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TOPOLOGICAL TOTALLY CONVEX SPACES, II

by Heinrich KLEISLI and Hans-Peter KÜNZI

Résumé: La catégorie des espaces complets totalement convexes et des morphismes continus est équivalente à la catégorie des espaces de Saks complets et de leurs morphismes. Contrairement à une croyance répandue elle n'est pas auto-duale. Pour obtenir une catégorie auto-duale il faut se borner à ne considérer que la sous-catégorie pleine des espaces totalement convexes qui sont complets et cocomplets. La question si la dernière est déjà une catégorie fermée s'impose mais reste ouverte.

1. Introduction

The authors consider it as a privilege to dedicate this paper to Professor D. Pumplün. It was Nico Pumplün who together with Helmut Röhrl started the study of the algebraic part of the category of Banach spaces and contracting linear maps in a series of fundamental papers centered around the algebraic concept of a totally convex space. They also suggested to study topologies on totally convex spaces in the same way as it has been done for Banach spaces, endowing them with a Saks space structure.

We have already investigated topologies on a totally convex space in a previous paper (see [7]) and have chosen the name "topological totally

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convex space" for the topologized algebraic objects which were studied there. It has been found that the introduction of topologies not only allowed generalizations of classical theorems, but also rendered some constructions in the category of totally convex spaces more transparent.

In this paper we investigate some full subcategories of the category of topological totally convex spaces (abbreviated TTCS) and continuous morphisms in view of establishing a duality theory. First, we study complete TTCS and show that the corresponding full subcategory is equivalent to the category of complete Saks spaces and their morphisms (Corollary (3.5)). Unfortunately that category is not self-dual as it is claimed in [5] and in [6]. Two counter-examples are given at the end of the paper (in Section 5), namely an example of a complete TTCS whose dual space is not complete (Example 1) and an example of a complete TTCS which is not reflexive (Example 2).

In order to understand those counter-examples better, we introduce a generalization of compact TTCS, called cocomplete spaces. It turns out that the dual spaces of complete TTCS are cocomplete, and vice versa (Propositions (4.9) and (4.10)). Finally, we study bicomplete spaces, i.e., TTCS which are complete and cocomplete, and observe that the corresponding full subcategory is a self-dual category (Theorem (4.12)). That category does not seem to be large enough in order to be a closed category and thus a $*$-autonomous category in the sense of Barr [1]. However, at the time being we do not know an appropriate counter-example. Many of the open problems listed in this paper are related to that problem.

2. The category of topological totally convex spaces

In order to make the paper reasonably self-contained, we first recall some of the definitions and theorems of [7], however, without repeating all proofs.

Let Ban_1 denote the category of complex Banach spaces and contracting linear maps, and let $O : Ban_1 \to Set$ be the unit ball functor. It has been shown in [9] that the category of Eilenberg-Moore algebras of the functor O is equationally presentable. Its objects are given by non-empty sets X together with operations $\underline{\alpha}_X : X^{\mathbb{N}} \to X$ for every $\underline{\alpha} = (\alpha_n)_{n\geq 0}$ in $O\ell_1\mathbb{N}$ (i.e., $\underline{\alpha}$ is a sequence of complex numbers such that $\sum_{n=0}^{\infty} |\alpha_n| \leq 1$). If we write the operations $\underline{\alpha}_X$ as infinite formal sums $\underline{\alpha}_X(\underline{x}) = \sum_{n=0}^{\infty} \alpha_n x_n$ for all $\underline{x} = (x_n)_{n\geq 0}$ in $X^{\mathbb{N}}$, then the defining relations are given by

$$(2.1) \qquad \sum_{n=0}^{\infty} \delta_n^m x_n = x_m \text{ for } \underline{\delta}^m = (\delta_n^m)_{n\geq 0} \text{ and all } \underline{x} = (x_n)_{n\geq 0},$$

$$(2.2) \qquad \sum_{n=0}^{\infty} \alpha_n \Big(\sum_{m=0}^{\infty} \beta_m^n x_m\Big) = \sum_{m=0}^{\infty} \Big(\sum_{n=0}^{\infty} \alpha_n \beta_m^n\Big) x_m$$

for all $\underline{\alpha} = (\alpha_n)_{n\geq 0}, \underline{\beta}^1 = (\beta_m^1)_{m\geq 0}, \ldots, \underline{\beta}^i = (\beta_m^i)_{m\geq 0}, \ldots$ in $O\ell_1\mathbb{N}$ and $\underline{x} = (x_m)_{m\geq 0}$ in $X^{\mathbb{N}}$.

Definitions. The objects introduced above are called *totally convex spaces*, abbreviated TCS. Given two TCS X and Y, a *morphism* $f : X \to Y$ is a map between the underlying sets satisfying the condition

$$f\Big(\sum_{n=0}^{\infty} \alpha_n x_n\Big) = \sum_{n=0}^{\infty} \alpha_n f(x_n)$$

for all $\underline{\alpha}$ in $O\ell_1\mathbb{N}$ and $\underline{x}$ in $X^{\mathbb{N}}$. The resulting category is called the *category of totally convex spaces* and will be denoted by $\mathcal{TC}$.

The full and faithful comparison functor $\hat{O} : Ban_1 \to \mathcal{TC}$ associates with each Banach space B its unit ball OB, and the infinite sums $\sum_{n=0}^{\infty} \alpha_n x_n$ in $\hat{O}B$ are defined as convergent series with respect to the norm-topology of B. We shall call the resulting totally convex spaces $\hat{O}B$ *Banach balls.*

Definitions. Let X be a TCS. A function $\varphi : X \to O\mathbb{R} = [-1, 1]$ is called a *semi-norm* if it satisfies the following conditions:

$$(2.3) \qquad \varphi(\alpha x) = |\alpha|\varphi(x) \text{ for all } \alpha \text{ in } O\mathbb{C} \text{ and } x \text{ in } X,$$

(2.4) $\varphi(\sum_{n=0}^{\infty} \alpha_n x_n) \leq \sum_{n=0}^{\infty} |\alpha_n| \varphi(x_n)$ for all $\underline{\alpha}$ in $O\ell_1\mathbb{N}$ and $\underline{x}$ in $X^{\mathbb{N}}$.

A semi-norm φ is said to be a *norm* if, in addition, we have

(2.5) $\varphi(x) > 0$ for all $x \neq 0$.

Observe that conditions (2.3) and (2.4) imply that

$$|\frac{1}{2}\varphi(x) - \frac{1}{2}\varphi(y)| \leq \varphi(\frac{1}{2}x - \frac{1}{2}y)$$

whenever x, y in X.

Example 1. If φ is a semi-norm on a Banach space $(B, \| \, \|)$ (i.e., if (2.3) is replaced by $\varphi(\alpha x) = |\alpha|\varphi(x)$ for all α in $\mathbb{C}$ and (2.4) by $\varphi(\alpha_1 x_1 + \alpha_2 x_2) \leq |\alpha_1|\varphi(x_1) + |\alpha_2|\varphi(x_2)$ for all α_1, α_2 in $\mathbb{C}$ and x_1, x_2 in X) and if $\varphi(x) \leq \|x\|$ for all x in X, then the restriction $\varphi|_{OB}$ is a semi-norm on $\hat{O}B$. It can be shown that every semi-norm on a Banach ball $\hat{O}B$ can be obtained that way (see Lemma (3.2) below).

Definitions. Let $\mathcal{S}$ be a family of semi-norms on a TCS X. For any φ in $\mathcal{S}$, define a pseudo-metric d_φ on X by setting

$$d_\varphi(x, y) = 2\varphi(\frac{1}{2}x - \frac{1}{2}y) \text{ for all } x, y \text{ in } X.$$

The uniformity on X determined by the family $(d_\varphi)_{\varphi \in \mathcal{S}}$ of pseudo-metrics and its induced topology are called the *uniformity and the topology generated by the family $\mathcal{S}$ of semi-norms*, respectively. A topology on a TCS X is called *locally convex* provided that it can be generated by a family of semi-norms on X (cf. Proposition (2.6) below).

By a *topological totally convex space*, abbreviated TTCS, we understand a TCS endowed with a locally convex topology. The TTCS together with the continuous morphisms form a category which will be denoted by $\mathcal{TTC}$.

The following propositions will be stated without proof.

(2.6) **Proposition.** *Let φ be a semi-norm on a TCS X, x_0 an element of X and $\varepsilon > 0$. Then, the open balls*

$$B_{\varepsilon,\varphi}(x_0) = \{x \in X; d_\varphi(x_0, x) < \varepsilon\}$$

are convex sets, and the open balls $B_{\varepsilon,\varphi}(0)$ centered at the origin 0 are totally convex subspaces.

(2.7) **Proposition.** *Let X and Y be TTCS. The following properties of a morphism $f : X \to Y$ are equivalent:*

(i) *f is uniformly continuous,*

(ii) *f is continuous,*

(iii) *f is continuous at* 0.

(2.8) **Proposition.** *The operations on a TTCS are continuous.*

(2.9) **Remark.** On every TTCS X there is a largest family of semi-norms which generates the topology $\mathcal{T}$ on X, namely the family of all continuous semi-norms. Therefore, the uniformity does not depend on the particular family of semi-norms generating $\mathcal{T}$.

Examples

2. For each TCS X, there is a coarsest and a finest locally convex topology. The first is given by the semi-norm φ_0 with constant value 0, the second by the family of all semi-norms (in [7] that topology was referred to as the strong topology).

It is important to note that the strong topology on a TTCS X can be induced by a single semi-norm, namely

$$\|\ \|_s = \sup\{\varphi; \varphi \text{ is a semi-norm on } X\}.$$

On the other hand, there exist TTCS whose topologies are generated by a single semi-norm, but which do not carry the strong topology:

We define a TTCS X by setting

$$X = \{f \in O\mathbb{C}^{\mathbb{N}}; |f(n)| \leq \frac{1}{n+1} \text{ for all } n \in \mathbb{N}\},$$

and by endowing the set X with the pointwise totally convex structure and the topology generated by the semi-norm $\varphi(f) = \sup_{n\in\mathbb{N}} |f(n)|$ for all f in X.

Observe that, for all integers $n \geq 0$, the semi-norms φ_n, given by

$$\varphi_n(f) = (n+1)|f(n)| \text{ for all } f \text{ in } X,$$

are continuous.

Assume that the semi-norm $\| \|_s$ of X were continuous. Then there exists a $\delta > 0$ such that, for all f in X, $\varphi(f) < \delta$ implies $\|f\|_s < 1$. Let m be an integer ≥ 0 such that $\frac{1}{m+1} < \delta$ and consider the element f in X given by

$$f(n) = \begin{cases} \dfrac{1}{m+1} & \text{if } n = m, \\ 0 & \text{otherwise}. \end{cases}$$

Then $\varphi(f) < \delta$ so that $\|f\|_s < 1$. On the other hand, $\varphi_m(f) = 1$. Hence, $\|f\|_s < \varphi_m(f)$ in contradiction to the definition of the semi-norm $\| \|_s$.

3. Let $(B, \| \|, \mathcal{T})$ be a Saks space, i.e., $(B, \| \|)$ is a complex Banach space and $\mathcal{T}$ an additional locally convex Hausdorff topology on B such that the unit ball OB is closed and bounded (see [4], page 28, for a slightly more general definition). Then the totally convex space $\hat{O}B$ with the trace-topology induced by $\mathcal{T}$ is a TTCS, called a *Saks ball*. It should be noted that a Banach ball $\hat{O}B$, when considered as TTCS, always carries the norm-topology and is therefore a trivial Saks ball. Of course, the norm-topology of $\hat{O}B$ is the strong topology on $\hat{O}B$.

Observe that the category of Saks spaces and Saks morphisms is isomorphic to the full subcategory of $\mathcal{TTC}$ generated by the Saks balls. Indeed, a Saks morphism $f : B \to C$ is defined as a contracting linear map such that the restriction $f|_{OB}$ is continuous, so that $f|_{OB} : \hat{O}B \to \hat{O}C$ is a continuous morphism of TTCS. Conversely, every continuous morphism $g : \hat{O}B \to \hat{O}C$ is of that form.

