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ON ANALYTIC MODELS OF SYNTHETIC DIFFERENTIAL GEOMETRY

by Eduardo J. DUBUC and Jorge G. ZILBER

Résumé. Soit $\mathcal{M}$ une catégorie de variétés ou d'espaces analytiques complexes. Les objets de $\mathcal{M}$ se construisent par recollement de certain objets de base, ce qui impose qu'un modèle biem adapté de la Géométrie Différentielle Synthétique (GDS) $i : \mathcal{M} \rightarrow \mathcal{T}$, comme foncteur dans un topos $\mathcal{T}$, doit préserver ces recouvrements. Egalement aussi, pour permettre des applications de caractére local (et non uniquement infinitésimal), en plus de l'axiome original de Représentation de Jets ([**8**],[**2**]), l'axiome plus puissant de Représentation de Germs ([**4**], [**2**]) doit être valide. La validité de cet exiome requiert que dans la modèle, l'object des infinitesimaux de Penon soit un faisceau représentable. Dans le cas réel C^{∞} cela sóbtient au moyen du concept d'idéal de caractère local, introduit dans [**3**] . Cependant cette solution ne s'applique pas dans le cas analytique complexe. Ici nous introduisons le concept de Schème Analytique [définition 1.3] au moyen duquel nous parvenons à la solution de ce problème. Les schèmes analytiques sont des objets géométriques (strictement) plus généraux que les espaces analytiques et sont déterminés par un ouvert en C^n et deux faisceaux cohérents d'idéaux.

Cet article est une suite de [**6**] et de [**12**] où les notions d'anneau analytique et d'anneau analitique local ont été introduites. Ces travaux culminent ici par la construction d'un modèle de la GDS bien adapté àl'etude des variétés et des espaces analytiques complexes.

Introduction.

This article is a sequel of [**6**] and [**12**], where we introduced the notions of analytic and (local) analytic rings, and developed their spectral theory. These works culminate here, where the construction of a model of Synthetic Differential Geometry (S.D.G.) well adapted to the study of complex analytic varieties and spaces is achieved.

Let $\mathcal{M}$ be a category of *algebraic varieties*, or *(complex) analytic* or *(real) differentiable manifolds.* Objects in $\mathcal{M}$ are build up pasting together basic objects. This implies that a *well adapted model* of SDG, $i : \mathcal{M} \rightarrow \mathcal{T}$, as a functor into a topos $\mathcal{T}$ should preserve *all open covers.* This leads to the classical notions of

Zariski Topology and *Local ring* (the Zariski topology works because it consist of all open covers, and it is characteristic of the algebraic case that it is generated by a pretopology of finite covers), as well as to the respective open cover topologies and their related notions of C^∞ *-local ring* ([3], [1]), (not the same that a C^∞-ring which is local in the algebraic sence) and *local analytic ring* ([6], [12]).

Besides the preservation of open covers and the original axiom of *Jet Representability* ([8], [2]), the stronger axiom of *Germ Representability* ([4], [2]) has to hold for the applicability of SDG to results of *local* (and not only *infinitesimal*) character (see for example [7]). The validity of the axiom of germ representability is reflected in the model by the fact that the object of infinitesimals $\Delta = [[x \in \mathbf{Line}^n \mid \neg\neg(x = 0)]]$ should be a representable sheaf. This fact is achieved in the differentiable case by means of the notion of *germ determined ideal* (introduced in [3], the axiom of germ representability proved later in [5]). This solution does not apply to the analytic case.

We introduce here the notion of *Analytic Scheme* (definition 1.3), key to the solution of the problem. Analytic Schemes are (strictly) more general than usual Analytic Spaces (as defined for example in [9]), and as such they have no counterpart in the differentiable case (they are determined by an open set in $\mathbf{C}^n$ and two coherent sheaves of ideals).

In section 1 we consider the notion of analytic model $\mathcal{M} \to \mathcal{T}$ of S.D.G. ($\mathcal{M}$ denotes the category of analytic complex manifolds), and construct a topos $\mathcal{T}$ which is such a model (1.10, 1.11 and 1.12). In sections 2 and 3 we study the intrinsic (Penon) topology in $\mathcal{T}$. In section 4 (theorem 4.5), we prove the axiom of germ representability in $\mathcal{T}$, and in section 5 (theorem 5.5), we prove that it furthermore satisfy (*the postulate of*) Δ *-Infinitesimal integration* [2], [5]*)*.

