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REINHARD BÖRGER
RALF KEMPER

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THERE IS NO COGENERATOR FOR TOTALLY CONVEX SPACES

by Reinhard BÖRGER & Ralf KEMPER

Dedicated to the memory of Jan Reiterman

Résumé. Nous démontrons que la catégorie des espaces totalement convexes ne possède pas un cogénérateur.

Totally convex spaces were introduced by Pumplün and Röhrl [5] (cf. also [6]) as the Eilenberg-Moore-algebras induced by the unit ball functor $\bigcirc$: $\mathbf{Ban}_1 \longrightarrow \mathbf{Set}$ and its left adjoint l_1 : $\mathbf{Set} \longrightarrow \mathbf{Ban}_1$, where $\mathbf{Ban}_1$ is the category of Banach spaces and linear operators of norm ≤ 1. Pumplün and Röhrl characterized a totally convex space as a non-empty set X together with a map $X^{\mathbb{N}} \longrightarrow X, (x_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} \alpha_n x_n$ for all $(\alpha_n)_{n\in\mathbb{N}} \in \Omega := \{(\alpha_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}} \mid \sum_{n\in\mathbb{N}} |\alpha_n| \leq 1\}$, (where $K \in \{\mathbb{R},\mathbb{C}\}$) subject to the following two axioms:

(TC1) $\sum_{n\in\mathbb{N}} \delta_{nm} x_n = x_m$ for all $m \in \mathbb{N}$, $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ and δ the Kronecker symbol.

(TC2) $\sum_{n\in\mathbb{N}} \alpha_n(\sum_{m\in\mathbb{N}} \beta_{nm} x_m) = \sum_{m\in\mathbb{N}}(\sum_{n\in\mathbb{N}} \alpha_n\beta_{nm})x_m$ whenever

$(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}, (\alpha_n)_{n\in\mathbb{N}} \in \Omega$, and $(\beta_{nm})_{m\in\mathbb{N}} \in \Omega$ for all $n \in \mathbb{N}$.

Note that in (TC2) the right-hand side makes sense because $(\sum_{n\in\mathbb{N}} \alpha_n\beta_{nm})_{m\in\mathbb{N}} \in \Omega$. **TC** denotes the category of totally convex spaces, where morphisms are maps preserving the above operations.
Later Pumplün ([3], [4]) introduced the category **PC** of positively convex spaces and the category **SC** of superconvex spaces. A positively convex space is a non-empty set X together with operations $(x_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} \alpha_n x_n$ for all $(\alpha_n)_{n\in\mathbb{N}} \in \Omega^+ := \{(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid \forall_n\ \alpha_n \geq 0 \text{ and } \sum_{n\in\mathbb{N}} \alpha_n \leq 1\}$, where X satisfies (TC1) and the

restriction of (TC2) to Ω^+. A superconvex space is defined similarly by restricting the operations and axioms to $\Omega_{SC} := \{(\alpha_n)_{n\in\mathbb{N}} \in \Omega^+ \mid \sum_{n\in\mathbb{N}} \alpha_n = 1\}$ and allowing $X = \emptyset$. For totally convex or positively convex spaces the empty space can be excluded because of the nullary operation corresponding to $(0)_{n\in\mathbb{N}} \in \Omega^+\backslash\Omega_{SC} \subset \Omega$. It has been an open problem whether the category **TC** (**PC**, **SC** resp.) has a cogenerator, i.e. a (small !) set $\mathcal{C}$ of objects such that for all pairs of distinct morphisms $f, g : D' \longrightarrow D$ there is a morphism $h : D \longrightarrow C$ with $C \in \mathcal{C}$ and $hf \neq hg$. Obviously this is equivalent to saying that for all $D \in |\mathbf{TC}|$ ($|\mathbf{PC}|$, $|\mathbf{SC}|$ resp.), $d_0, d_1 \in D$ with $d_0 \neq d_1$ there is a morphism $h : D \longrightarrow C$ with $C \in \mathcal{C}$ and $h(d_0) \neq h(d_1)$. In [2] we showed that the "finitary versions" of **TC** and **SC** (i.e. the categories obtained by restriction to finitary operations) have cogenerators. Here we give negative answers for the infinitary cases. □