4. The dual space $X^* = \{f : X \to \hat{O}\mathbb{C}; f \text{ a morphism}\}$ of a TCS X has a pointwise defined totally convex structure. A topology on X^* which is weaker than the strong topology is the topology generated by the family of semi-norms of the form

$$\varphi(f) = \max\{|f(x_1)|, \ldots, |f(x_k)|\} \text{ for all } f \text{ in } X^*,$$

where $x_1, \ldots, x_k$ are given points in X. We shall call that topology the *weak $*$-topology* (since it has that name in the case where X^* is the Saks ball of a dual Banach space).

5. The circled absorbing convex subset K of a Waelbroeck space (see [3], chap. I, § 2) is an example of a TTCS with compact topology.

Definition. Let X and Y be TTCS. We denote by $[X, Y]$ the set of all continuous morphisms $f : X \to Y$ and endow it with the pointwise totally convex structure and the *topology of compact convergence*, i.e., the topology $\mathcal{T}$ generated by the family of semi-norms of the form

$$\psi(f) = \sup_{x \in K} \eta(f(x)) \text{ for all } f \text{ in } [X, Y],$$

where K is a compact subset of X and η a continuous semi-norm of Y. Observe that $\mathcal{T}$ is the compact-open topology.

(2.10) **Proposition.** *$[X, Y]$ is a well-defined TTCS.*

Proof. We have to verify that, for any $\underline{\alpha}$ in $O\ell_1\mathbb{N}$ and any $\underline{f}$ in $[X, Y]^{\mathbb{N}}$, the morphism $f = \sum_{n=0}^{\infty} \alpha_n f_n$, given by

$$f(x) = \sum_{n=0}^{\infty} \alpha_n f_n(x) \text{ for all } x \text{ in } X,$$

is continuous. By Proposition (2.7) it suffices to verify continuity at 0. Let $\varepsilon > 0$ and η be a continuous semi-norm on Y. There exists an integer $N \geq 0$ such that $\sum_{n>N} |\alpha_n| < \varepsilon$. Moreover, for every n, $0 \leq n \leq N$, there exist a continuous semi-norm ξ_n on X and $\delta_n > 0$ such that, for all x in X, $\xi_n(x) < \delta_n$ implies $\eta(f_n(x)) < \varepsilon$. We define a continuous semi-norm ξ on X, by setting $\xi(x) = \max\{\xi_0(x), \ldots, \xi_N(x)\}$ for all x in X, and we set $\delta = \min\{\delta_0, \ldots, \delta_N\}$. If $\xi(x) < \delta$, then $\xi_n(x) < \delta_n$ for all $n = 0, \ldots, N$. Hence,

$$\eta(f(x)) = \eta(\sum_{n=0}^{\infty} \alpha_n f_n(x)) \leq \sum_{n\leq N} |\alpha_n| \eta(f_n(x)) + \sum_{n>N} |\alpha_n| \eta(f_n(x))$$
$$\leq \sum_{n\leq N} |\alpha_n| \varepsilon + \varepsilon \leq 2\varepsilon,$$

i.e., f is continuous at 0. □

(2.11) **Proposition.** *Let X, Y and Z be TTCS and $f : X \to Y$ a continuous morphism. Then, the maps*
$[f, Z] : [Y, Z] \to [X, Z]$ where $[f, Z](g) = g \circ f$ for all g in $[Y, Z]$, and
$[Z, f] : [Z, X] \to [Z, Y]$ where $[Z, f](g) = f \circ g$ for all g in $[Z, X]$, are continuous morphisms.

Proof. The verification that the maps $[f, Z]$ and $[Z, f]$ are well-defined morphisms is straightforward. The continuity of the map $[f, Z]$ follows from the relation

$$\sup_{x\in K} \zeta([f, Z](g))(x) = \sup_{x\in K} \zeta(g(f(x))) = \sup_{y\in f(K)} \zeta(g(y)) \text{ for all } g \text{ in } [Y, Z],$$

where ζ is a continuous semi-norm on Z and K a compact subset of X, and from the fact that the image $f(K)$ is a compact subset of Y. The continuity of $[Z, f]$ follows from the relation

$$\sup_{z\in K} \eta([Z, f](g))(z) = \sup_{z\in K} \eta(f(g(z))) \text{ for all } g \text{ in } [Z, X],$$

where η is a continuous semi-norm on Y and K a compact subset of Z, and from the fact that $x \mapsto \eta(f(x))$ is a continuous semi-norm on X.□

Definitions. For a TTCS X, we denote the TTCS $[X, \hat{O}\mathbb{C}]$ by X' and call it the *dual space of* X. Likewise, for a continuous morphism $f : X \to Y$, we denote the continuous morphism $[f, \hat{O}\mathbb{C}]$ by $f' : Y' \to X'$ and speak of the *dual morphism of* f. By iteration we obtain the *bidual space* X'' and the *bidual morphism* $f'' : X'' \to Y''$.

A straightforward verification shows that the functions $X \mapsto X'$ and $f \mapsto f'$ define a contravariant endofunctor of the category $\mathcal{TTC}$, called the *duality functor.*

Example 6. Let B and C be Saks spaces. Consider the Saks space given by the complex vector space $Saks(B, C) = \{f : B \to C;\ f$ a bounded linear map such that $f|_{OB}$ is continuous$\}$ endowed with the sup-on-the-unit-ball norm and the topology of uniform convergence on compact subsets of OB. Observe that the dual space B' of a Saks space B is the Saks space $Saks(B, \mathbb{C})$. The map $h : \hat{O}Saks(B, C) \to [\hat{O}B, \hat{O}C]$, given by $h(f) =$ restriction $f|_{OB}$ for all f in $\hat{O}\, Saks(B, C)$, is an isomorphism in $\mathcal{TTC}$. That allows us to identify the Saks ball $\hat{O}(B'')$ of the Saks space bidual B'' with the bidual TTCS $(\hat{O}B)''$.

Definitions. Let X be a TTCS. The *evaluation map* $e_X : X \to X''$ is given by

$$e_X(x)(f) = f(x) \text{ for all } x \text{ in } X \text{ and } f \text{ in } X'.$$

It is a morphism. If it is a topological isomorphism (i.e., an isomorphism in the category $\mathcal{TTC}$), we shall call X a *reflexive* TTCS; if it is only an isomorphism, then we shall speak of a *semi-reflexive* TTCS.

It is readily checked that the dual X' of any reflexive TTCS X is reflexive.

(2.12) **Proposition.** *Every Banach ball $\hat{O}B$ (endowed with the norm-topology) is reflexive.*

Proof. This is a direct consequence of the following well-known result. If the Banach space is considered as a trivial Saks space (i.e., the "additional" topology being the norm-topology) and if we denote the Saks space bidual by B'', the evaluation map $e_B : B \to B''$ is an isomorphism. We then identify $\hat{O}(B'')$ and $(\hat{O}B)''$ by means of the isomorphism h given in Example 6 above. A "direct proof" of the reflexivity of $\hat{O}B$ can be found in [7] (Proof of Lemma 3.3). □

Remark. We hope that our notion of reflexivity is not misleading. It should not be confounded with the classical notion of reflexive locally convex vector spaces, where the bidual space is defined by endowing the dual space with the strong topology (see, for instance, [10], chapt. IV, § 5).

We continue to recall two further results from [7] on TTCS carrying the strong topology.

(2.13) **Theorem.** *Let X be a totally convex space endowed with the strong topology. Then the dual space X' is a compact Hausdorff space.*

(2.14) **Theorem.** *Let X be a TTCS for the strong topology. Then the bidual space X'' is a Banach ball $\hat{O}BX$. Furthermore, the evaluation map $e : X \to X''$ is a generalized completion, i.e., for every continuous morphism $f : X \to Y$ into a complete Hausdorff TTCS Y, there exists a unique continuous morphism $g : X'' \to Y$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{e} & X'' \\ & \searrow^{f} & \downarrow{g} \\ & & Y \end{array}$$

is commutative. Finally, $e(X)$ is a dense subspace of the complete Hausdorff TTCS X''.

(2.15) **Remark.** It has been pointed out by D. Pumplün that the following relation holds for $e(X) : \mathring{O}BX \subseteq e(X) \subseteq \hat{O}BX$. Here is a proof. Denote by E the closure of $e(X)$ in $\hat{O}BX$. Consider an element y in $\overset{\circ}{O}BX$. We construct inductively a sequence $(x_n)_{n\geq 1}$ in X such that, for each integer $n \geq 1$, $\|y - \sum_{k=1}^{n-1} \frac{1}{2^k} e(x_k)\| < 2^{-(n-1)}$. Observe that the condition is satisfied for $n = 1$. For $n \geq 1$, assume that x_k has been defined for all $k < n$. Set $z_n = 2^{n-1}(y - \sum_{k=1}^{n-1} \frac{1}{2^k} e(x_k))$. By the induction hypothesis, $\|z_n\| < 1$. Choose $\varepsilon_n > 0$ such that $\|z_n\| + 4\varepsilon_n < 1$. Since z_n belongs to $\overset{\circ}{O}BX \subseteq E$, there is an x_n in X such that $\|z_n - e(x_n)\| < \varepsilon_n$. Therefore,

$$\|z_n - \frac{1}{2}e(x_n)\| \leq \|z_n - e(x_n)\| + \|\frac{1}{2}e(x_n)\| < \varepsilon_n + \frac{1}{2}(\|z_n\| + \varepsilon_n) < \frac{1}{2},$$

whence $\|(y - \sum_{k=1}^{n-1} \frac{1}{2^k} e(x_k)) - \frac{1}{2^n} e(x_n)\| < 2^{-n}$. That completes the induction. We conclude that

$$y = \sum_{k=1}^{\infty} \frac{1}{2^k} e(x_k) = e(\sum_{k=1}^{\infty} \frac{1}{2^k} x_k), \text{ i.e., } y \text{ belongs to } e(X). \quad \square$$

3. Complete totally convex spaces and Saks spaces

(3.1) From now on, the topology of all TTCS is assumed to satisfy the Hausdorff axiom. Note that the underlying TCS of such a TTCS X is separated in the sense of Pumplün and Röhrl: For x, y in X and α in $\mathbb{C}$, $\alpha x = \alpha y$ implies $x = y$ or $\alpha = 0$ (see [9], Lemma (11.2), where the in the meantime abolished terminology "separable TCS" is used). Indeed, assume that there exist x, y in X, $x \neq y$ and $\alpha \neq 0$ such that $\alpha x = \alpha y$. Then $|\alpha|\varphi(\frac{1}{2}x - \frac{1}{2}y) = \varphi(\frac{1}{2}\alpha x - \frac{1}{2}\alpha y) = 0$ for all semi-norms φ on X. Hence, X cannot satisfy the Hausdorff axiom.

Definition. A TTCS is called *complete* (from now on) if the underlying space is a complete Hausdorff space.

Examples

1. Every Banach ball is a complete TTCS.

2. The dual space $(\hat{O}B)'$ of a Banach ball has an underlying space which is a compact Hausdorff space, and thus a complete space.

3. Let Y be a locally compact (Hausdorff) space. We denote by $C^\infty(Y)$ the vector space of bounded, continuous $\mathbb{C}$-valued functions on Y and by $\| \|_\infty$ the sup-norm. Then, $(C^\infty(Y), \| \|_\infty)$ is a Banach space, and we define X to be the TCS $\hat{O}(C^\infty(Y), \| \|_\infty)$ endowed with the topology of compact convergence.

We claim that X is complete. Clearly, the topology of X is Hausdorff. Let $(f_\delta)_{\delta \in D}$ be a Cauchy net in X. Then (f_δ) converges pointwise to a map $f : Y \to O\mathbb{C}$, the convergence being uniform on each compact subset K of Y. Hence, the restriction $f|_K : K \to O\mathbb{C}$ is continuous. Let U be a closed subset of $O\mathbb{C}$. Then, $f^{-1}(U) \cap K$ is closed for each compact subset K of Y. Since X is locally compact, $f^{-1}(U)$ is closed, too. Otherwise, there would exist an accumulation point x of $f^{-1}(U)$ which does not belong to $f^{-1}(U)$. But then, for a compact neighbourhood $K(x)$ of x, $f^{-1}(U) \cap K(x)$ would not be closed.