1 Analytic Schemes

1.1 Theorem. *Let U be an open subset of $\mathbf{C}^n$ and let $\mathcal{J}, \mathcal{L}$ $\mathcal{L} \subset \mathcal{J}$, be any two coherent sheaves of ideals in $\mathcal{O}_U$. Let*

$$E = supp(\mathcal{O}_U/\mathcal{J}) = \{p \in U \mid h(p) = 0 \ \forall h|_p \in \mathcal{J}_p\} \ \ and$$

$$\mathcal{O}_{\mathcal{L}E} = (\mathcal{O}_U/\mathcal{L})|_E = \text{ the restriction of } \mathcal{O}_U/\mathcal{L} \text{ to } E.$$

Then $\mathcal{O}_{\mathcal{L}E}$ is a local analytic ring [12] in Sh_E, the category of sheaves over E, which defines an A-Ringed Space [6] denoted $(E, \mathcal{O}_{\mathcal{L}E})$. Furthermore:

The inclusion $i : E \to U$, together with the quotient $q : \mathcal{O}_{n,p} \to \mathcal{O}_{\mathcal{L}E,p}$ define a morphism of A-Ringed Spaces: $(E, \mathcal{O}_{\mathcal{L}E}) \to (U, \mathcal{O}_U)$ which is characterized by the universal property sketched in the following diagram:

Given any $(X, \mathcal{O}_X)$ *and* $(f, \phi) : (X, \mathcal{O}_X) \to (U, \mathcal{O}_U)$ *such that:*

$$\forall x \in X,\ f(x) \in E \quad and \quad \forall h_{f(x)} \in \mathcal{L}_{f(x)} \quad \phi_x(h_{f(x)}) = 0;$$

we have:

$$\begin{array}{ccc} (X, \mathcal{O}_X) & & \\ {\scriptstyle \exists!(g,\psi)} \downarrow & \searrow {\scriptstyle (f,\phi)} & \\ (i,q) : (E, \mathcal{O}_{\mathcal{L}E}) & \longrightarrow & (U, \mathcal{O}_U). \end{array}$$

Proof. The same proof that the one given in [6 theorem 2.10.] for the particular case $\mathcal{J} = \mathcal{L}$ works also here. □

If $\mathcal{J} = \mathcal{L}$, according to the usual notation we write $(E, \mathcal{O}_{\mathcal{L}E}) = (E, \mathcal{O}_E)$, and in the case $\mathcal{L} = 0$, we write $(E, \mathcal{O}_{0E}) = (E, \mathcal{O}_U)$. From the previous theorem we have $(E, \mathcal{O}_E) \to (E, \mathcal{O}_{\mathcal{L}E}) \to (E, \mathcal{O}_U) \subset (U, \mathcal{O}_U)$. The first two arrows can be considered to be infinitesimal extensions of $(E, \mathcal{O}_E)$ inside $(U, \mathcal{O}_U)$.

1.2 Example. Let $U = \mathbf{C}^n$, $\mathcal{J} = (z_1, z_2, \ldots, z_n)$, and $\mathcal{L} = 0$. Then $(\{0\}, \mathcal{O}_{\mathbf{C}^n})$ is the largest (n-dimensional) infinitesimal extension of the point $\{0\}$. It consist of a singleton, structured (as an A-Ringed Space) with the ring of germs of holomorphic functions on n variables, (a (very) fat point).

1.3 Definition (Affine Analytic Schemes). *Let U be an open subset of $\mathbf{C}^n$ and let $\mathcal{J}, \mathcal{L}$, $\mathcal{L} \subset \mathcal{J}$, be any two coherent sheaves of ideals in $\mathcal{O}_U$. The A-Ringed Space constructed in Theorem 1.1 will be called an (Affine) Analytic Scheme. We will denote by $\mathcal{H}$ the category determined by considering as arrows the morphisms of A-Ringed Spaces.*

1.4 Remark. The category of (local) models (see [9]) considered in [6] and [12], is by definition a full subcategory of $\mathcal{H}$, its objects being those affine analytic schemes for which $\mathcal{J} = \mathcal{L}$.

