\mathfrak{C}_L h : \mathfrak{C}L \longrightarrow \mathfrak{C}_{h[L]}M$ is closed.*

(1) If h is an embedding, so is $h_\theta : L/\theta \longrightarrow M/\mathfrak{C}h(\theta)$ for every $\theta \in \mathfrak{C}L$.

(2) If h is closed, so is $h_\theta : L/\theta \longrightarrow M/\mathfrak{C}h(\theta)$ for every $\theta \in \mathfrak{C}L$.

PROOF. Notice that $\mathfrak{C}h : \mathfrak{C}L \longrightarrow \mathfrak{C}M$ is factored as $\mathfrak{C}L \xrightarrow{\mathfrak{C}_L h} \mathfrak{C}_{h[L]}M \longrightarrow \mathfrak{C}M$ where $\mathfrak{C}_{h[L]}M \longrightarrow \mathfrak{C}M$ is the identical embedding.

(1) When h is an embedding, $\mathfrak{C}h$ is dense, implying $\mathfrak{C}_L h$ is dense, and then $\mathfrak{C}_L h$ is one-one since $\mathfrak{C}_L h$ is closed. It turns out that $\mathfrak{C}h$ is one-one, which is equivalent to the fact that $h_\theta : L/\theta \longrightarrow M/\mathfrak{C}h(\theta)$ is one-one for every $\theta \in \mathfrak{C}L$.

(2) $\mathfrak{C}_L h : \mathfrak{C}L \longrightarrow \mathfrak{C}_{h[L]}M$ is closed means that

$$(h \times h)^{-1}(\mathfrak{C}h(\theta) \vee \psi) = \theta \vee (h \times h)^{-1}(\psi) \text{ for any } \theta \in \mathfrak{C}L \text{ and } \psi \in \mathfrak{C}_{h[L]}M.$$

In particular, for any $\theta \in \mathfrak{C}L$ and $u \in M$,

$$(h \times h)^{-1}(\mathfrak{C}h(\theta) \vee \nabla_u) = \theta \vee (h \times h)^{-1}(\nabla_u) \quad \text{since } \nabla_u \in \mathfrak{C}_{h[L]}M.$$

When h is also closed, we have

$$(h \times h)^{-1}(\mathfrak{C}h(\theta) \vee \nabla_u) = \theta \vee \nabla_{h_*(u)},$$

which indicates that $h_\theta : L/\theta \longrightarrow M/\mathfrak{C}h(\theta)$ is closed by Proposition 2.3. ∎

Lemma 3.3 *If $h : L_1 \longrightarrow L_2$ is coherent and one-one, then $h \oplus id_N : L_1 \oplus N \longrightarrow L_2 \oplus N$ is one-one for every frame N.*

PROOF. Take an arbitrary $U \in L_1 \oplus N$. Put $V = \downarrow\{(h(x), y) | (x, y) \in U\}$, then $h \oplus id_N(U) = \pi(V)$. It is easy to check that V is fixed by π_1. Then

$$\pi_2(\hat{\pi}_1(V)) = \pi_2(V) = \downarrow\{(\bigwedge_{(x,y)\in K} h(x), \bigvee_{(x,y)\in K} y) | K \subseteq U\}.$$

Consider $a \oplus b \in L_1 \oplus N$ such that a is compact and $h(a) \oplus b \leq h \oplus id_N(U) = \pi(V)$. We have $(h(a), b) \in \pi_2(\hat{\pi}_1(V))$ since $h(a)$ is compact and Propositon 1.2, so there is a $K \subseteq U$ such that $h(a) \leq \bigwedge\{h(x) | (x,y) \in K\}$ and $b \leq \bigvee\{y | (x,y) \in K\}$, hence $a \leq x$ for every $(x,y) \in K$ since h is one-one. Then $(a,b) \in U$, that is, $a \oplus b \leq U$. This proves that $h \oplus id_N$ is one-one. ∎

Lemma 3.4 *If $h : L \longrightarrow M$ is closed and L has a basis B such that $h(B)$ consists of some compact elements of M, then $h \oplus id_N : L \oplus N \longrightarrow M \oplus N$ is closed for every frame N.*

PROOF. See Proposition 4.3 in [10]. ∎

Now we are ready for the main result of this section.