$\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$ (where $\mathbb{R}_+ := \{x \in \mathbb{R} | x \geq 0\}$ is a positively convex space in the usual way (with $0 \cdot \infty := 0$). For any set J, a congruence relation $\sim$ can be defined on the cartesian power $\overline{\mathbb{R}}_+^J$ by:

$$x \sim y :\Longleftrightarrow x = y \text{ or } \exists\ k, j \in J \quad x(k) = y(j) = \infty.$$

(We consider the elements of $\overline{\mathbb{R}}_+^J$ as maps from J to $\overline{\mathbb{R}}_+$). Then $S_J := \overline{\mathbb{R}}_+^J\backslash\sim$ can be identified with $\mathbb{R}_+^J \cup \{\infty\}$ in the canonical way, and we denote the constant map $J \longrightarrow \mathbb{R}_+$ with value 1 by $u \in S_J$.

Lemma: *Let J be an infinite set, C a positively convex space of cardinality $\sharp C < \sharp J$ and $f : S_J \longrightarrow C$ a morphism of positively convex spaces. Then $f(u) = f(\infty)$.*

PROOF: For $k \in J$, define $e_k \in \mathbb{R}_+^J \subset S_J$ by $e_k(k) := k$ and $e_k(j) = 0$ for $j \neq k$. Since $\sharp C < \sharp J$, there are a $c \in C$ and a sequence of distinct elements $k_n \in J$ with $f(e_{k_n}) = c$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, define $x_n \in \mathbb{R}_+^J \subset S_J$ by $x_n(k_n) := 2^{n+1}$ and $x_n(j) := 0$ for $j \neq k_n$. Then in S_J we have $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} x_n = v$ with $v(k_n) = 1$ for all $n \in \mathbb{N}$ and $v(j) = 0$ for $j \notin \{k_n | n \in \mathbb{N}\}$. Moreover, for $n \in \mathbb{N}$ define $y_n \in \mathbb{R}_+^J \subset S_J$ by $y_n(k_1) := 2^{n+1}$ and $y_n(j) := 0$ for $j \neq k_1$. Then for every $n \in \mathbb{N}$ we have $\frac{1}{2^{n+1}} x_n = e_{k_n}$ and $\frac{1}{2^{n+1}} y_n = e_{k_1}$, hence $\frac{1}{2^{n+1}} f(x_n) = f(\frac{1}{2^{n+1}} x_n) = f(e_{k_n}) = c = f(e_{k_1}) = \frac{1}{2^{n+1}} f(y_n)$, and from ([1], Theorem 1.1) we get $f(\frac{1}{2} x_n) = \frac{1}{2} f(x_n) = \frac{1}{2} f(y_n) = f(\frac{1}{2} y_n)$. Since $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} y_n = \infty$ we obtain $f(v) = f(\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} x_n) = f(\sum_{n=1}^{\infty} \frac{1}{2^n}(\frac{1}{2} x_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(\frac{1}{2} x_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(\frac{1}{2} y_n) = f(\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} y_n) = f(\infty)$. Now define $w \in \mathbb{R}_+^J \subset S_J$ by $w(j) := 1$ for $j \in \{k_n \mid n \in \mathbb{N}\}$ and $w(j) := 2$ otherwise. Then we get $\frac{1}{2} v + \frac{1}{2} w = u$ and $\frac{1}{2}\infty + \frac{1}{2} w = \infty$, hence $f(u) = \frac{1}{2} f(v) + \frac{1}{2} f(w) = \frac{1}{2} f(\infty) + \frac{1}{2} f(w) = f(\infty)$. □