1.5 For the record (Analytic Schemes). Although we shall not need this concept here, it is clear that an *Analytic Scheme* should be an A-Ringed Space where every point has an open neighborhood such that the corresponding subobject (structured with the restriction sheaf) is isomorphic to an (Affine) Analytic Scheme.

1.6 Remark. Since an object $(E, \mathcal{O}_{\mathcal{L}E}) \in \mathcal{H}$ is a A-Ringed Space, there is a bijection $[(E, \mathcal{O}_{\mathcal{L}E}), (\mathbf{C}, \mathcal{O}_{\mathbf{C}})] \approx \Gamma(E, \mathcal{O}_{\mathcal{L}E})$, where $[(E, \mathcal{O}_{\mathcal{L}E}), (\mathbf{C}, \mathcal{O}_{\mathbf{C}})]$ is the set of arrows in $\mathcal{H}$ $(E, \mathcal{O}_{\mathcal{L}E}) \to (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$ and $\Gamma(E, \mathcal{O}_{\mathcal{L}E})$ is the set of global sections of the sheaf $\mathcal{O}_{\mathcal{L}E}$ (see [6] corollary 2.9).

Hence, if $(f, \phi) : (E, \mathcal{O}_{\mathcal{L}E}) \to (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$, we have that for each $p_0 \in E$, there exists an open neighborhood W of p_0 in $\mathbf{C}^n$, $W \subset U$, and an holomorphic function $g : W \to \mathbf{C}$ such that:

$$\begin{array}{lll} g(p) = f(p) & \forall\, p \in W \cap E & \textit{and} \\ \phi_p(t_{f(p)}) = [(t \circ g)_p]_{\mathcal{L}} & \forall\, p \in W \cap E, \ \forall\, t_{f(p)} \in \mathcal{O}_{1,f(p)} & \end{array}$$

(where by $[\ ]_{\mathcal{L}}$ we indicate class modulo $\mathcal{L}$).
That is, the following diagrams commute (for $p \in W \cap E$) :

$$\begin{array}{ccc} W \cap E & \longrightarrow & W \\ f \searrow & & \swarrow g \\ & \mathbf{C} & \end{array} \qquad\qquad \begin{array}{ccc} g^* : \mathcal{O}_{1,f(p)} & \longrightarrow & \mathcal{O}_{n,p} \\ \phi_p \searrow & & \swarrow \\ & \mathcal{O}_{\mathcal{L}E,p} & \end{array}$$

1.7 Observation. *An arrow $1 \to (E, \mathcal{O}_{\mathcal{L}E})$ in $\mathcal{H}$ is the same thing that a point $p \in E$.*

Proof. In fact, the terminal object 1 in $\mathcal{H}$ is $1 = (\mathbf{C}^0, \mathcal{O}_{\mathbf{C}^0}) \approx (1, \mathbf{C})$. It consist of a singleton, structured (as an A-Ringed Space) with the field of complex numbers (a simple point). If $(E, \mathcal{O}_{\mathcal{L}E})$ is an object in $\mathcal{H}$, to give an arrow $1 \to (E, \mathcal{O}_{\mathcal{L}E})$ is equivalent to fix a point p in E and a morphism of analytic rings $\mathcal{O}_{\mathcal{L}E,p} \to \mathbf{C}$. But since $\mathcal{O}_{\mathcal{L}E,p}$ is a local analytic ring there is a (unique) morphism $\mathcal{O}_{\mathcal{L}E,p} \to \mathbf{C}$ [12 theorem 1.6]. □

1.8 Finite Limits in $\mathcal{H}$. Let U and V be open subsets of $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively, let $\mathcal{J}, \mathcal{L}$ $\mathcal{L} \subset \mathcal{J}$, be coherent sheaves of ideals in $\mathcal{O}_U$ and $\mathcal{J}', \mathcal{L}'$, $\mathcal{L}' \subset \mathcal{J}'$ be coherent sheaves of ideals in $\mathcal{O}_V$. Let $(E, \mathcal{O}_{\mathcal{L}E})$ and $(E', \mathcal{O}_{\mathcal{L}'E'})$ be the objects in $\mathcal{H}$ defined by $\mathcal{J}$, $\mathcal{L}$ and $\mathcal{J}'$, $\mathcal{L}'$ respectively. Then, the product in