We shall now give a characterization of the complete TTCS as limits of Banach balls in $\mathcal{TTC}$. For Saks spaces such a characterization is well known (see [4], Proposition I.3.8) so that we shall proceed by a reduction to that case.

(3.2) **Lemma.** *Let X be a TCS which is a radial subset of a $\mathbb{C}$-vector space V. Then every semi-norm φ on X can be extended to a semi-norm $\tilde{\varphi}$ on V.*

Proof. We define $\tilde{\varphi}$ by setting

$$\tilde{\varphi}(x) = \frac{1}{|\alpha|}\varphi(\alpha x) \quad \text{for every} \quad x \quad \text{in} \quad V,$$

where α is a complex number $\neq 0$ such that αx belongs to X. It is easy to see that we obtain a well-defined map $\tilde{\varphi} : V \to \mathbb{R}$. Moreover,

$$\tilde{\varphi}(\lambda x) = \frac{1}{|\alpha|}\varphi(\alpha(\lambda x)) = \frac{|\lambda|}{|\alpha\lambda|}\varphi((\alpha\lambda)x) = |\lambda|\tilde{\varphi}(x), \quad \text{where} \quad \alpha(\lambda x) \in X,$$

and

$$\tilde{\varphi}(x_1 + x_2) = \frac{1}{|\alpha|}\varphi(\alpha(x_1 + x_2)) = \frac{2}{|\alpha|}\varphi(\frac{1}{2}\alpha x_1 + \frac{1}{2}\alpha x_2) \leq$$
$$\frac{2}{|\alpha|}[\varphi(\frac{1}{2}\alpha x_1) + \varphi(\frac{1}{2}\alpha x_2)] = \frac{1}{|\alpha|}\varphi(\alpha x_1) + \frac{1}{|\alpha|}\varphi(\alpha x_2) = \tilde{\varphi}(x_1) + \tilde{\varphi}(x_2),$$

where $2\alpha x_i \in X$ $(i = 1, 2)$. □

(3.3) **Lemma.** *Let E be a bounded absolutely convex subset of a locally convex space $(V, \mathcal{T})$. We denote by B the subspace of V generated by E and by $\| \ \|$ the norm on B given by the Minkowski functional of E. If E is complete, then $(B, \| \ \|)$ is a Banach space.*

For a proof, see [4], Lemma I.1.2 and Definition I.1.1.

(3.4) **Theorem.** *Every (Hausdorff) TTCS X is a dense subset of a complete Saks ball $\hat{O}\tilde{S}X$. Moreover, every continuous semi-norm on X admits a unique extension to a continuous semi-norm on $\tilde{S}X$.*

Proof. Consider the underlying TCS $|X|$ of X endowed with the finest locally convex topology. Hence, by Theorem (2.14), the evaluation map $e : |X| \to |X|''$ is a completion, i.e., $|X|$ is (topologically) isomorphic to $e(|X|)$ and the latter is a dense subset of the Banach ball $|X|'' = \hat{O}S(|X|)$. Let BX be the linear subspace of $S(|X|)$ generated by the subset $e(|X|)$. Then $e(|X|)$ is a radial subset of BX. By Lemma (3.2), we can extend every continuous semi-norm φ on X to a semi-norm of the $\mathbb{C}$-vector space BX and thus obtain a locally convex topology $\mathcal{T}$ on BX such that X can be considered as a topological subspace of the locally convex space $(BX, \mathcal{T})$.

Denote by $(\hat{B}X, \hat{\mathcal{T}})$ the completion of the locally convex space $(BX, \mathcal{T})$ and write E for the closure of $e(|X|)$ in $\hat{B}X$ and $\tilde{S}X$ for the subspace of

$\hat{B}X$ generated by E. Clearly, E is an absolutely convex subset of $\hat{B}X$. If $\|\ \|$ is the norm on $\tilde{S}X$ given by the Minkowski functional of E and $\tilde{\mathcal{T}}$ the locally convex topology induced by $\hat{\mathcal{T}}$ on $\tilde{S}X$, then $(\tilde{S}X, \|\ \|, \tilde{\mathcal{T}})$ is a complete Saks space such that $O\tilde{S}X = E$. Indeed, the unit-ball is complete and bounded, and Lemma (3.3) implies that $(\tilde{S}X, \|\ \|)$ is a Banach space (such that the restrictive definition of a Saks space we are using is satisfied - see Example 2.3). Hence, with the identification $X = e(|X|)$, X is a dense subset of the complete Saks ball $\hat{O}\tilde{S}X$.

The second assertion of the theorem follows immediately from the construction of $\tilde{S}X$ given above. □

(3.5) **Corollary.** *The function $\hat{O}$ which associates with each Saks space $(E, \|\ \|, \mathcal{T})$ its unit ball $(\hat{O}E, \mathcal{T}\big|_{\hat{O}E})$ extends to an equivalence of the category of complete Saks spaces to the category of complete TTCS.*

Indeed, it is obvious how $\hat{O}$ extends to a full and faithfull functor, and by Theorem (3.4) each complete TTCS X is isomorphic to $\hat{O}\tilde{S}X$ in $\mathcal{TTC}$.

(3.6) **An explicit construction of the limit of an inverse system.** Let $\mathbb{D} = (p_{\alpha\beta} : X_\beta \to X_\alpha)_{\alpha \le \beta}$ be an inverse system in $\mathcal{TTC}$ with directed index set D. We shall present a standard construction of a limit X of $\mathbb{D}$, which will be denoted $X = \lim \mathbb{D}$. The underlying set of the limit is given by

$$\lim \mathbb{D} = \{(x_\alpha)_{\alpha \in D} \in \prod_{\alpha \in D} X_\alpha;\ x_\alpha = p_{\alpha\beta}(x_\beta) \text{ for all } \alpha \le \beta \text{ in } D\}.$$

The totally convex structure is defined termwise and the resulting TCS is endowed with the product topology, i.e., the topology generated by the semi-norms of the form

$$\varphi((x_\alpha)_{\alpha \in D}) = \varphi_\beta(x_\beta) \text{ for all } (x_\alpha)_{\alpha \in D} \text{ in } \prod_{\alpha \in D} X_\alpha,$$

where β is an index in D and φ_β a continuous semi-norm on X_β. The continuous morphisms $p_\beta : \lim \mathbb{D} \to X_\beta$ $(\beta \in D)$ of the limit cone are given by the restriction to $\lim \mathbb{D}$ of the canonical projections $\prod_{\alpha \in D} X_\alpha \to X_\beta$. The verification that this way we obtain a limit of the given inverse system $\mathbb{D}$ is straightforward and therefore left to the reader.

(3.7) **Proposition.** *Let* $\mathbb{D} = (p_{\alpha\beta} : X_\beta \to X_\alpha)_{\alpha \leq \beta}$ $(\alpha, \beta \in D)$ *be an inverse system such that the TTCS* X_α $(\alpha \in D)$ *are complete. Then,* $\lim \mathbb{D}$ *is a complete TTCS.*

Proof. The product uniformity on the product $\prod_{\alpha \in D} X_\alpha$ of the family $(X_\alpha)_{\alpha \in D}$ of complete Hausdorff spaces X_α yields a complete Hausdorff space. It suffices therefore to verify that $\lim \mathbb{D}$ is a closed subset of the topological space $\prod_{\alpha \in D} X_\alpha$. But this follows from the observation that $\lim \mathbb{D}$ is the kernel of the continuous map

$$p : \prod_{\alpha \in D} X_\alpha \to \prod_{\beta \leq \gamma} X_{\beta\gamma} \text{ (where } X_{\beta\gamma} = X_\beta \text{ for all } \beta \leq \gamma \text{ in } D),$$

given by

$$p((x_\alpha)_{\alpha \in D}) = (\frac{1}{2} x_\beta - \frac{1}{2} p_{\beta\gamma}(x_\gamma))_{\beta \leq \gamma} \text{ for all } (x_\alpha)_{\alpha \in D} \text{ in } \prod_{\alpha \in D} X_\alpha. \quad \square$$

(3.8) **Corollary.** *If* $\mathbb{D}$ *is an inverse system of Banach balls, then* $\lim \mathbb{D}$ *is a complete TTCS.* □

(3.9) **The construction of the completion of a TTCS as the limit of an inverse system.** Let X be a (Hausdorff) TTCS. We shall present a standard construction of a completion $q : X \to \hat{X}$, i.e., a dense embedding of X into a complete TTCS $\hat{X}$.

Let $\mathcal{S}$ denote the set of all continuous semi-norms of X directed with the pointwise ordering. By Theorem (3.4), X is a dense subspace of the unit ball $\hat{O}\tilde{S}X$ of a complete Saks space, and every continuous semi-norm φ on X can be considered as the restriction

$\tilde{\varphi}\big|_X$ of a continuous semi-norm $\tilde{\varphi}$ of the complete Saks space $\tilde{S}X$. Observe that the topology $\mathcal{T}$ on $\tilde{S}X$ generated by the semi-norms $\tilde{\varphi}$ is the mixed topology, so that, by Proposition I.1.26 of [4], the locally convex space $(\tilde{S}X, \mathcal{T})$ is complete (and not only quasi-complete). If we denote by $B_{\tilde{\varphi}}$ the completion of the normed space $\tilde{S}X/\ker\tilde{\varphi}$ with the norm $\| \ \|$ given by $\|[x]\| = \tilde{\varphi}(x)$ for all x in $\tilde{S}X$, then the projection $q_{\tilde{\varphi}} : \tilde{S}X \to B_{\tilde{\varphi}}$ is a continuous, contracting linear map. If, for each pair (φ, ψ) of continuous semi-norms on X such that $\varphi \leq \psi$, we denote by $q_{\tilde{\varphi},\tilde{\psi}}$ the natural contraction from $B_{\tilde{\psi}}$ into $B_{\tilde{\varphi}}$, we obtain an inverse system of Banach spaces such that the diagrams

$$\begin{array}{ccccc} & & \tilde{S}X & & \\ & \swarrow^{q_{\tilde{\psi}}} & & \searrow^{q_{\tilde{\varphi}}} & \\ B_{\tilde{\psi}} & & \xrightarrow{q_{\tilde{\varphi},\tilde{\psi}}} & & B_{\tilde{\varphi}} \end{array}$$

are commutative. It follows from a standard argument concerning limits of locally convex spaces that the cone thus obtained is a limit cone. By Corollary (3.5), the cone

$$\begin{array}{ccccc} & & \hat{O}\tilde{S}X & & \\ & \swarrow^{\hat{O}q_{\tilde{\psi}}} & & \searrow^{\hat{O}q_{\tilde{\varphi}}} & \\ \hat{O}B_{\tilde{\psi}} & & \xrightarrow{\hat{O}q_{\tilde{\varphi},\tilde{\psi}}} & & \hat{O}B_{\tilde{\varphi}} \end{array}$$

is still a limit cone. We denote the Banach balls $\hat{O}B_{\tilde{\varphi}}$ by $\hat{X}_\varphi$, the morphisms $\hat{O}q_{\tilde{\varphi}}$ and $\hat{O}q_{\tilde{\varphi},\tilde{\psi}}$ by $\hat{q}_\varphi$ and $\hat{q}_{\varphi,\psi}$ respectively, and by $\mathbb{B}$ the inverse system $(\hat{q}_{\varphi,\psi} : \hat{X}_\psi \to \hat{X}_\varphi)_{\varphi \leq \psi}$.

For later use we observe that, since X is a radial subspace of $\hat{O}S(|X|)$ (2.15), it follows from the construction above that the dual morphism $\hat{q}'_\varphi$ is injective for any $\varphi \in \mathcal{S}$.