Proposition 3.4 *Let L, M and $h : L \longrightarrow M$ be coherent.*

(1) If h is closed, then h is stably closed.

(2) If h is one-one, then the pushout of h along arbitrary homomorphism $g : L \longrightarrow N$ is one-one.

PROOF. First, we have a pushout square in **Frm**:

$$\begin{array}{ccc} L & \longrightarrow & \mathfrak{C}_{KL}L \\ h\downarrow & & \downarrow h_k \\ M & \longrightarrow & \mathfrak{C}_{h[KL]}M \end{array}$$

Since $h[KL] \subseteq KM$, $\mathfrak{C}_{h[KL]}M$ is a subframe of $\mathfrak{C}_{KM}M$. Then $\mathfrak{C}_{h[KL]}M$ is compact. It follows that h_k is stably closed by Proposition 2.10.

Let N be an arbitrary frame. Consider the following pushout squares:

$$\begin{array}{ccccccc} L \oplus N & \longrightarrow & \mathfrak{C}_{KL}L \oplus N & \longrightarrow & \mathfrak{C}L \oplus N & \longrightarrow & \mathfrak{C}(L \oplus N) \\ h \oplus id_N \downarrow & & h_k \oplus id_N \downarrow & & \mathfrak{C}_L h \oplus id_N \downarrow & & \downarrow f \\ M \oplus N & \longrightarrow & \mathfrak{C}_{h[KL]}M \oplus N & \longrightarrow & \mathfrak{C}_{h[L]}M \oplus N & \longrightarrow & P \end{array}$$

Since $h_k \oplus id_N$ is the pushout of h_k along $\mathfrak{C}_{KL}L \longrightarrow \mathfrak{C}_{KL}L \oplus N$, it follows that $f = \mathfrak{C}_{L\oplus N}h \oplus id_N$ is also a pushout of h_k. Therefore f is closed since h_k is stably closed.

Now, for arbitrary homomorphism $g : L \longrightarrow N$, the pushout of h along g is the same as the pushout of $h \oplus id_N$ along $G(g) : L \oplus N \longrightarrow N$, as shown in the following diagram:

$$\begin{array}{ccccc} L & \xrightarrow{q_1} & L \oplus N & \xrightarrow{G(g)} & N \\ h\big\downarrow & & h \oplus id_N \big\downarrow & & \big\downarrow \bar{h} \\ M & \longrightarrow & M \oplus N & \longrightarrow & P \end{array}$$

(1) If h is closed, $h \oplus id_N$ is also closed by Lemma 3.4. By Lemma 3.2, the pushout of $h \oplus id_N$ along $G(g) : L \oplus N \longrightarrow N$ is closed.

(2) If h is one-one, $h \oplus id_N$ is also one-one by Lemma 3.3. Lemma 3.2 indicates that the pushout of $h \oplus id_N$ along $G(g) : L \oplus N \longrightarrow N$ is one-one. ∎

Remark: In Proposition 3.4 (2), the homomorphism g is arbitrary, hence this result is stronger than the known result that *pushout preserves monomorphisms in the category* **CohFrm**. Also our argument is choice-free.

4 Regular Continuous Frames

Recall that, on any complete lattice L, $a \ll b$ means that $b \leq \bigvee S$ implies $a \leq T$ for some finite $T \subseteq S$ and that L is called continuous if $x = \bigvee\{y \in L | y \ll x\}$ for all $x \in L$. For the background of regular continuous frames, we refer to Banaschewski [4]. In this section, we characterize perfect (= stably closed by Proposition 2.6) homomorphisms between regular continuous frames in terms of the relation $\ll$.

For a regular continuous frame L, we will use $\overline{L}$ to denote the subframe of the ideal frame $\mathfrak{J}L$ consisting of all $\lhd$-ideals, where the relation "$\lhd$" on L

is defined by:

$$x \lhd y \text{ iff } x \prec y \text{ and (1) } \uparrow x^* \text{ is compact, or (2) } \uparrow y \text{ is compact.}$$

An important role of $\overline{L}$ is that it provides the frame counterpart of the 1-point compactification of a locally compact Hausdorff space.