$\mathcal{H}$, $(E, \mathcal{O}_{\mathcal{L}E}) \times (E', \mathcal{O}_{\mathcal{L}'E'})$, is given by the coherent sheaves of ideals $(\mathcal{J}, \mathcal{J}')$ and $(\mathcal{L}, \mathcal{L}')$ in $\mathcal{O}_{U \times V}$, $(\mathcal{L}, \mathcal{L}') \subset (\mathcal{J}, \mathcal{J}')$. That is, we have:

$$(E, \mathcal{O}_{\mathcal{L}E}) \times (E', \mathcal{O}_{\mathcal{L}'E'}) = (E \times E', \mathcal{O}_{(\mathcal{L},\mathcal{L}')E \times E'})$$

In fact, it is clear that $supp(\mathcal{O}_U/\mathcal{J}) \times supp(\mathcal{O}_V/\mathcal{J}') = supp(\mathcal{O}_{U \times V}/(\mathcal{J}, \mathcal{J}'))$. The rest is straightforward, it follows essentially from [6 Proposition 1.19].

The construction of equalizers in $\mathcal{H}$ is similar to the case $\mathcal{J} = \mathcal{L}$ considered in [6, section 2]. It follows that $\mathcal{H}$ has all finite limits, and that they are actually limits in the whole category of A-Ringed Spaces. Thus, from [Corollary 2.9] we have:

1.9 Proposition. *The full embedding $\mathcal{C} \to \mathcal{H}$ (from the category of open subsets of $\mathbf{C}^n$ (all n) and holomorphic functions) is an Analytic ring in $\mathcal{H}$. That is, it preserves terminal objects and transversal pullbacks.* □

1.10 Definition-Proposition. *We consider in $\mathcal{H}$ the Grothendieck topology given by the open corerings. It is straightforward to see that this topology is subcanonical. We will denote by $\mathcal{T}$ the topos of sheaves on $\mathcal{H}$ for this topology. There is a full embedding $\mathcal{H} \to \mathcal{T}$. We consider the functor $i : \mathcal{M} \to \mathcal{T}$, (where $\mathcal{M}$ denotes the category of analytic complex manifolds) given by:*

$$iM(E, \mathcal{O}_{\mathcal{L}E}) = [(E, \mathcal{O}_{\mathcal{L}E}), (M, \mathcal{O}_M)], \text{ for } (E, \mathcal{O}_{\mathcal{L}E}) \in \mathcal{H} \text{ and } M \in \mathcal{M}$$

Here $[(E, \mathcal{O}_{\mathcal{L}E}), (M, \mathcal{O}_M)]$ is the set of arrows $(E, \mathcal{O}_{\mathcal{L}E}) \to (M, \mathcal{O}_M)$ in the category of A-Ringed Spaces. Recall that $(M, \mathcal{O}_M)$ is the A-Ringed Space given by the sheaf $\mathcal{O}_M$ of germs of holomorphic functions in M. In general, $(M, \mathcal{O}_M) \notin \mathcal{H}$, but it is an Analytic Scheme in the sense of 1.5 above. This fact, together with the fact that covers are universal in the whole category of A-Ringed Spaces, implies that the functor $i : \mathcal{M} \to \mathcal{T}$ is a full embedding. Furthermore, from 1.9 it follows that it preserves terminal object and transversal pull-backs. □

1.11 Definition. *An analytic Model of S.D.G. is a topos $\mathcal{E}$ together with a full embedding $i : \mathcal{M} \to \mathcal{E}$ which preserves open coverings, terminal object and transversal pullbacks. Furthermore, it is required that $i\mathbf{C}$ be a ring object of line type* [**8**].

Equivalently, an analytic model of S.D.G. is a local analytic ring [**12**] *of line type in a topos $\mathcal{E}$, which, in addition, is full and faithful as a functor.*

A first analytic model of S.D.G. is the classifying topos $\mathcal{L}$ of the theory of local analytic rings (cf [12, 3.10]). However this model is not satisfactory since the ring of germs of holomorphic functions (on $\mathbf{C}^n$) is not of finite presentation, thus a point equipped with this ring as structure sheaf is not a local model of analytic space [9], and therefore it is not in the site of definition of the topos (notably, this is not the case with the topos in 1.10, as it follows from 1.2). A consequence of this is that the axiom of germ representability (see introduction) does not hold in the topos $\mathcal{L}$.