Finally we define the TTCS $\hat{X}$ as the limit $\lim \mathbb{B}$ as constructed in (3.6). Then $\hat{X}$ is naturally isomorphic to $\hat{O}\tilde{S}X$ in $\mathcal{TTC}$, the isomorphism being given by the map $\hat{q} : \hat{O}\tilde{S}X \to \hat{X}$ of the form $\hat{q}(x) = (\hat{q}_\psi(x))_{\psi \in \mathcal{S}}$ for all $x \in \hat{O}\tilde{S}X$. Since the TTCS X is a dense subspace of $\hat{O}\tilde{S}X$, the restriction of $\hat{q}$ to X gives the desired completion $q : X \to \hat{X}$. Thus, we have proved the following

(3.10) **Proposition.** *Let X be a Hausdorff TTCS. Then, the continuous morphism $q : X \to \hat{X}$ is a completion, i.e., a dense embedding of X into the complete TTCS $\hat{X} = \lim \mathbb{B}$.*

Combining Proposition (3.10) and Corollary (3.8) we obtain the following structure theorem for complete TTCS.

(3.11) **Theorem.** *A TTCS is complete if and only if it is the limit of an inverse system of Banach balls.* □

For a complete TTCS X carrying the strong topology, we know that the evaluation map $e : X \to X''$ is a topological isomorphism (see Theorem (2.14)). If X is an arbitrary complete TTCS, we can only prove the following result.

(3.12) **Theorem.** *If X is a complete TTCS, then the evaluation map $e : X \to X''$ is an open isomorphism.*

Proof. Since X is a complete TTCS, we can write it as the summit of a limit cone $(\hat{q}_\varphi : X \to \hat{X}_\varphi)_{\varphi \in \mathcal{S}}$ (see (3.9)). For every φ in $\mathcal{S}$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{e_X} & X'' \\ \downarrow{\scriptstyle \hat{q}_\varphi} & & \downarrow{\scriptstyle \hat{q}''_\varphi} \\ \hat{X}_\varphi & \xrightarrow{e_{\hat{X}_\varphi}} & \hat{X}''_\varphi \end{array}$$

is commutative and the morphism $e_{\hat{X}_\varphi}$ is a topological isomorphism. Hence, X'' is the summit of the cone $(e^{-1}_{\hat{X}_\varphi} \circ \hat{q}''_\varphi : X'' \to \hat{X}_\varphi)$ with the same basis as the limit cone above. Therefore, there exists a continuous morphism $r : X'' \to X$ such that, for all φ in $\mathcal{S}$, the diagram

$$\begin{array}{ccc} X'' & \xrightarrow{r} & X \\ \downarrow{\scriptstyle \hat{q}''_\varphi} & & \downarrow{\scriptstyle \hat{q}_\varphi} \\ \hat{X}''_\varphi & \xrightarrow{e^{-1}_{\hat{X}_\varphi}} & \hat{X}_\varphi \end{array}$$

is commutative, too. Hence,

$$\hat{q}_\varphi \circ r \circ e_X = e^{-1}_{\hat{X}_\varphi} \circ \hat{q}''_\varphi \circ e_X = e^{-1}_{\hat{X}_\varphi} \circ e_{\hat{X}_\varphi} \circ \hat{q}_\varphi = \hat{q}_\varphi \text{ for all } \varphi \text{ in } \mathcal{S},$$

so that $r \circ e_X = \text{id}X$. If we can show that the map r is injective, then $e_X = r^{-1}$, i.e., e_X is an open isomorphism.

It follows from the construction of X (see (3.9)) that the map r can be given by the formula

$$r(\lambda) = (e^{-1}_{\hat{X}_\varphi} \circ \hat{q}''_\varphi(\lambda))_{\varphi \in \mathcal{S}} \text{ for all } \lambda \text{ in } X''.$$

Let $r(\lambda_1) = r(\lambda_2)$, i.e., $\hat{q}''_\varphi(\lambda_1) = \hat{q}''_\varphi(\lambda_2)$, resp. $\lambda_1 \circ \hat{q}'_\varphi = \lambda_2 \circ \hat{q}'_\varphi$ whenever φ in $\mathcal{S}$. Now, $X' = \bigcup_{\varphi \in \mathcal{S}} \hat{q}'_\varphi(\hat{X}'_\varphi)$, so that $\lambda_1 = \lambda_2$. Indeed, let $f : X \to \hat{O}\mathbb{C}$ be an element of X'. Then the function $x \mapsto |f(x)|$ is a continuous semi-norm ψ on X. The morphism $\hat{q}_\psi : X \to \hat{X}_\psi$ used in the construction of the limit cone of X is then given as follows: $\hat{X}_\psi$ is isomorphic to $\{0\}$ or $\hat{O}\mathbb{C}$, and $f = j_\psi \circ \hat{q}_\psi$ where j_ψ is the embedding of $\hat{X}_\psi$ into $\hat{O}\mathbb{C}$. □

4. Cocomplete totally convex spaces as a generalization of compact totally convex spaces

Theorem (3.12) implies that a complete TTCS is semi-reflexive. However, a complete TTCS need not be reflexive (see Example 5.2 below). Therefore, we try to find additional properties a complete TTCS has to have in order to be reflexive. Theorem (3.12) says that we must look for properties of a complete TTCS X which render the evaluation map $e : X \to X''$ continuous.

Definition. A TTCS is called *compact* if the underlying topological space is a compact Hausdorff space.

Clearly, every compact TTCS is complete.

Examples

1. The dual space X' of a TTCS carrying the strong topology is compact (see Theorem (2.13)), in particular, if B is a Banach space, then $(\hat{O}B)'$ is compact.

2. Using the same arguments as in the proof of Theorem (3.11) we can show that a TTCS is compact if and only if it is the limit of an inverse system of finite-dimensional Banach balls.

We adopt the terminology of R. Brown (see [2]) and call a map $f : X \to Y$ between topological spaces X and Y *k-continuous* provided that $f|_K$ is continuous whenever K is a compact subset of X. Clearly the composition of two k-continuous maps is k-continuous.

(4.1) **Proposition.** *For any TTCS X the evaluation map $e_X : X \to X''$ is k-continuous.*

Proof. Let K be a compact subset of X and let $(x_\delta)_{\delta \in D}$ be a net in K converging to some point $x \in K$. Consider an arbitrary compact subset C of X'. Then the set $\{f|_K : f \in C\}$ is compact in K' and thus equicontinuous by Ascoli's Theorem. Therefore, we have convergence $\sup_{f \in C} |e_X(x_\delta)(f) - e_X(x)(f)| \to 0$. We conclude that e_X is k-continuous. □

(4.2) **Corollary.** *Each compact TTCS is reflexive. In fact, any semi-reflexive TTCS X that is a k-space (i.e., is compactly generated) is reflexive.*

Proof. Indeed, the evaluation map $e_X : X \to X''$ is continuous, since X is a k-space (Proposition (4.1)). Let $i : X \to \hat{X}$ be the canonical embedding where $\hat{X}$ denotes the completion of X. By naturality of e we have $(e_{\hat{X}}^{-1} \circ i'' \circ e_X)(d) = i(d) = d$ whenever $d \in X$. Thus $e_X^{-1} = e_{\hat{X}}^{-1} \circ i''$ by semi-reflexivity. We conclude that the evaluation map of any semi-reflexive TTCS is open. □

Let *Ban* denote the full subcategory of the category $\mathcal{TTC}$ consisting of Banach balls, and *Comp* the full subcategory of $\mathcal{TTC}$ consisting of compact TTCS. Then the duality functor $D : \mathcal{TTC}^{op} \to \mathcal{TTC}$ (for the definition, see Section 2) allows a restriction $D : Ban^{op} \to Comp$ (Theorem (2.13)). Likewise, it allows a restriction $D : Comp^{op} \to Ban$, since the dual space $X' = [X, \hat{O}\mathbb{C}]$ of a compact TTCS X is easily seen to be a Banach ball.

(4.3) **Theorem.** *The functors $D : Ban^{op} \to Comp$, $D : Comp^{op} \to Ban$ define a duality between the categories Ban and Comp.*

Proof. By Proposition (2.12) every Banach ball $\hat{O}B$ is reflexive, i.e., the evaluation map $e_{\hat{O}B} : \hat{O}B \to (\hat{O}B)''$ is a topological isomorphism. By Corollary (4.2), every compact TTCS X is reflexive, so that the evaluation map $e_X : X \to X''$ is likewise a topological isomorphism. A straightforward calculation shows that we obtain that way two natural isomorphims $e : Id(Ban^{op}) \to D^{op} \circ D$ and $e : Id(Comp) \to D \circ D^{op}$. Hence, the functors $D : Ban^{op} \to Comp$ and $D^{op} : Comp \to Ban^{op}$ define an equivalence between the categories Ban^{op} and *Comp*. □

(4.4) **Remark.** Using Corollary (3.5) we could have obtained Theorem (4.3) also from the well-known duality between the category Ban_1 of Banach spaces and contracting linear maps and the category $\mathcal{C}_1$ of compact Saks spaces and their morphisms (cf. [3], I.2.8, where compact Saks spaces are presented as Waelbroeck spaces, or [11], Proposition 13.18, where compact Saks spaces are called γ-compact MT-spaces).

The following corollary to Theorem (4.3) will be needed in the sequel.

(4.5) **Corollary.** *Let $\hat{O}B$ be a Banach ball, C a compact TTCS and $g : (\hat{O}B)' \to C'$ a continuous morphism. Then, there exists a unique continuous morphism $f : C \to \hat{O}B$ such that $g = f'$.*

Proof.

- Uniqueness: Let $f_1' = f_2'$ for two continuous morphisms $f_i : C \to \hat{O}B$ $(i = 1, 2)$. The diagrams

$$\begin{array}{ccc} C & \xrightarrow{e_C} & C'' \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_i''} \\ \hat{O}B & \xrightarrow{e_{\hat{O}B}} & (\hat{O}B)'' \end{array}$$

are commutative and the morphisms e_C and $e_{\hat{O}B}$ are topological isomorphisms. Therefore, it follows from $f_1'' = f_2''$ that $f_1 = f_2$.

- Existence: We define $f : C \to \hat{O}B$ by setting $f = e_{\hat{O}B}^{-1} \circ g' \circ e_C$. Then the following diagram is commutative

$$\begin{array}{ccc} C & \xrightarrow{e_C} & C'' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g'} \\ \hat{O}B & \xrightarrow{e_{\hat{O}B}} & (\hat{O}B)'' \end{array}$$

Dualizing, we obtain the commutative diagrams

$$\begin{array}{ccc} C' & \xleftarrow{e_C'} & C''' \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle g''} \\ (\hat{O}B)' & \xleftarrow{e_{\hat{O}B}'} & (\hat{O}B)''' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C' & \xrightarrow{e_{C'}} & C''' \\ \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle g''} \\ (\hat{O}B)' & \xrightarrow{e_{\hat{O}B'}} & (\hat{O}B)''' \end{array}$$

A direct verification shows that $e_C' = e_{C'}^{-1}$ and $e_{\hat{O}B}' = e_{\hat{O}B'}^{-1}$. Thus,

$$g = e_{C'}^{-1} \circ g'' \circ e_{\hat{O}B'} = e_C' \circ g'' \circ (e_{\hat{O}B}')^{-1} = f' \circ e_{\hat{O}B}' \circ (e_{\hat{O}B}')^{-1} = f'. \quad \square$$

The following class of TTCS larger than the class of compact TTCS, which still has the property that the evaluation map is continuous, suggests itself: colimits of compact TTCS. We shall study such colimits in the remainder of this section, and we shall call them cocomplete TTCS (thinking of the characterization of the complete TTCS as limits of Banach balls).

Definition. A Hausdorff TTCS X is called *cocomplete* if it is the colimit of a direct system of compact TTCS.

There is a connection between *k-spaces* (i.e., compactly generated TTCS) and cocomplete spaces which will be discussed in due course.

(4.6) **Theorem.** *The following properties of a TTCS X are equivalent:*

(i) *For any TTCS Y and any morphism $f : X \to Y$, the map f is continuous whenever it is k-continuous.*

(ii) *For any semi-norm $\varphi : X \to O\mathbb{R}$, the map φ is continuous whenever it is k-continuous.*

Those properties are satisfied in a cocomplete space.

Proof.

(ii) $\Rightarrow$ (i): Let $f : X \to Y$ be a k-continuous morphism. By (ii), for any continuous semi-norm ψ on Y, $\psi \circ f$ is a continuous semi-norm on X.

