CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

H. HU
M. MAKKAI

Accessible embeddings and the solution-set condition

Cahiers de topologie et géométrie différentielle catégoriques, tome 35, nº 2 (1994), p. 99-108
<http://www.numdam.org/item?id=CTGDC_1994__35_2_99_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XXXV (1994)

ACCESSIBLE EMBEDDINGS AND THE SOLUTION-SET CONDITION

by H. HU and M. MAKKAI

Dedicated to the memory of Jan Reiterman

Résumé. Pour chaque catégorie localement présentable, on démontre qu'une sous-catégorie accessiblement plongée d'une catégorie localement présentable est accessible si et seulement si elle satisfait la condition de l'ensemble-solution.

Introduction

The connection between accessibility and the solution set condition has been pointed out in several papers, including [7] and [9]. J. Adámek and J. Rosický have recently proved that each locally presentable category has the following property: a full subcategory closed under products and κ-filtered limits (for some infinite regular cardinal κ) is accessible if the embedding satisfies the solution set condition, see [3]. It is well-known that a category is locally presentable iff it is accessible and complete, or equivalently cocomplete (cf. [7]), but no assumption is made of the existence of any particular limits for accessible categories. One is impelled to ask: does the above mentioned result hold for any full subcategory closed under κ-filtered colimits? In this paper we provide an affirmative answer to this question. We will show that, for any locally presentable category $\mathbf{B}$, a full subcategory $\mathbf{A}$ of $\mathbf{B}$ closed under κ-filtered colimits is accessible iff the embedding satisfies the solution set condition. The proof of this result is a modification of the proof of the above mentioned result of J. Adámek and J. Rosický in [3].

1 Generalities on Accessible Categories

Let κ be an infinite regular cardinal. Recall that a category $\mathbf{A}$ is κ-filtered if for any graph $\mathbf{G}$ of cardinality less than κ, any diagram $D : \mathbf{G} \longrightarrow \mathbf{A}$ has a cocone on it. $\mathbf{A}$ has κ-filtered colimits, if $\mathbf{A}$ has colimits of all diagrams whose domain is a κ-filtered category. Another concept is limit of κ-diagram, where a κ-diagram is a diagram whose domain category is of size less than κ.

An object A of a category $\mathbf{A}$ is said to be κ-presentable if the representable functor $\mathbf{A}(A, -)$ preserves κ-filtered colimits existing in $\mathbf{A}$. The full subcategory of $\mathbf{A}$ whose objects are the κ-presentable ones is denoted by $\mathbf{A}_\kappa$. Note that for any category, a κ-colimit of a diagram in which the objects are κ-presentable is κ-presentable itself (see [5] and [7]).

Definition 1.1 *([7]) A category* $\mathbf{A}$ *is* κ*-accessible if:*

(i) $\mathbf{A}$ *has* κ*-filtered colimits;*

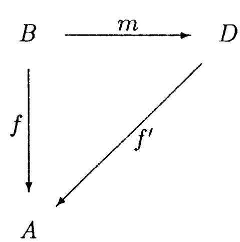
(ii) There is a small full subcategory $\mathbf{C}$ *of* $\mathbf{A}_\kappa$ *such that every object of* $\mathbf{A}$ *is a* κ*-filtered colimit of a diagram of objects in* $\mathbf{C}$.

A category is accessible if it is κ*-accessible for some infinite regular cardinal* κ.

Recall from [4] that a category is called locally κ-presentable if it is locally small, cocomplete, and has a small strong generator consisting of κ-presentable objects. A theorem (Theorem 6.1.4. in [7]) says that an accessible category is complete iff it is cocomplete. P. Gabriel and F. Ulmer have shown in [4] that a category $\mathbf{B}$ is locally κ-presentable if it is equivalent to the category of the form $L_\kappa(\mathbf{C}, \mathbf{Set})$, the category of all functors preserving κ-limits; here $\mathbf{C}$ is a small category with κ-limits, and $\mathbf{Set}$ denotes the category of small sets.

Let $\mathbf{B}$ be a κ-accessible category, and $\mathbf{A}$ a full subcategory of $\mathbf{B}$. $\mathbf{A}$ is said to be κ-accessibly embedded if $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$. $\mathbf{A}$ is accessibly embedded if it is λ-accessibly embedded for some regular cardinal λ with $\lambda \geq \kappa$.