Recall that, in a regular continuous frame,
(1) $x \ll y$ if and only if $x \prec y, \uparrow x^*$ is compact.
(2) $x \ll e$ implies $x \lhd e$.

Lemma 4.1 *For separated L, continuous M and surjective $h : L \longrightarrow M$, there exist $s \leq m$ in L such that h restricted to $[s, m]$ is an isomorphism.*

PROOF. See Proposition 5.1 of [10]. ∎

Lemma 4.2 *For a regular continuous frame L, let $m_L = \{x \in L | x \ll e\}$. Then m_L is a maximal element of $\overline{L}$ and $\downarrow m_L \cong L$.*

PROOF. For the map $v_L : \overline{L} \longrightarrow L$ taking each $\lhd$-ideal to its join in L, it is easy to check that $m_L = \{x \in L | x \ll e\}$ is the smallest $\lhd$-ideal sent to $e \in L$. By Lemma 4.1, $\downarrow m_L \cong L$. The maximality of m_L is proved in [4]. Therefore m_L is the only possible non-top element of $\overline{L}$ with e as its join in L. ∎

Lemma 4.3 *In a continuous frame, $\uparrow a$ is compact iff $a \vee c = e$ for some $c \ll e$.*

PROOF. Because $\{c | c \ll e\}$ is updirected and $\bigvee\{c | c \ll e\} = e$, $\uparrow a$ compact implies $a \vee c = e$ for some $c \ll e$. Conversely, assume $a \vee c = e$ for some $c \ll e$. Then, for any $X \subseteq \uparrow a$ with $\bigvee X = e$, there exists a finite subset $E \subseteq X$ such that $c \leq \bigvee E$, then $\bigvee E \geq a \vee c = e$. This shows that $\uparrow a$ is compact. ∎

Proposition 4.1 *For regular continuous frames L and M, and a homomorphism $h : L \longrightarrow M$, the following conditions are equivalent:*

(1) *h is perfect.*

(2) *h preserves $\ll$, i.e. $x \ll y$ implies $h(x) \ll h(y)$.*

(3) *h can be extended to a homomorphism $\overline{h} : \overline{L} \longrightarrow \overline{M}$ in the sense that the square:*

$$\begin{array}{ccc} \overline{L} & \xrightarrow{\overline{h}} & \overline{M} \\ {\scriptstyle v_L}\downarrow & & \downarrow{\scriptstyle v_M} \\ L & \xrightarrow[h]{} & M \end{array}$$

is a pushout.

PROOF. $(1 \Longrightarrow 2)$. If h is perfect, then, for any $a \in L$, the induced homomorphism $h^a : \uparrow a \longrightarrow \uparrow h(a)$ is perfect. Suppose $x \ll y$, that is $x \prec y$ and $\uparrow x^*$ compact, which implies $h(x) \prec h(y)$, and $\uparrow h(x^*)$ is compact since $\uparrow x^* \longrightarrow \uparrow h(x^*)$ is perfect, hence $\uparrow h(x)^*$, as a subframe of $\uparrow h(x^*)$, must be compact, therefore $h(x) \ll h(y)$.

$(2 \Longrightarrow 3)$. Since $\overline{L}$ and $\overline{M}$ are the subframes of $\mathfrak{J}L$ and $\mathfrak{J}M$, respectively, consisting of all $\lhd$-ideals, and any $h : L \longrightarrow M$ induces a homomorphism $\mathfrak{J}h : \mathfrak{J}L \longrightarrow \mathfrak{J}M$, we first claim that h preserves $\lhd$, which implies that $\mathfrak{J}h$ preserves $\lhd$-ideals, therefore induces a homomorphism $\overline{h} : \overline{L} \longrightarrow \overline{M}$.