(i) $\Rightarrow$ (ii): Every semi-norm $\varphi : X \to O\mathbb{R}$ can be factored as follows:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & O\mathbb{R} \\ & {\scriptstyle q_\varphi}\searrow \quad \nearrow{\scriptstyle \|\,\|} & \\ & X_\varphi & \end{array}$$

where q_φ is the canonical projection onto the quotient-space X_φ for the equivalence relation

$$x_1 \sim x_2 \Longleftrightarrow \varphi(\frac{1}{2}x_1 - \frac{1}{2}x_2) = 0 \text{ for all } x_1, x_2 \text{ in } X,$$

and endowed with the norm

$$\|[x]\| = \varphi(x) \text{ for all } x \text{ in } X,$$

where $[x]$ denotes the equivalence class of x. Clearly, the semi-norm φ is continuous if and only if the morphism q_φ is continuous.

Let X be cocomplete and let $(p_\alpha : X_\alpha \to X)_{\alpha \in D}$ be the colimit cone of a basis consisting of compact TTCS X_α. Then X is also the colimit of the direct system consisting of the compact subspaces $C_\alpha = p_\alpha(X_\alpha)$ of X, whence the stated properties follow, since X can be identified with the TCS generated by $\bigcup_{\alpha \in D} p_\alpha(X_\alpha)$. □

Definition. For any TTCS X, the *κ-modification* X^κ of X denotes that TTCS with underlying TCS X the topology of which is generated by the set of all k-continuous semi-norms on X. Obviously X and X^κ have the same compact subsets. In particular $(X^\kappa)^\kappa = X^\kappa$. Furthermore, for any k-continuous morphism $f : X \to Y$, it is readily verified that $f^\kappa : X^\kappa \to Y^\kappa$ is continuous, where f and f^κ denote the same set map.

Clearly $X = X^\kappa$ for any cocomplete TTCS X. The authors do not know whether the converse obtains.

(4.7) **Corollary.** *For any TTCS X, the evaluation map $e_{X^\kappa} : X^\kappa \to (X^\kappa)''$ is continuous. In particular, if X^κ is semi-reflexive, then it is reflexive. Thus, any cocomplete and complete TTCS is reflexive.*

Proof. See the proof of Corollary (4.2) and use Proposition (4.1) and Theorem (4.6). □

(4.8) **Proposition.** *Let X be a TTCS having at least one of the following properties:*
(i) *Any compact subset of X is contained in some compact subspace of X.*
(ii) *Any sequentially continuous morphism $f : X^\kappa \to Y$ into an arbitrary TTCS Y is continuous.*
Then X^κ is cocomplete.

In the proof of Proposition (4.8) we shall make use of the following simple, but useful lemma.

(4.8.1) **Lemma.** *Let $(x_n)_{n\geq 0}$ be a sequence in a TTCS X converging to 0. Then the totally convex subspace K of X generated by $\{x_n; n \geq 0\}$ is compact.*

Proof. Let $(y_\delta)_{\delta\in D}$ be a net in K where $y_\delta = \sum_{i=0}^{\infty} \alpha_{i\delta} x_i$ and $(\alpha_{i\delta})_{i\geq 0}$ in $O\ell_1\mathbb{N}$ for $\delta \in D$. The auxiliary net $(c_\delta)_{\delta\in D}$ where $c_\delta = (\alpha_{0\delta}, \alpha_{1\delta}, \cdots)$ whenever $\delta \in D$ has a cluster point $(\alpha_i)_{i\geq 0}$ in the product space $O\mathbb{C}^{\mathbb{N}}$. We want to show that $z := \sum_{i=0}^{\infty} \alpha_i x_i$ is a cluster point of $(y_\delta)_{\delta\in D}$ in K. Since $\sum_{i=0}^{\infty} |\alpha_i| \leq 1$, z is clearly a well-defined element of K. Let φ be an arbitrary continuous semi-norm on X, $\varepsilon > 0$ and $\delta \in D$. There is $n_0 \in \mathbb{N}$ such that $\varphi(x_i) < \varepsilon/2$ whenever $i \in \mathbb{N}$ and $i > n_0$. Furthermore, there is $\delta' \in D$ such that $\delta' \geq \delta$ and such that $|\alpha_{i\delta'} - \alpha_i| < \frac{\varepsilon}{n_0+1}$ whenever $i \leq n_0$, since $(\alpha_i)_{i\geq 0}$ is a cluster point of $(c_\delta)_{\delta\in D}$. Then

$$\begin{aligned}
&\varphi(\frac{y_{\delta'}}{2} - \frac{z}{2}) = \varphi(\sum_{i=0}^{\infty}(\frac{\alpha_{i\delta'} - \alpha_i}{2})x_i) \\
&\leq \frac{1}{2}\sum_{i=0}^{n_0} |\alpha_{i\delta'} - \alpha_i|\varphi(x_i) + \frac{1}{2}\sum_{i=n_0+1}^{\infty} |\alpha_{i\delta'} - \alpha_i|\varphi(x_i) \\
&\leq \frac{1}{2}\frac{\varepsilon}{n_0+1}(n_0+1)\cdot 1 + \frac{1}{2}(2\cdot\frac{\varepsilon}{2}) = \varepsilon.
\end{aligned}$$

We have shown that z is a cluster point of $(y_\delta)_{\delta\in D}$. Thus K is compact.
□

Proof of (4.8). Let K denote the set of all compact (totally convex) subspaces of X. Observe first that in an arbitrary TTCS the totally convex subspace generated by the union of finitely many compact totally convex subspaces is compact. Hence the set $\mathcal{K}$ is directed for the ordering by inclusion, and the embeddings $p_{KL} : C_K \to C_L$ $(K \leq L)$ form a direct system $\mathbb{D}$ of compact TTCS. Moreover, the embeddings $p_K : C_K \to X^\kappa$ form a cone with summit X^κ and basis $\mathbb{D}$. We claim that it is a colimit cone. Since $X = \bigcup_{K\in\mathcal{K}} C_K$, for every cone $(f_K : C_K \to Y)_{K\in\mathcal{K}}$ with the same basis $\mathbb{D}$, there exists a unique map

$f : X^{\kappa} \to Y$ such that $f \circ p_K = f_K$ for all K in $\mathcal{K}$. Clearly f is defined by $f(x) = f_K(x)$ for any $K \in \mathcal{K}$ such that $x \in C_K$.

We next show that f is a morphism. Let $(x_n)_{n\geq 0}$ in $X^{\mathbb{N}}$ and $(\alpha_n)_{n\geq 0}$ in $O\ell_1\mathbb{N}$. For each $n \geq 0$ consider the subspace K_n of X generated by $\{x_0, x_1, \dots, x_n, \sum_{i=0}^{\infty} \alpha_i x_i, 0\}$. Furthermore, let K denote the subspace generated by the sequence $(\frac{1}{2}\sum_{i=0}^{\infty}\alpha_i x_i - \frac{1}{2}\sum_{i=0}^{n}\alpha_i x_i)_{n\geq 0}$. Since K and each K_n ($n \geq 0$) are compact (Lemma (4.8.1)), we have

$$\frac{1}{2}f(\sum_{i=0}^{\infty}\alpha_i x_i) - \frac{1}{2}\sum_{i=0}^{n}\alpha_i f(x_i) = f(\frac{1}{2}\sum_{i=0}^{\infty}\alpha_i x_i - \frac{1}{2}\sum_{i=0}^{n}\alpha_i x_i) \to 0$$

and thus $f(\sum_{i=0}^{\infty}\alpha_i x_i) = \sum_{i=0}^{\infty}\alpha_i f(x_i)$.
Note that f is sequentially continuous: If $x_n \to x$ in X, then $f(\frac{1}{2}x_n - \frac{1}{2}x) \to 0$, i.e., $f(x_n) \to f(x)$, by Lemma (4.8.1). Therefore f is continuous under assumption (ii). Under assumption (i) f is k-continuous. Hence, f is continuous by Theorem (4.6). □

The following two propositions support our terminology "complete TTCS" and "cocomplete TTCS" since they show that the duality functor transports complete TTCS to cocomplete TTCS, and vice versa.

(4.9) **Proposition.** *If X is a complete TTCS, then the dual space X' is cocomplete.*

Proof. Since X is a complete TTCS, we can write it as the summit of a limit cone $(\hat{q}_\varphi : X \to \hat{X}_\varphi)_{\varphi\in\mathcal{S}}$ of an inverse system $\mathbb{D}$ of Banach balls (cf. (3.9)). We claim that the family $(\hat{q}'_\varphi)_{\varphi\in\mathcal{S}}$ of dual morphisms $\hat{q}'_\varphi : \hat{X}'_\varphi \to X'$ is a colimit of the dual direct system $\mathbb{D}'$. Since the objects of the latter are compact TTCS (Theorem (2.13)), the TTCS X' is cocomplete.

Let $(g_\varphi : \hat{X}'_\varphi \to Y)_{\varphi\in\mathcal{S}}$ be another cone with basis $\mathbb{D}'$ and summit Y. We have seen that $X' = \bigcup_{\psi\in\mathcal{S}} \hat{q}'_\psi(\hat{X}'_\psi)$ (cf. proof of Theorem (3.12)), so that we can define a map $g : X' \to Y$ by setting

$g(f) = g_\varphi(f_\varphi)$ for any $\varphi \in \mathcal{S}$ and any f_φ in $\hat{X}'_\varphi$ such that $\hat{q}'_\varphi(f_\varphi) = f$.

Since each $\hat{q}'_\varphi$ is injective, g is a well-defined map satisfying $g \circ \hat{q}'_\psi = g_\psi$ for all ψ in $\mathcal{S}$, and even the only one.

It remains to be shown that g is a continuous morphism. Observe that it suffices to verify this when Y is a Banach ball $\hat{O}B$. Indeed, g is a continuous morphism if and only if the composition $q \circ g : X' \to Y \to \hat{Y}$, where $q : Y \to \hat{Y}$ is the completion of Y, is a continuous morphism, and by Theorem (3.11) $\hat{Y}$ is a limit of an inverse system of Banach balls.

By Theorem (4.3), we may write $Y = C'$ for a compact TTCS C, and $\hat{X}_\psi = \hat{O}B_\psi$ for a Banach space B_ψ. By Corollary (4.5), each continuous morphism $g_\psi : (\hat{O}B_\psi)' \to C'$ is of the form $g_\psi = f'_\psi$ for a unique continuous morphism $f_\psi : C \to \hat{O}B_\psi$. The uniqueness implies that the family $(f_\psi)_{\psi \in \mathcal{S}}$ defines a cone with basis $\mathbb{D}$ and summit C. Therefore, there exists a unique continuous morphism $f : C \to X$ such that $\hat{q}_\psi \circ f = f_\psi$ for all ψ in $\mathcal{S}$. By the definition of g and the results obtained so far, we have

$$g \circ \hat{q}'_\varphi = g_\varphi = f'_\varphi = (\hat{q}_\varphi \circ f)' = f' \circ \hat{q}'_\varphi \text{ for all } \varphi \text{ in } \mathcal{S}.$$

Since $X' = \bigcup_{\psi \in \mathcal{S}} \hat{q}'_\psi(\hat{X}'_\psi)$, we conclude that $g = f'$, and hence g is a continuous morphism. □

Example 3. A classical result of R.C. Buck for the Banach space $C^\infty(Y)$ of Example 3.3, endowed with the mixed topology associated with the topology of compact convergence τ_K, says that the dual of the resulting locally convex space is the space $M(Y)$ of bounded $\mathbb{C}$-valued Radon measures on the locally compact space Y (cf. [4], II.3.). The mixed topology is the strongest locally convex topology among those which coincide with τ_K on $OC^\infty(Y)$. Hence, $\hat{O}M(Y) \cong (\hat{O}C^\infty(Y))'$ and, since $\hat{O}C^\infty(Y)$ is a complete TTCS, the Radon measures on Y of norm ≤ 1 form a cocomplete TTCS (which in general is not compact).