Recall from [3] that a full subcategory $\mathbf{A}$ of $\mathbf{B}$ is said to be cone-reflective if the inclusion functor $\mathbf{A} \to \mathbf{B}$ satisfies the solution set condition, i.e., for each object B of $\mathbf{B}$ there exists a small cone $< r_i : B \to A_i >_{i \in I}$ with $A_i \in \mathbf{A}$ such that for any $A \in \mathbf{A}$, every morphism $B \to A$ factors through some r_i. Let $\mathbf{D}$ be a set of objects of $\mathbf{A}$. We say that $\mathbf{D}$ weakly reflects B (in $\mathbf{A}$) if for every $A \in \mathbf{A}$ and $f : B \to A$ there is $D \in \mathbf{D}$ and a factorization



where m and f' are some morphisms. Note that $\mathbf{A}$ is cone-reflective in $\mathbf{B}$ iff for every $B \in \mathbf{B}$, there is a small set $\mathbf{D} \subset \mathbf{A}$ weakly reflecting B. If $\mathbf{B}$ is accessible, then this is equivalent to saying that for every $B \in \mathbf{B}$ there is κ such that $\mathbf{D}_\kappa = \mathbf{A} \bigcap \mathbf{B}_\kappa$ weakly reflects B. Note that, of course, if $\kappa < \kappa'$ and $\mathbf{D}_\kappa$ weakly reflects B, so does $\mathbf{D}_{\kappa'}$.

The following lemma can be found in [7] (Lemma 1.1.2.).

Lemma 1.2 *Suppose that* $\mathbf{J}$ *is* κ*-filtered and the functor* $F : \mathbf{I} \to \mathbf{J}$ *satisfies that for every* $J \in \mathbf{J}$*, there exists* I *in* $\mathbf{I}$ *and a morphism* $J \to F(I)$*. If* F *is full and faithful, then* $\mathbf{I}$ *is* κ*-filtered and* F *is final, i.e., for any diagram* $\Sigma : \mathbf{J} \to \mathbf{A}$*,* $colim\Sigma$ *exists if and only if* $colim\Sigma \circ F$ *exists and the canonical morphism* $colim\Sigma(F) \to colim\Sigma$ *is an isomorphism.*

Proposition 1.3 *Let* $\mathbf{B}$ *be a* κ*-accessible category, and* $\mathbf{A}$ *a* κ*-accessibly embedded subcategory of* $\mathbf{B}$*. If every* $B \in \mathbf{B}_\kappa$ *is weakly reflected in* $\mathbf{A}$ *by* $\mathbf{D} = \mathbf{A} \bigcap \mathbf{B}_\kappa$*, then* $\mathbf{A}$ *is* κ*-accessible.*

Proof: $\mathbf{A}$ has κ-filtered colimits, by assumption. For any $A \in \mathbf{A}$, we have a canonical diagram $G : \mathbf{B}_\kappa/A \to \mathbf{B}$, and $A = colimG$, by Proposition 2.1.5. in [7]. This colimit is κ-filtered. Let $\mathbf{D} = \mathbf{A} \bigcap \mathbf{B}_\kappa$. Since $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$, all objects in $\mathbf{D}$ are κ-presentable in $\mathbf{A}$. We have a full and faithful functor $F : \mathbf{D}/A \to \mathbf{B}_\kappa/A$. Let $G' : \mathbf{D}/A \to \mathbf{B}$ be the canonical diagram. Given an object $f : B \to A$ in $\mathbf{B}_\kappa/A$, by assumption, there is a factorization $f = f' \circ m$, with $f' : D \to A$ in $\mathbf{D}$/A. By Lemma 1.2, A is the κ-filtered colimit $colimG'$ in $\mathbf{B}$, and as a consequence, also in $\mathbf{A}$. Thus, $\mathbf{A}$ is κ-accessible.

2 Main Theorem and Some Remarks

In what follows, κ, λ and subscripted variants of them always denote infinite cardinals.

The proof of the following theorem follows closely the lines of the corresponding proof in [3].