Consider $x \lhd y$ in L. If (i) $x \prec y$ and $\uparrow x^*$ is compact, which means $x \ll y$, then $h(x) \ll h(y)$. If (ii) $x \prec y$ and $\uparrow y$ is compact, the latter means $y \vee c = e$ for some $c \ll e$ in L by Lemma 4.3 , then $h(y) \vee h(c) = e$ with $h(c) \ll e$, which means $\uparrow h(y)$ is compact again by Lemma 4.3. In all, (i) and (ii) show that $h(x) \lhd h(y)$.

To see the corresponding square is a pushout, it is enough to show $\overline{h}(m_L) = m_M$ since $L \cong \downarrow m_L$ and $M \cong \downarrow m_M$: Indeed, $\overline{h}(m_L) = \{y \in M | y \leq h(x) \text{ for some } x \ll e\} \subseteq m_M$ since h preserves $\ll$; and, on the other hand, $\bigvee \overline{h}(m_L) = e$ implies $\overline{h}(m_L) = m_M$ since m_M is the smallest $\lhd$-ideal whose join is e.

(3 $\Longrightarrow$ 1). Apply Proposition 2.10. ∎

5 Completely Regular Frames

We refer to Banaschewski-Mulvey [9] for the background of completely regular frames. The Stone-Čech compactification of a completely regular frame L will be denoted by βL.

Lemma 5.1 *Consider a commuting square:*

$$\begin{array}{ccc} U & \xrightarrow{\bar{h}} & V \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ L & \xrightarrow[h]{} & M \end{array}$$

where V is separated, i and j are dense onto. If $h : L \longrightarrow M$ is perfect, then the square is a pushout.

PROOF. Suppose we have the pushout square:

$$\begin{array}{ccc} U & \xrightarrow{\bar{h}} & V \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_2} \\ L & \xrightarrow[p_1]{} & N \end{array}$$

then there exists a homomorphism $g : N \longrightarrow M$ such that $g \circ p_1 = h$ and $g \circ p_2 = j$. As a pushout of i, p_2 is onto, hence N is separated since V is, therefore g is perfect by Proposition 2.8. On the other hand, g must be dense onto since j is dense onto and p_2 is onto. Therefore g is an isomorphism since any dense closed homomorphism is one-one. ∎

Proposition 5.1 *For completely regular frames L and M, the square*

$$\begin{array}{ccc} \beta L & \xrightarrow{\beta h} & \beta M \\ \downarrow & & \downarrow \\ L & \xrightarrow[h]{} & M \end{array}$$

is a pushout if and only if h is perfect.

PROOF. By Lemma 5.1 and Proposition 2.10. ∎

Finally, we present (without proofs) an application of the concept of perfect homomorphisms. As an analogue of the projectives for completely regular spaces, we can study the injectives in the category **CRegFrm** of completely regular frames as follows, using unexplained terminology as in [5].

Definition (1) *In* **CRegFrm**, *L is perfect-injective if, for any perfect embedding $h : M \longrightarrow N$ and any homomorphism $f : M \longrightarrow L$, there exists a homomorphism $g : N \longrightarrow L$ such that $g \circ h = f$.*

(2) *A dense homomorphism $h : L \longrightarrow M$ is called essentially dense if, for any frame homomorphism $f : M \longrightarrow N$, f is dense whenever $f \circ h$ is.*

(3) *For $L \in$* **CRegFrm**, *its Gleason envelope in* **CRegFrm** *is defined to be a completely regular deMorgan frame $G(L)$ together with an essential dense perfect embedding $\gamma_L : L \longrightarrow G(L)$.*

By applying the Stone-Čech compactification and the results in **KRegFrm** by Banaschewski [5], together with the characterization of perfect homomorphism, we obtain

Proposition: (1) *Every frame of* **CRegFrm** *has a unique Gleason envelope in* **CRegFrm**.

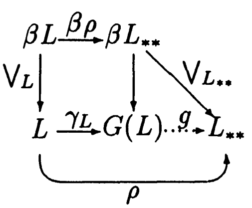
(2) *For any $L \in$* **CRegFrm**, *$\beta G(L) = G(\beta L)$.*

(3) *Sikorski Theorem that* a Boolean algebra is injective iff it is complete *holds if and only if the completely regular deMorgan frames are precisely the injectives in* **CRegFrm**.