(4.10) **Proposition.** *For any TTCS X, the dual space $(X^\kappa)'$ is complete. In particular, the dual of a cocomplete TTCS is complete.*

Proof. Let $(f_\delta)_{\delta\in D}$ be a Cauchy net in $(X^\kappa)'$ with respect to the uniformity of compact convergence. The net converges pointwise to a morphism $f : X^\kappa \to \hat{O}\mathbb{C}$, the convergence being uniform on each compact subset K of X. Hence, the restriction $f|_K : K \to O\mathbb{C}$ is continuous. The result follows from Theorem (4.6). □

(4.11) **Remark.** We shall see later that it is not necessary for a (complete) TTCS X to be cocomplete in order that the dual space X' is complete (see Example 5.2).
Clearly X' is also complete for a TTCS X that is a k-space. Hence, a reflexive TTCS X that is a k-space is cocomplete, since X'' is cocomplete. This observation suggests the following problem.

Problem. Is there a TTCS that is a k-space, but not cocomplete? (By Proposition (4.8) any sequential TTCS is cocomplete.)

Definition. A TTCS is called *bicomplete* if it is complete and cocomplete.

Example 4. Any complete TTCS that is a k-space is bicomplete. Indeed it is a semi-reflexive k-space and thus reflexive and cocomplete (see Corollary (4.2) and Remark (4.11)).

Let $\mathcal{V}$ denote the full subcategory of the category $\mathcal{TTC}$ consisting of bicomplete TTCS. By Propositions (4.9) and (4.10), the duality functor $D : \mathcal{TTC}^{op} \to \mathcal{TTC}$ allows a restriction $D : \mathcal{V}^{op} \to \mathcal{V}$.

(4.12) **Theorem.** *The functor $D : \mathcal{V}^{op} \to \mathcal{V}$ defines a self-duality of the category $\mathcal{V}$.*

Proof. By Corollary (4.7) every bicomplete TTCS X is reflexive, i.e., the evaluation map $e_X : X \to X''$ is a topological isomorphism. We obtain therefore natural isomorphisms $e : Id(\mathcal{V}^{op}) \to D^{op} \circ D$ and

$e : Id(\mathcal{V}) \to D \circ D^{op}$. Hence, the functor $D : \mathcal{V}^{op} \to \mathcal{V}$ defines an equivalence between the categories $\mathcal{V}^{op}$ and $\mathcal{V}$. □

(4.13) Compact subsets and compact subspaces in a complete TTCS.

Definition. Let X be a TCS and M a subset of X. We define the *convex circled hull* $\Gamma(M)$ *of* M *in* X as the set

$$\Gamma(M) = \{\sum_{n=0}^{N} \alpha_n x_n; \sum_{n=0}^{N} |\alpha_n| \leq 1, x_n \in M \text{ for } n = 0, \ldots, N \text{ and } N \in \mathbb{N}\}.$$

(4.13.1) **Proposition.** *Let X be a TTCS. If M is a precompact subset of X, then $\Gamma(M)$ is again a precompact subset.*

Proof. In the special case where M is a finite set, say $M = \{x_0, \ldots, x_N\}$, $\Gamma(M)$ is the image of the compact subset $\Delta_N = \{\underline{\alpha} \in \mathbb{C}^{N+1}; \sum_{n=0}^{N} |\alpha_n| \leq 1\}$ of $\mathbb{C}^{N+1}$ by the continuous map $(\alpha_0, \ldots, \alpha_N) \mapsto \sum_{n=0}^{N} \alpha_n x_n$, and therefore compact. Let now M be any precompact subset of X, i.e., for each $\varepsilon > 0$ and every continuous semi-norm φ on X, there exists a finite set M_0 such that $M \subseteq \bigcup_{m \in M_0} B_{\varepsilon,\varphi}(m)$, where $B_{\varepsilon,\varphi}(m) = \{x \in X; d_\varphi(m, x) < \varepsilon\}$. Since $\Gamma(M_0)$ is compact, we have $\Gamma(M_0) \subseteq \bigcup_{m \in N_0} B_{\varepsilon,\varphi}(m)$ for a finite set N_0. Consider $y = \sum_{n=0}^{N} \alpha_n x_n$ with $N \in \mathbb{N}$, $\sum_{n=0}^{N} |\alpha_n| \leq 1$ and $x_n \in M$ for $n = 0, \ldots, N$. For each n there exists $y_n \in M_0$ such that $x_n \in B_{\varepsilon,\varphi}(y_n)$. Furthermore, there is a $z \in N_0$ such that $2\varphi(\frac{1}{2} \sum_{n=0}^{N} \alpha_n y_n - \frac{1}{2} z) < \varepsilon$. Thus,

$$\varphi(\frac{1}{2}y - \frac{1}{2}z) \leq \varphi(\frac{1}{2}y - \frac{1}{2}\sum_{n=0}^{N} \alpha_n y_n) + \varphi(\frac{1}{2}\sum_{n=0}^{N} \alpha_n y_n - \frac{1}{2}z)$$
$$< \varphi(\sum_{n=0}^{N} \alpha_n (\frac{x_n}{2} - \frac{y_n}{2})) + \frac{\varepsilon}{2} < \varepsilon.$$

We conclude that $\Gamma(M)$ is precompact. $\square$

(4.13.2) **Corollary.** *Let X be a complete TTCS. If M is a compact subset of X, then the closure $\overline{\Gamma(M)}$ of $\Gamma(M)$ in X is again a compact subset.* $\square$

(4.13.3) **Proposition.** *Let X be a TTCS and M a subset of X. Then the closure $\overline{\Gamma(M)}$ of $\Gamma(M)$ in X is a totally convex subspace.*

Proof. We have to verify that, for every $\underline{\alpha}$ in $O\ell_1\mathbb{N}$ and $\underline{x}$ in $\overline{\Gamma(M)}^{\mathbb{N}}$, the sum $\sum_{n=0}^{\infty} \alpha_n x_n$ belongs to the closure $\overline{\Gamma(M)}$. Let $\varepsilon > 0$ and φ be a continuous semi-norm on X. There exists an integer $N \geq 0$ such that $\sum_{n>N} |\alpha_n| < \varepsilon$. Hence,

$$\varphi(\frac{1}{2}\sum_{n=0}^{\infty} \alpha_n x_n - \frac{1}{2}\sum_{n\leq N} \alpha_n x_n) = \varphi(\frac{1}{2}\sum_{n>N} \alpha_n x_n) \leq \frac{1}{2}\sum_{n>N} |\alpha_n| < \frac{\varepsilon}{2}.$$

Furthermore, there exist points $y_1, \ldots, y_N$ in $\Gamma(M)$ (i.e., $y_n = \sum_{m=0}^{L} \beta_{nm} z_m$, where $z_0, \ldots, z_L$ are points in M) such that $\varphi(\frac{1}{2}x_n - \frac{1}{2}y_n) < \frac{\varepsilon}{2}$. Hence,

$$\begin{aligned}
&2\varphi(\tfrac{1}{2}\sum_{n=0}^{\infty} \alpha_n x_n - \tfrac{1}{2}\sum_{n\leq N} \alpha_n y_n) \\
&= 4\varphi(\tfrac{1}{4}\sum_{n=0}^{\infty} \alpha_n x_n - \tfrac{1}{4}\sum_{n\leq N} \alpha_n x_n + \tfrac{1}{4}\sum_{n\leq N} \alpha_n x_n - \tfrac{1}{4}\sum_{n\leq N} \alpha_n y_n) \\
&\leq 2\varphi(\tfrac{1}{2}\sum_{n=0}^{\infty} \alpha_n x_n - \tfrac{1}{2}\sum_{n\leq N} \alpha_n x_n) + 2\sum_{n\leq N} |\alpha_n| \varphi(\tfrac{1}{2}x_n - \tfrac{1}{2}y_n) < 2\varepsilon\,,
\end{aligned}$$

and $\sum_{n\leq N} \alpha_n y_n = \sum_{n\leq N} \alpha_n (\sum_{m=0}^{L} \beta_{nm} z_m) = \sum_{m=0}^{L} (\sum_{n=0}^{N} \alpha_n \beta_{nm}) z_m$ is an element of $\Gamma(M)$, i.e., $\sum_{n=0}^{\infty} \alpha_n x_n$ belongs to $\overline{\Gamma(M)}$. $\square$

Problem. Let K be a compact subset of a TTCS X. When is the closure of the totally convex subspace generated by K compact in X?

(4.14) **Proposition.** *For any complete TTCS X, the κ-modification X^κ is cocomplete. In particular any complete TTCS X satisfying $X = X^\kappa$ is bicomplete.*

Proof. The assertion follows from (4.8), (4.13.2) and (4.13.3). We shall give a different proof of this result later (Proposition (5.4)). □

Example 5. *A bicomplete TTCS whose underlying space is not a k-space.*

Let B be an infinite dimensional complex Banach space and I an uncountable set. Set $X = (\hat{O}B)^I$ endowed with the pointwise totally convex structure and the product topology. Then X is a complete TTCS. Since each factor of $(\hat{O}B)^I$ is a first countable space, by Theorem 5.6 of [8], every map into a regular space whose restriction to each compact subset is continuous is already continuous. Hence, by Proposition (4.14), X is a bicomplete TTCS. Since no factor of $(\hat{O}B)^I$ is countably compact, by Proposition 5.5 of [8], the underlying space of X is not a k-space.

Problem. The product of finitely many bicomplete TTCS is readily seen to be bicomplete. What can be said about products of infinitely many such spaces?

5. Completion and duality

In the sequel we shall use the notation X^* for the dual space $(X^\kappa)'$ of the κ-modification X^κ of a TTCS X (cf. Section 4). Moreover we distinguish between the evaluation map $e_X : X \to X''$ and the evaluation map $\hat{e}_X : X \to X^{**}$ which is similarly defined by setting

$$\hat{e}_X(x)(f) = f(x) \quad \text{for all } x \text{ in } X \text{ and } f \text{ in } X^*.$$

An obvious modification of the proof of Proposition (4.1) establishes that $\hat{e}_X : X \to X^{**}$ is k-continuous for an arbitrary TTCS X.

It will follow from the results presented below that $\hat{e}_X : X \to X^{**}$ is a homeomorphism if and only if the TTCS X is bicomplete.

(5.1) **Lemma.** *Let X be a TTCS and let $i : X' \to X^*$ be the obvious embedding. Then $\hat{e}_X(X) \subseteq X^{*\prime}$ and $i'(\hat{e}_X(x)) = e_X(x)$ whenever $x \in X$.*

Proof. Let $x \in X$. Suppose that $(f_\delta)_{\delta \in D}$ is a net in X^* converging to 0. Then $\hat{e}_X(x)(f_\delta) = f_\delta(x) \to 0$, since $\{x\}$ is a compact subset of X^κ. Thus $\hat{e}_X(x) \in X^{*\prime}$, and therefore $i'(\hat{e}_X(x)) = \hat{e}_X(x) \circ i \in X''$. We conclude that $\hat{e}_X(x) \circ i = e_X(x)$, because $(\hat{e}_X(x) \circ i)(h) = h(x) = e_X(x)(h)$ for any $h \in X'$. □

(5.2) **Lemma.** *Let X be a TTCS containing a dense subset D such that the restriction to D of the evaluation map $e_X : X \to X''$ is continuous. Then the completion $\hat{X}$ of X is reflexive.*

Proof. By Theorem (3.12) the evaluation map $e_{\hat{X}} : \hat{X} \to (\hat{X})''$ is an open isomorphism, since $\hat{X}$ is complete. Let $i : X \to \hat{X}$ be the canonical embedding. By our assumption $i'' \circ e_X|_D$ is continuous on D. Since $(\hat{X})''$ is complete, there exists a continuous extension $f : \hat{X} \to (\hat{X})''$ of $i'' \circ e_X|_D$ to $\hat{X}$. It suffices to show that $(e_{\hat{X}}^{-1} \circ f)(x) = x$ whenever $x \in D$, because then, by continuity of f and $e_{\hat{X}}^{-1}$, and by the density of D in $\hat{X}$ it follows that $e_{\hat{X}}^{-1} \circ f = \mathrm{id}_{\hat{X}}$ and thus $e_{\hat{X}} = f$. Let $x \in D$ and $h \in (\hat{X})'$. We have $e_{\hat{X}}(x)(h) = h(x)$ and $[(i'' \circ e_X)(x)](h) = (e_X(x) \circ i')(h) = e_X(x)(h \circ i) = h(x)$. Therefore $e_{\hat{X}}(x) = (i'' \circ e_X)(x)$ and thus, by the definition of f, $(e_{\hat{X}}^{-1} \circ f)(x) = e_{\hat{X}}^{-1}[(i'' \circ e_X)(x)] = x$. We have shown that $\hat{X}$ is reflexive. □