Theorem 2.1 *Let* $\mathbf{B}$ *be a locally presentable category, and* $\mathbf{A}$ *an accessibly embedded subcategory of* $\mathbf{B}$*. If* $\mathbf{A}$ *is cone-reflective, then it is accessible.*

Proof: We may assume that $\mathbf{B}$ is a functor category ($\mathbf{C}$, $\mathbf{Set}$), for some small category $\mathbf{C}$. The reason is that every locally presentable category is a reflective subcategory of a functor category $(\mathbf{C}, \mathbf{Set})$ with $\mathbf{C}$ small, and the inclusion functor is accessibly embedded. If λ is a regular cardinal bigger than the cardinality of $\mathbf{C}$ and $\aleph_0$, then a functor $F \in \mathbf{B}$ is λ-presentable in $\mathbf{B}$ iff the cardinality of $\coprod_{C \in \mathbf{C}} F(C)$ is less than λ. It easily follows that if $\mu = sup_{i<\nu}\kappa_i$ with $\kappa_i \leq \kappa_j$ for $i < j < \nu$, and $B \in \mathbf{B}_{\mu^+}$, then we can write B as a colimit of a ν-chain, $B = colim_{i<\nu} B_i$, with $B_i \in \mathbf{B}_{\kappa_i}$.

Let κ be a regular cardinal such that $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$. Let us define κ_i for $i < \kappa$ by transfinite induction. Let $\kappa_0 = \kappa$. Given $0 < i < \kappa$, having defined κ_j for all $j < i$, for i limit, let κ_i be a regular cardinal bigger than κ_j

for all $j < i$, and for $i = j+1 < \kappa$, let κ_{j+1} be a regular cardinal $\geq \kappa_j$ such that all objects in $\mathbf{B}_{\kappa_j}$ have a weak reflection in $\mathbf{D}_{\kappa_{j+1}}$; since each $\mathbf{B}_{\kappa'}$ is small, and since every $B \in \mathbf{B}$ has a weak reflection in $\mathbf{D}_{\kappa'}$, for some κ', such κ_{j+1} clearly exists.

Let $\mu = sup_{i<\kappa}\kappa_i$, and $\lambda = \mu^+$. We claim that every $B \in \mathbf{B}_\lambda$ has a weak reflection in $\mathbf{D}_\lambda$. Since $\mathbf{B}$ is clearly κ'-accessible for all $\kappa' \geq \aleph_0$, in particular, for $\lambda = \kappa'$, and $\lambda > \kappa$, by Proposition 1.3, this will suffice for the proof of the theorem.

Let $B \in \mathbf{B}_\lambda$. As above, let us represent B as the colimit of a κ-chain $(b_{i,j} : B_i \rightarrow B_j)_{i<j<\kappa}$, with $B_i \in \mathbf{B}_{\kappa_i}$. Let $\phi_i : B_i \rightarrow B$ be the colimit coprojection. Let $A \in \mathbf{A}$ and $f : B \rightarrow A$ be arbitrary; we want to find $A^* \in \mathbf{D}_\lambda$ with a factorization

$$\begin{array}{ccc} B & \xrightarrow{f^*} & A^* \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle a^*} & \\ A & & \end{array}$$

By transfinite induction on $i < \kappa$, we will define objects $A_i \in \mathbf{D}_{\kappa_{i+1}}$ and morphisms $a_{i,j} : A_i \rightarrow A_j$, $f_i : B_i \rightarrow A_i$, $\psi_i : A_i \rightarrow A$ such that the morphisms $a_{i,j}$ form a chain, the morphisms ψ_i form a compatible cocone, and the following diagram

$$\begin{array}{ccccc} B_i & \xrightarrow{b_{i,j}} & B_j & \xrightarrow{\phi_j} & B \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} & & \downarrow{\scriptstyle f} \\ A_i & \xrightarrow[a_{i,j}]{} & A_j & \xrightarrow[\psi_i]{} & A \end{array}$$

commutes.

For $i = 0$, we let $A_0 \in \mathbf{D}_{\kappa_1}$, $f_0 : B_0 \rightarrow A_0$ and $\psi_0 : A_0 \rightarrow A$ such that

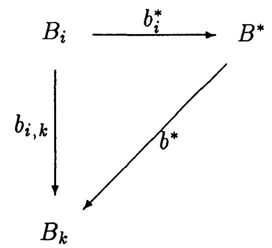
$$\begin{array}{ccc} B_0 & \xrightarrow{\phi_0} & B \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ A_0 & \xrightarrow[\psi_0]{} & A \end{array}$$

commutes; these items are obtained from a suitable factorization of the morphism $f \circ \phi_0 : B_0 \longrightarrow A$, possible by the choice of κ_1 and $B_0 \in \mathbf{D}_{\kappa_0}$.