Concerning Gleason envelopes, we have the following observation: Setting $\rho = (.)^{**} : L \longrightarrow L_{**}$, and applying the functor β, we have $\beta\rho : \beta L \longrightarrow \beta L_{**}$, which is given by

$$\beta\rho(A) = \{x^{**} | x \in A\} = \{x \in L_{**} | x \in A\} \text{ for each } A \in \beta L.$$

Now βL_{**} together with $\beta\rho$ is actually the Gleason envelope of βL. Then the Gleason envelope $\gamma_L : L \longrightarrow G(L)$ is the pushout of $\beta\rho$ along $\bigvee_L : \beta L \longrightarrow L$ as shown in the following diagram:



We close with an open question: How can one describe γ_L and $G(L)$ directly without going through βL?

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References

[1] B. Banaschewski, Singly Generated Frame Extensions, *J. Pure Appl. Algebra*, 83 (1992) 1-21.

[2] B. Banaschewski, Bourbaki's Fixpoint Lemma Reconsidered, to appear at *Comm. Math. Universitatis Carolinae.*

[3] B. Banaschewski, On pushing out frames, *Comm. Math. Universitatis Carolinae* 31, 1(1990) 13-21.

[4] B. Banaschewski, Compactification of Frames, *Math. Nachr.* 149 (1990) 105-116.

[5] B. Banaschewski, Compact regular frames and the Sikorski theorem, *Kyungpook Math. J.* 28 (1988) 1-14.

[6] B. Banaschewski, Another look at the localic Tychonoff theorem, *Comm. Math. Universitatis Carolinae* 26 (1985) 619-630.

[7] B. Banaschewski, Coherent frames, *Continuous Lattices*, Proceedings Bremen 1979. Springer LNM 871 (1981) 1-11.

[8] B. Banaschewski and G.C.L. Brümmer, Stably continuous frame, *Math. Proc. Camb. Phil. Soc.* 104 (1988) 7-19.

[9] B. Banaschewski and C.J. Mulvey, Stone-Čech compactification of locales I, *Houston J. Math.* 6 (1980) 302-312.

[10] X. Chen, On Binary Coproducts of Frames, *Comm. Math. Universitatis Carolinae* 33, 4(1992).

[11] X. Chen, On local connectedness of frames, *J. Pure Appl. Algebra* 79 (1992) 35-43.

[12] X. Chen, *Closed Frame Homomorphisms,* Doctoral Dissertation, McMaster University, 1991.

[13] C. H. Dowker and D. Papert, Paracompact frames and closed maps, *Symp. Math.* 16 (1975) 93-116.

[14] J. R. Isbell, Atomless parts of spaces, *Math. Scand.* 31 (1972) 5-32.

[15] M. Jibladze and P. T. Johnstone, The frame of fibrewise closed nuclei, *Cahiers de Top. et Géom. Diff. Cat.* XXX III-I (1991) 99 - 112.

[16] P. T. Johnstone, Fibrewise Separation Axioms for locales, *Math. Proc. Camb. Phil. Soc.* 108 (1990) 247 - 255.

[17] P. T. Johnstone, *Stone spaces*, Cambridge University Press 1982.

[18] P. T. Johnstone, *Gleason cover of a topos, I*, J. Pure Appl. Alg. 19 (1980), 171-192.

[19] P. T. Jonhstone, *Gleason cover of a topos, II*, J. Pure Appl. Alg. 22 (1981), 229-247.

[20] P. T. Johnstone, *Factorization and pullback theorems for localic geometric morphisms*, Univ. Cath. de Louvain, Sém. de Math. Pure, Rapport no. 79.

[21] A. Pultr and A. Tozzi, Notes on Kuratowski-Mrówka theorems in point-free context, *Cahiers de Top. et Géom. Diff. Cat.* XXX III-I (1992) 3 - 14.

[22] J. J. C. Vermeulen, Some constructive results related to compactness and the (strong) Hausdorff property for locales, *Category Theory*, Proceedings Como 1990, Springer LNM 1488 (1991) 401-409.

Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1