(5.2.1) **Corollary.**

(a) *The completion of any reflexive TTCS is reflexive.*

(b) *For any TTCS X, the completion of X^κ is reflexive.* □

(5.3) **Theorem.** *Let X be a semi-reflexive TTCS. Then X^* is the completion of X'.*

Proof. Since X and X^κ have the same compact subsets, X' is clearly a subspace of X^*. Let E be the closure of X' in X^*. Since X^* is complete (Proposition (4.10)), E is the completion of X'. Hence, the evaluation map $e_E : E \to E''$ is an open bijection (Theorem (3.12)). Let $p : X' \to E$ be the embedding. Note that $p' : E' \to X''$ is a bijection. Moreover, let $\hat{e}_X : X \to X^{**}$ denote the evaluation map. Since X is semi-reflexive, the evaluation map $e_X : X \to X''$ is an open bijection (see the proof of Corollary (4.2)).
Consider $f \in X^*$. Since $e_X^{-1} \circ p'$ is continuous, $f \circ e_X^{-1} \circ p' : E' \to \hat{O}\mathbb{C}$ is k-continuous. Because E' is cocomplete (Proposition (4.9)), we see with the help of Theorem (4.6) that $f \circ e_X^{-1} \circ p' \in E''$. Since e_E is bijective, there is $h \in E$ such that $e_E(h) = f \circ e_X^{-1} \circ p'$. Fix $x \in X$. Note that

$$e_E(h)\left[((p')^{-1} \circ e_X)(x)\right] = (f \circ e_X^{-1} \circ p')((p')^{-1} \circ e_X)(x)) = f(x).$$

On the other hand, since by Lemma (5.1) $\hat{e}_X(x)$ is a continuous extension of $e_X(x)$ to X^*, we have

$$e_E(h)\left[((p')^{-1} \circ e_X)(x)\right] = \left(((p')^{-1} \circ e_X)(x)\right)(h) = (\hat{e}_X(x)|_E)(h).$$

Thus $f = h \in E$ and $E = X^*$. □

(5.3.1) **Corollary.** *If X is a complete or a reflexive TTCS, then X^* is the completion of X'. In either case X^* is reflexive.* □

(5.3.2) **Corollary.** *For any reflexive TTCS X, the space X'^* yields the completion of X with canonical embedding $i \circ e_X$, where $e_X : X \to X''$ denotes the evaluation map and $i : X'' \to X'^*$ the obvious embedding.* □

Problem. Find an example of a reflexive TTCS that is not complete. (Note that the dual of such a space would be a reflexive space that is not cocomplete.)

(5.4) **Proposition.** *If X is a semi-reflexive TTCS, then X^κ and $X^{*\prime}$ are isomorphic (under the isomorphism $\dot{e}_X^\kappa$ defined below). In particular, X^κ is cocomplete.*

Proof. As above, $e_X : X \to X''$ denotes the evaluation map and $i : X' \to X^*$ the obvious embedding. It will be helpful to remember that $X^{*\prime}$ is cocomplete (Propositions (4.9) and (4.10)). Since the corestriction $\dot{e}_X : X \to X^{*\prime}$ of the evaluation map $\hat{e}_X : X \to X^{**}$ is k-continuous, $\dot{e}_X^\kappa : X^\kappa \to X^{*\prime}$ is continuous. By Theorem (5.3) $i' : X^{*\prime} \to X''$ is a bijection. We recall that $i' \circ \dot{e}_X = e_X$ (Lemma (5.1)). Since e_X is bijective, $\dot{e}_X$ is bijective. Furthermore $\dot{e}_X^{-1} = e_X^{-1} \circ i' : X^{*\prime} \to X$ is continuous and by the cocompleteness of $X^{*\prime}$, we see that $(\dot{e}_X^\kappa)^{-1} : X^{*\prime} \to X^\kappa$ is continuous. Thus, $\dot{e}_X^\kappa : X^\kappa \to X^{*\prime}$ is an isomorphism for any semi-reflexive TTCS X. □

(5.4.1) **Corollary.** *For any reflexive or complete TTCS X, the space X^κ is reflexive.* □

Remark. It follows from Proposition (5.4) that for a semi-reflexive TTCS X, the family of all semi-norms $\varphi_K(x) = \sup_{f \in K} |\hat{e}_X(x)(f)|$, where K is a compact subset of X^*, generates the topology of the space X^κ.

(5.5) **Lemma.** *If X is complete, X^* is bicomplete, and if D is a dense subset of X such that the restriction of the evaluation map $\hat{e}_X : X \to X^{**}$ to D is continuous, then X is bicomplete.*

Proof. Since X is semi-reflexive, by the proof of Proposition (5.4) the corestriction $\dot{e}_X : X \to X^{*\prime}$ of the evaluation map $\hat{e}_X : X \to X^{**}$ is an open k-continuous bijection. By our assumption, the restriction

$\dot{e}_X|_D$ is continuous. The continuous extension $e : X \longrightarrow X^{*\prime}$ of $\dot{e}_X|_D$ exists, because $X^{*\prime}$ is complete. Since $\dot{e}_X^{-1} \circ \dot{e}_X|_D = \mathrm{id}_D$, we see that $\dot{e}_X^{-1} \circ e = \mathrm{id}_X$ by the density of D in X. Therefore

$$e = (\dot{e}_X \circ \dot{e}_X^{-1}) \circ e = \dot{e}_X \circ (\dot{e}_X^{-1} \circ e) = \dot{e}_X.$$

We conclude that $\dot{e}_X$ is a homeomorphism and thus X is cocomplete, since $X^{*\prime}$ is cocomplete. □

(5.6) **Proposition.** *Let X be a complete TTCS. Then the following conditions are equivalent:*

(i) *X^{**} is bicomplete.*

(ii) *X^* is bicomplete.*

(iii) *X^κ is bicomplete.*

Proof.

(i) $\Longrightarrow$ (ii): Apply the preceding lemma to X^* and $D = X'$ and use that D is cocomplete.

(ii) $\Longrightarrow$ (iii): The dual $X^{*\prime}$ of X^* is bicomplete. But $X^{*\prime}$ is isomorphic to X^κ.

(iii) $\Longrightarrow$ (i): X^κ is isomorphic to $X^{*\prime}$ and X^{**} is the completion of $X^{*\prime}$. □

Problem. Let X be a complete TTCS. Is X^* always bicomplete? By Lemma (5.5), this question is easily seen to be equivalent to the problem whether the completion of each cocomplete TTCS is bicomplete.

(5.7) **Remark.** We are finally ready to answer the following questions: Is there a complete TTCS which is not reflexive? (Cf. the assertion preceding Theorem (3.12).) Is there a complete TTCS X such that the dual space X' is not complete? The following two examples, each of which gives a positive answer to one of those questions, are due to the second author. He thus saved the first author, who believed that the category of complete TTCS was a $*$-autonomous category, from a major embarrassment.

Example 1. *A complete TTCS X such that the dual space X' is not complete.*

Let Y be an uncountable set. For each subset C of Y, we denote by $\mathcal{F}(C)$ the set of finite subsets of C. Then we define a TTCS X by setting

$$X = O\ell_1 Y = \{g \in O\mathbb{C}^Y;\ \sup_{F\in\mathcal{F}(Y)} \sum_{y\in F} |g(y)| \leq 1\},$$

and by endowing that TCS with the topology generated by the seminorms of the form

$$\varphi_C(g) = \sup_{F\in\mathcal{F}(C)} \sum_{y\in F} |g(y)| \text{ for all } g \text{ in } X,$$

where C is a countable subset of Y.

(5.8.1) *The TTCS X is complete.*

Proof. Clearly, the topology on X is Hausdorff. Let $(g_\delta)_{\delta\in D}$ be a Cauchy net in X, i.e., for $\varepsilon > 0$ and any countable subset C of Y, there exists a δ_0 in D such that, for all $\delta', \delta'' \geq \delta_0$, $\sup\limits_{F\in\mathcal{F}(C)} \sum\limits_{y\in F} |\frac{1}{2}g_{\delta'}(y) - \frac{1}{2}g_{\delta''}(y)| < \varepsilon$. Then for any point y in Y, $(g_\delta(y))_{\delta\in D}$ is a Cauchy net in $O\mathbb{C}$ and thus converges to some element $g(y)$ in $O\mathbb{C}$. It follows that, for $\varepsilon > 0$ and any countable subset C of Y, there exists a δ_0 in D such that

$$\sup_{F\in\mathcal{F}(C)} \sum_{y\in F} |\frac{1}{2}g_\delta(y) - \frac{1}{2}g(y)| \leq \varepsilon \text{ whenever } \delta \geq \delta_0 \text{ in } D,$$

i.e., the net $(g_\delta)_{\delta\in D}$ converges to the function $y \mapsto g(y)$. It remains to be shown that the latter belongs to X, i.e., $\sum\limits_{y\in F} |g(y)| \leq 1$ for all F in $\mathcal{F}(Y)$. Assume the contrary: there is an F in $\mathcal{F}(Y)$ such that $\sum\limits_{y\in F} |g(y)| > 1$.

Since $g_\delta(y)$ converges to $g(y)$ whenever y belongs to F, there exists a $\tilde{\delta}$ in D such that $\sum\limits_{y \in F} |g_{\tilde{\delta}}(y)| > 1$, which contradicts that $g_{\tilde{\delta}}$ belongs to X.
□

Definition of an "unpleasant" morphism $f : X \to \hat{O}\mathbb{C}$.

Observe that $\mathcal{F}(Y)$ is a directed set for the ordering by inclusion, and consider the net $(\sum\limits_{y \in F} g(y))_{F \in \mathcal{F}(Y)}$ where $g \in O\mathbb{C}^Y$. It is a Cauchy net in $\hat{O}\mathbb{C}$. Indeed, given $\varepsilon > 0$, we choose F_0 in $\mathcal{F}(Y)$ such that $\sum\limits_{y \in F_0} |g(y)| > \sup\limits_{F \in \mathcal{F}(Y)} \sum\limits_{y \in F} |g(y)| - \frac{\varepsilon}{2}$. Then for all F', F'' in $\mathcal{F}(Y)$ such that $F_0 \subseteq F', F''$ (i.e., $F_0 \leq F', F''$ in $\mathcal{F}(Y)$),

$$|\sum_{y \in F'} g(y) - \sum_{y \in F''} g(y)| \leq \sum_{y \in F' \setminus F''} |g(y)| + \sum_{y \in F'' \setminus F'} |g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We define now $f(g)$ as the unique limit of that Cauchy net and thus obtain a function $f : X \to O\mathbb{C}$.