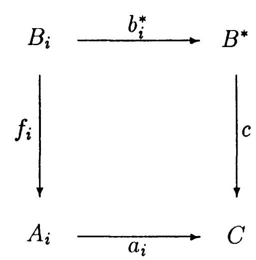
Fix k, $0 < k < \kappa$, and assume that all items with indices $< k$ have been defined. Let $C = colim(a_{i,j} : A_i \longrightarrow A_j)_{i<j<k}$ with coprojections $a_i : A_i \longrightarrow C$, and $B^* = colim(b_{i,j} : B_i \longrightarrow B_j)_{i<j<k}$ with coprojections $b_i^* : B_i \longrightarrow B^*$.

Since $A_i \in \mathbf{B}_{\kappa_i} \subset \mathbf{B}_{\kappa_k}$, and $\mathbf{B}_{\kappa_k}$ is closed under $< \kappa \leq \kappa_k$-sized colimits, $C \in \mathbf{B}_{\kappa_k}$. Similarly, $B_i \in \mathbf{B}_{\kappa_{i+1}} \subset \mathbf{B}_{\kappa_k}$, and so $B^* \in \mathbf{B}_{\kappa_k}$.

By the universal property of B^* , we have $b^* : B^* \longrightarrow B_k$ such that

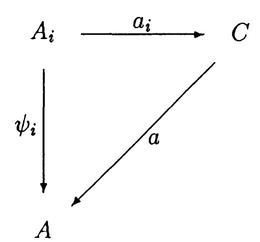


commute, and $c : B^* \longrightarrow C$ such that

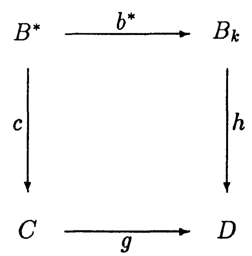


commute, for all $i < k$.

By the universal property of C, we have $a : C \longrightarrow A$ such that



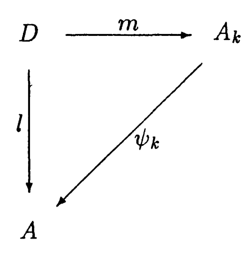
commute for all $i < k$. We form the pushout of c and b^*:



Since $B^*, B_k, C \in \mathbf{B}_{\kappa_k}$, we have $D \in \mathbf{B}_{\kappa_k}$. We prove that $a \circ c = f \circ \phi_k \circ b^*$ by showing that

$$\begin{aligned} a \circ c \circ b_i^* &= a \circ a_i \circ f_i \\ &= \psi_i \circ f_i \\ &= f \circ \phi_i \\ &= f \circ \phi_k \circ b^* \circ b_i^* \end{aligned}$$

for each projection b_i^*. By using the universal property of pushout D, we have a unique morphism $l : D \to A$ such that $a = l \circ g$ and $f \circ \phi_k = l \circ h$. Since $D \in \mathbf{B}_{\kappa_k}$, and every object in $\mathbf{B}_{\kappa_k}$ is weakly reflected by $\mathbf{D}_{\kappa_{k+1}}$, there is $A_k \in \mathbf{D}_{\kappa_{k+1}}$ with $\psi_k : A_k \to A$ such that the diagram



commutes. We have defined the items A_k and ψ_k.

Next, we define $f_k = m \circ h : B_k \to A_k$ and $a_{i,k} = m \circ g \circ a_i : A_i \to A_k$. Note that the diagrams

$$\begin{array}{ccccc} B_i & \xrightarrow{b_i^*} & B^* & \xrightarrow{b^*} & B_k \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle h} \\ A_i & \xrightarrow[a_i]{} & C & \xrightarrow[g]{} & D \end{array}$$

commute for all $i < k$; and $b_{i,k} = b^* \circ b_i^*$. Then the diagrams

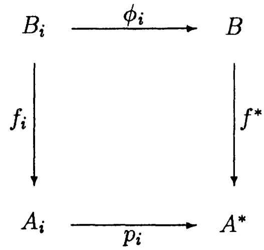
$$\begin{array}{ccc} B_i & \xrightarrow{b_{i,k}} & B_k \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_k} \\ A_i & \xrightarrow[a_{i,k}]{} & A_k \end{array}$$

commute for all $i < k$, and the diagram

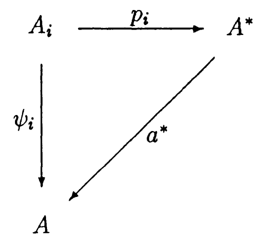
$$\begin{array}{ccc} B_k & \xrightarrow{\phi_k} & B \\ {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle f} \\ A_k & \xrightarrow[\psi_k]{} & A \end{array}$$

commutes. It is clear that $\psi_i = a_{i,k} \circ \psi_k$ and $a_{i,k} = a_{i,j} \circ a_{j,k}$ hold for all $i < j < k$. This completes the construction.