Theorem: *None of the categories* **TC**, **PC**, **SC** *has a cogenerator.*

PROOF: If $\mathcal{C}$ were a cogenerator of **PC**, then there would be an infinite set I with $\sharp I > \sharp C$ for all $C \in \mathcal{C}$. Hence by the Lemma there could not be a **PC**-morphism $f : S_I \longrightarrow C$ with $C \in \mathcal{C}$ and $f(u) \neq f(\infty)$, disproving the cogenerator property.
Now assume that $\hat{\mathcal{C}}$ is a cogenerator of **SC**. For every $C \in \hat{\mathcal{C}}$, $c \in C$ there is a unique positively convex structure on C inducing the original superconvex structure and having c as zero element (cf. [2], 1.2). Moreover, a map between positively convex spaces is a **PC**-morphism if and only if it is an **SC**-morphism preserving the zero element. Let $\mathcal{C}$ be the set of all positively convex spaces whose underlying superconvex space belongs to $\hat{\mathcal{C}}$. Then for every $D \in |\mathbf{PC}|$, $x, y \in D$, $x \neq y$ there exist $C \in \hat{\mathcal{C}}$ and an **SC**-morphism $f : D \longrightarrow C$ with $f(x) \neq f(y)$, and f even becomes a **PC**-morphism if C is equipped with the positively convex structure extending the given superconvex structure and having $f(0)$ as zero element (where 0 is the zero element of D). Thus $\mathcal{C}$ is a cogenerator of **PC**, contradicting our previously proven result.

Finally, assume that **TC** has a cogenerator $\tilde{\mathcal{C}}$. We claim that the underlying superconvex spaces of the elements of $\tilde{\mathcal{C}}$ form a cogenerator $\hat{\mathcal{C}}$ of **SC**, contradicting our previous result. For $D \in |\mathbf{SC}|$ fixed, define $\overline{D} \in |\mathbf{TC}|$ in the following way: the underlying set of $\overline{D}$ is $(U \times D) \,\dot{\cup}\, \{0\}$, where $U := \{\gamma \in K \mid |\gamma| = 1\}$ and $K \in \{\mathbb{R}, \mathbb{C}\}$ is the base field. For $(\alpha_n)_{n \in \mathbb{N}} \in \Omega$, $(x_n)_{n \in \mathbb{N}} \in \overline{D}^{\mathbb{N}}$, $I := \{n \in \mathbb{N} \mid x_n \neq 0\}$, $(\gamma_n, a_n) := x_n$ for $n \in I$ (where $\gamma_n \in U$, $a_n \in D$ for $n \in I$) define

$$\sum_{n \in \mathbb{N}} \alpha_n x_n := \begin{cases} \left(\sum_{n \in I} \alpha_n \gamma_n, \sum_{n \in I} |\alpha_n| a_n\right), & \text{if } \sum_{n \in I} \alpha_n \gamma_n \in U \\ 0 & \text{otherwise.} \end{cases}$$

Note that this definition makes sense, because in the first case we have $1 = |\sum_{n \in I} \alpha_n \gamma_n| \leq \sum_{n \in I} |\alpha_n| \leq 1$, hence $\sum_{n \in I} |\alpha_n| = 1$. For $I = \emptyset$ we obviously have $\sum_{n \in I} \alpha_n \gamma_n = 0 \notin U$, hence $\sum_{n \in I} \alpha_n x_n = 0$. Now it is readily checked that $\overline{D} \in |\mathbf{TC}|$, and $s : D \longrightarrow \overline{D}$, $a \mapsto (1, a)$ for all $a \in D$ is an **SC**-morphism. For all $a, b \in D$ with $a \neq b$ we have $s(a) \neq s(b)$, and by hypothesis there is a **TC**-morphism $f : \overline{D} \longrightarrow C$ with $C \in \tilde{\mathcal{C}}$, $f\, s(a) \neq f\, s(b)$. But then $f\, s$ is a **TC**-morphism, proving that $\hat{\mathcal{C}}$ is a cogenerator of **SC**. □

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Reinhard Börger, Ralf Kemper
Fachbereich Mathematik
Fernuniversität–GHS
D-58084 Hagen
Germany