(5.8.2) *The function f is a morphism.*

Proof. Fix $\underline{\alpha} = (\alpha_n)_{n \geq 0}$ in $O\ell_1\mathbb{N}$ and $\underline{g} = (g_n)_{n \geq 0}$ in $X^{\mathbb{N}}$. The net $((\sum\limits_{y \in F} g_n(y))_{n \geq 0})_{F \in \mathcal{F}(Y)}$ in the topological product $O\mathbb{C}^{\mathbb{N}}$ converges to $(f(g_n))_{n \geq 0}$. Since the operation $\underline{\alpha}_{\hat{O}\mathbb{C}} : \hat{O}\mathbb{C}^{\mathbb{N}} \to \hat{O}\mathbb{C}$ is continuous, the net $(\sum\limits_{n=0}^{\infty} \alpha_n (\sum\limits_{y \in F} g_n(y)))_{F \in \mathcal{F}(Y)}$ converges to $\sum\limits_{n=0}^{\infty} \alpha_n f(g_n)$. On the other hand, the latter net can be written $(\sum\limits_{y \in F} (\sum\limits_{n=0}^{\infty} \alpha_n g_n(y)))_{F \in \mathcal{F}(Y)}$, and therefore converges to $f(\sum\limits_{n=0}^{\infty} \alpha_n g_n)$. Hence, $f(\sum\limits_{n=0}^{\infty} \alpha_n g_n) = \sum\limits_{n=0}^{\infty} \alpha_n f(g_n)$. □

(5.8.3) *The morphism f is not continuous.*

Proof. Assume that f were continuous at 0. Then, there exists a $\rho > 0$ and finitely many countable subsets $C_1, \ldots, C_N$ of Y such that, for all g in X with
$\max\{\varphi_{C_1}(g), \ldots, \varphi_{C_N}(g)\} < \rho$ we have $|f(g)| < 1$. We choose a point y_0 in $Y \setminus \bigcup_{n=1}^{N} C_n$ and define an element g in X by setting

$$g(y) = 1 \text{ if } y = y_0, \text{ and } g(y) = 0 \text{ otherwise.}$$

Then $\varphi_{C_n}(g) = 0$ for $n = 1, \ldots, N$ and $f(g) = 1$, which is absurd. □

(5.8.4) *For all compact subsets K of X, the restriction $f|_K$ is continuous.*

Proof. For any g in X and every integer $n \geq 1$, we consider the set

$$A_n(g) = \{y \in Y; |g(y)| \geq \frac{1}{n}\}.$$

Observe that each set $A_n(g)$ is finite. Let K be a compact subset of X and let $(g_\delta)_{\delta \in D}$ be a net in K converging in X to g. For every integer $n \geq 1$, we introduce the set

$$M_n = \{y \in Y; |\frac{1}{2}g_\delta(y) - \frac{1}{2}g(y)| \geq \frac{1}{n} \text{ for some } \delta \in D\}.$$

We claim that each set M_n is finite, too. Assume the contrary, i.e., that M_{n_0} is infinite for some integer $n_0 \geq 1$. Then we define inductively a sequence $(y_n)_{n\geq 0}$ in M_{n_0} and a sequence $(g_n)_{n\geq 0}$ in K in the following way. Suppose that y_m and g_m are defined for each $m < n$ such that y_m belongs to $A_{n_0}(\frac{1}{2}g_m - \frac{1}{2}g)$. Then choose y in $M_{n_0} \setminus \bigcup_{m<n} A_{n_0}(\frac{1}{2}g_m - \frac{1}{2}g)$ and an index δ in D such that $|\frac{1}{2}g_\delta(y) - \frac{1}{2}g(y)| \geq \frac{1}{n_0}$. Set $y_n = y$ and $g_n = g_\delta$. Hence, y_n belongs to $A_{n_0}(\frac{1}{2}g_n - \frac{1}{2}g)$, so that $y_n \neq y_m$ whenever $m < n$. Furthermore, $g_n \neq g_m$ for all $m < n$, because $|\frac{1}{2}g_m(y_n) - \frac{1}{2}g(y_n)| < \frac{1}{n_0}$. Since K is compact, the sequence $(g_n)_{n\geq 0}$ has a cluster point in K, say p. Since $\sum_{n=0}^{\infty} |\frac{1}{4}p(y_n) - \frac{1}{4}g(y_n)| \leq 1$, there is an integer $s \geq 1$ such that $\sum_{n\geq s} |\frac{1}{4}p(y_n) - \frac{1}{4}g(y_n)| < \frac{1}{4n_0}$. We consider the set $E = \{y_{s+k}; k \in \mathbb{N}\}$.

Since E is countable and p a cluster point of the sequence $(g_n)_{n\geq 0}$, there is an integer $t \geq s$ such that $\varphi_E(\frac{1}{4}p - \frac{1}{4}g_t) < \frac{1}{4n_0}$. Hence,

$$|\frac{1}{4}g_t(y_t) - \frac{1}{4}g(y_t)| \leq \varphi_E(\frac{1}{4}g_t - \frac{1}{4}g) \leq \varphi_E(\frac{1}{4}g_t - \frac{1}{4}p) + \varphi_E(\frac{1}{4}p - \frac{1}{4}g) < \frac{1}{2n_0},$$

which is in contradiction to the choice of the sequence $(g_n)_{n\geq 0}$.

We consider now the countable set $C = \bigcup_{n=1}^{\infty} M_n$. Observe that, for any δ in D, $g_\delta(y) = g(y)$ for all y in $Y \setminus C$, so that we have

$$\varphi_C(\frac{1}{2}g_\delta - \frac{1}{2}g) = \sup_{F\in\mathcal{F}(Y)} \sum_{y\in F} |\frac{1}{2}g_\delta(y) - \frac{1}{2}g(y)|.$$

The net $(\sum_{y\in F}(\frac{1}{2}g_\delta(y) - \frac{1}{2}g(y)))_{F\in\mathcal{F}(Y)}$ converges to $f(\frac{1}{2}g_\delta - \frac{1}{2}g)$, and therefore, $(|\sum_{y\in F}(\frac{1}{2}g_\delta(y) - \frac{1}{2}g(y))|)_{F\in\mathcal{F}(Y)}$ converges to $|f(\frac{1}{2}g_\delta - \frac{1}{2}g)|$. Hence,

$$|\frac{1}{2}f(g_\delta) - \frac{1}{2}f(g)| \leq \sup_{F\in\mathcal{F}(Y)} \sum_{y\in F} |\frac{1}{2}g_\delta(y) - \frac{1}{2}g(y)| = \varphi_C(\frac{1}{2}g_\delta - \frac{1}{2}g).$$

Since $\varphi_C(\frac{1}{2}g_\delta - \frac{1}{2}g)$ converges to 0, we conclude that $f(g_\delta)$ converges to $f(g)$. Thus, the restriction $f|_K$ is continuous. □

Summing up, Example 1 provides us with a complete TTCS X having a dual space X' which by Theorem (5.3) is not complete. Besides, the space X is not cocomplete (by Proposition (4.10) or by Theorem (4.6)).

Problem. Is X'' cocomplete for the space X of Example 1?

Example 2. *A complete TTCS which is not reflexive.*

We regard the same totally convex space X as in Example 1 but endowed with another topology. Observe that, for every subset C of Y, the function, given by $\varphi_C(g) = \sup_{y\in C} |g(y)|$ for all g in X, is a semi-norm on X. We choose the semi-norms φ_C, where C is a countable subset of Y, as generating family for the topology on X.

(5.9.1) *The TTCS X is complete.*

Proof. Clearly, the topology on X is Hausdorff. Let $(g_\delta)_{\delta \in D}$ be a Cauchy net in X. Then for any point y in Y, $(g_\delta(y))_{\delta \in D}$ is a Cauchy net in $O\mathbb{C}$ and thus converges to some element $g(y)$ in $O\mathbb{C}$. It follows that the net $(g_\delta)_{\delta \in D}$ converges to the function $y \mapsto g(y)$, so that it remains to verify that the latter belongs to X. □

(5.9.2) *The dual space X' is complete.*

Proof. We shall use Theorem (5.3). Let $f \in X^*$. For every point y_0 in Y, let g_{y_0} be the element of X given by

$$g_{y_0}(y) = 1 \text{ if } y = y_0, \text{ and } g_{y_0}(y) = 0 \text{ otherwise.}$$

We consider the set

$$C = \{y \in Y; f(g_y) \neq 0\}$$

and claim that it is countable. Assume the contrary. Then there exists an integer $n_0 \geq 1$ and an injective sequence $(y_n)_{n \geq 1}$ in Y such that one of the following four conditions hold: For all $n \geq 1$, we distinguish 1) $Re(f(g_{y_n})) > \frac{1}{n_0}$, 2) $Im(f(g_{y_n})) > \frac{1}{n_0}$, 3) $Re(f(g_{y_n})) < -\frac{1}{n_0}$, 4) $Im(f(g_{y_n})) < -\frac{1}{n_0}$. We consider only the first case, the treatment of the other cases is similar.

For every integer $m \geq 1$, set $g_m = \frac{1}{m} \sum_{n=1}^{m} g_{y_n}$. Then $(g_m)_{m \geq 1}$ is a sequence in X converging to 0. Hence, the set $K = \{g_m; m \geq 1\} \cup \{0\}$ is compact, so that $f|_K$ is continuous. Therefore, $f(g_m) = \frac{1}{m} \sum_{n=1}^{m} f(g_{y_n})$ converges to 0, in contradiction to $Re(f(g_m)) = \frac{1}{m} \sum_{n=1}^{m} Re(f(g_{y_n})) > \frac{1}{n_0}$ for all $m \geq 1$. We conclude that the set C is countable.

Now, we show that f is continuous. Assume the contrary: There is a net $(g_\delta)_{\delta \in D}$ in X such that (g_δ) converges to 0 but $f(g_\delta)$ does not. Hence, there is an $\varepsilon > 0$ such that, for all δ in D, there exists a δ' in D

with $\delta' \geq \delta$ and $|f(g_{\delta'})| \geq \varepsilon$. We define a sequence $(\delta_n)_{n\geq 1}$ as follows. For every integer $n \geq 1$, there exists a δ_* in D such that $\varphi_C(g_{\delta'}) < \frac{1}{n}$ whenever $\delta' \geq \delta_*$ in D. We choose δ'' in D such that $\delta'' \geq \delta_*$ and $|f(g_{\delta''})| \geq \varepsilon$. Then we set $\delta_n = \delta''$. Thus, $\varphi_C(g_{\delta_n}) < \frac{1}{n}$ and $|f(g_{\delta_n})| \geq \varepsilon$ for all $n \geq 1$.

For every g in X, we introduce the following elements g^C and $g^{\tilde{C}}$ of X:

$$g^C(y) = \begin{cases} g(y) & \text{if } y \in C, \\ 0 & \text{otherwise.} \end{cases} \qquad g^{\tilde{C}}(y) = \begin{cases} g(y) & \text{if } y \in Y \setminus C, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $g^{\tilde{C}} = \sum_{y \in Y\setminus C} g(y)g_y$ in X. Since f is a morphism and since $f(g_y) = 0$ whenever $y \in Y\setminus C$, we have $f(g^{\tilde{C}}) = 0$. Since $\frac{1}{2}g = \frac{1}{2}g^C + \frac{1}{2}g^{\tilde{C}}$, we deduce that $f(g) = f(g^C)$. The sequence $(g^C_{\delta_n})_{n\geq 1}$ converges to 0, because $\varphi_C(g^C_{\delta_n}) < \frac{1}{n}$ for all $n \geq 1$. Hence, the set $K = \{g^C_{\delta_n}; n \geq 1\} \cup \{0\}$ is compact, so that $f|_K$ is continuous. Therefore, $f(g^C_{\delta_n})$ converges to 0, in contradiction to $|f(g_{\delta_n})| \geq \varepsilon$ for all $n \geq 1$. We conclude that f is continuous. □

(5.9.3) *The semi-norm φ_Y, given by $\varphi_Y(g) = \sup_{y \in Y} |g(y)|$ for all g in X, is not continuous.*

(5.9.4) *For all compact subsets K of X, the restriction $\varphi_Y|_K$ is continuous.*

The proofs of (5.9.3) and (5.9.4) use the same arguments as those of (5.8.3) and (5.8.4), respectively, and are therefore left to the reader.

Remark. Since locally convex topologies generated by a single semi-norm are cocomplete, it is clear by (5.9.4) that the space X^κ is obtained by equipping the TCS X with the semi-norm φ_Y. Thus X^κ is a closed subspace of a Banach ball. Moreover, the space X^κ is isomorphic to $X^{*\prime} = X''$.

(5.9.5) *The TTCS X is not cocomplete.*

Indeed, the validity of (5.9.3) and (5.9.4) forbids X to be cocomplete (cf. Theorem (4.6)).

(5.9.6) *The TTCS X is not reflexive.*

Proof. The dual space X' is complete (5.9.2). By Proposition (4.9) the bidual space X'' is therefore cocomplete. If X were reflexive, then it would be cocomplete, which contradicts (5.9.5). □

Summing up, Example 2 provides us with a complete TTCS X which is not reflexive and in particular not cocomplete. On the other hand, X' is complete by Theorem (5.3). That shows that the validity of that property is not a sufficient condition for the bicompleteness of a complete TTCS.

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