Put $A^* = colim(a_{i,j} : A_i \to A_j)_{i<j<\kappa}$ with coprojections $p_i : A_i \to A^*$. Since $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$, $A^* \in \mathbf{A}$. Also, since $A_i \in \mathbf{B}_{\kappa_{i+1}} \subset \mathbf{B}_\lambda$ and $\kappa < \lambda$, we have that $A^* \in \mathbf{B}_\lambda$; that is, $A^* \in \mathbf{D}_\lambda$. By the construction above, we have $f^* : B \to A^*$ such that the diagrams



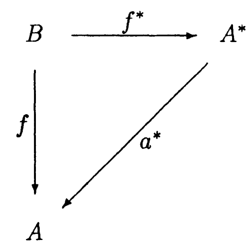
commute for all $i < \kappa$; also, we have $a^* : A^* \to A$ such that the diagrams



commute for all $i < \kappa$; hence we have that

$$f \circ \phi_i = \psi_i \circ f_i = a^* \circ p_i \circ f_i = a^* \circ f^* \circ \phi_i$$

for all $i < \kappa$. Since $< \phi_i >_{i<\kappa}$ is a colimit cocone, we conclude that the diagram



commutes. This completes the proof.

Definition 2.2 *For any locally presentable category* **B**, *an accessibly embedded subcategory* **A** *of* **B** *is accessible iff* **A** *is cone-reflective.*

Proof: By Theorem 2.1 and Proposition 2.1.8. in [7].

Remark 2.3 *Let* **A** *be an accessibly embedded subcategory of* **B** *closed under limits. J. Adámek and J. Rosický have shown in [2] that* **A** *is a reflective subcategory of* **B**. *Thus* **A** *is locally presentable.*

Remark 2.4 *Recall from [1] that Vopěnka's principle is the following statement: the category* **Gra** *of graphs does not have a large discrete full subcategory. It has been shown in [8] that Vopěnka's principle is equivalent to the following statement: every accessibly embedded subcategory of a locally presentable category is accessible. Thus, assuming Vopěnka's principle, every accessible embedding of a locally presentable category satisfies the solution set condition.*

Remark 2.5 *Let* **A** *be an accessible full subcategory of an accessible category* **B**. *Suppose that the inclusion functor from* **A** *to* **B** *satisfies the solution-set condition. J. Rosický and W. Tholen have recently proved that the inclusion functor is accessible (see Theorem 3.10 in [9]). Also, they have proved in [9] that Vopěnka's principle is equivalent to the the following statement: a functor between accessible categories is accessible if and only if it satisfies the solution-set condition.*

References

[1] J.Adámek , J.Rosický and V.Trnková , Are all limit-closed subcategories of locally presentable categories reflective?, Lecture Notes in Math., no.1348, Springer-Verlag,1988, 1-18

[2] J.Adámek and J.Rosický, Reflection in locally presentable categories, Arch. Math. (Brno) 25 (1989), 89-94

[3] J.Adámek and J.Rosický, On injectivity in locally presentable categories, to appear in Trans. Amer. Math. Soc.

[4] P.Gabriel and F.Ulmer, Lokal präsentierbare kategorrien, Lecture Notes in Math., no.221, Springer-Verlag, 1971

[5] H.Hu, Dualities for accessible categories, Canadian Math. Soc. Conference Proceeding, no.13, 1992, 211-242

[6] S.Mac Lane, Categories for the Working Mathematician, Springer-Verlag, 1971

[7] M.Makkai and R.Paré, Accessible categories: the foundations of categorical model theory, Contemporary Math. 104 (1990)

[8] J.Rosický, V.Trnková and J.Adámek, Unexpected properties of locally presentable categories, Algebra Universalis 27(1990), 153-170

[9] J.Rosický and W.Tholen, Accessibility and the solution set condition, to appear in J. Pure Appl. Algebra

HONGDE HU, DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, NORTH YORK, ONT., CANADA, M3J 1P3
E-mail address: hhu@mathstat.yorku.ca

MICHAEL MAKKAI, DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, MONTREAL, PQ, CANADA, H3A 2K6
E-mail address: Makkai@triples.math.mcgill.ca