1.12 Corollary. *The functor $i : \mathcal{M} \to \mathcal{T}$ defined in 1.10 is an analytic model of S.D.G.*

Proof. It only remains to check the condition of line type and that it preserves open coverings. This is easily seen by general categorical considerations. □

By properties of the site it is also clear that the following holds:

1.13 Remark. The topos $\mathcal{T}$ satisfies the Nullstellensatz. By this we mean:

$$\text{For any object } F \text{ in } \mathcal{T}, \quad F = \emptyset \Leftrightarrow \Gamma(F) = \emptyset \quad (\Gamma = \text{global sections})$$

□

2. Subobjects of iM

2.1 Proposition. *Let $M \in \mathcal{M}$, let N be an open subset of M, and let $(E, \mathcal{O}_{\mathcal{L}E}) \in \mathcal{H}$. Consider an arrow $f : (E, \mathcal{O}_{\mathcal{L}E}) \to iM$ in $\mathcal{T}$. If for every point $p : 1 \to (E, \mathcal{O}_{\mathcal{L}E})$, there is a point $1 \to iN$ such that the following square commutes:*

$$\begin{array}{ccc} 1 & \longrightarrow & (E, \mathcal{O}_{\mathcal{L}E}) \\ \big\downarrow & \swarrow & \big\downarrow f \\ iN & \longrightarrow & iM \end{array}$$

then f factors through iN as indicated by the diagonal arrow.

Proof. Recall 1.7 and 1.10; p and f as indicated above amount to a point $p \in E$ and an arrow $(f, \phi) : (E, \mathcal{O}_{\mathcal{L}E}) \to (M, \mathcal{O}_M)$ of A-Ringed spaces. Moreover, the arrow $f \circ p : 1 \to iM$ in $\mathcal{T}$ corresponds to the point $f(p) \in M$. The fact that $f \circ p$ factors through iN means that $f(p) \in N (\forall\, p \in E)$. Then, the continuous function $f : E \to M$ factors through a continuous function $g : E \to N \subset M$. Moreover, we have $\phi_p : \mathcal{O}_{M,f(p)} \to \mathcal{O}_{\mathcal{L}E,p}$ and $\mathcal{O}_{M,f(p)} \approx \mathcal{O}_{N,f(p)} = \mathcal{O}_{N,g(p)}$, $\forall\, p \in E$ (since N is open in M and $f(p) \in N \;\; \forall\, p \in E$). Then, we have an arrow $\eta_p : \mathcal{O}_{N,g(p)} \to \mathcal{O}_{\mathcal{L}E,p}, \forall\, p \in E$, and it follows that the diagram:

$$\begin{array}{ccc} (E, \mathcal{O}_{\mathcal{L}E}) & & \\ \Big\downarrow (g,\eta) & \searrow f,\phi) & \\ (N, \mathcal{O}_N) & \longrightarrow & (M, \mathcal{O}_M) \end{array} \qquad \text{commutes.}$$

That is, (f, ϕ) factors through $(N, \mathcal{O}_N)$. In other words, f factors through iN. □

2.2 Corollary. *Let $M \in \mathcal{M}$, let N be an open subset of M, and let $F \subset iM$ be any subobject in $\mathcal{T}$. If $\Gamma(F) \subset N$, then $F \subset iN$*

Proof. Observe that objects in $\mathcal{H}$ generate $\mathcal{T}$, then utilize Proposition 2.1. □

2.3 Theorem. *Given an object $(E, \mathcal{O}_{\mathcal{L}E}) \in \mathcal{H}$ and any pair of subobjects F, G in $\mathcal{T}, F \subset (E, \mathcal{O}_{\mathcal{L}E}), G \subset (E, \mathcal{O}_{\mathcal{L}E})$, we have: $F \cup G = (E, \mathcal{O}_{\mathcal{L}E})$ in $\mathcal{T}$ iff there exist $Y \subset E, Z \subset E$ open such that $Y \subset Z = E$, and $(Y, \mathcal{O}_{\mathcal{L}Y}) \subset F, (Z, \mathcal{O}_{\mathcal{L}Z}) \subset G$.*

Proof. Suppose that $F \cup G = (E, \mathcal{O}_{\mathcal{L}E})$. Then, there exist an open covering:

$$j_\alpha : (E_\alpha, \mathcal{O}_{\mathcal{L}E_\alpha}) \to (E, \mathcal{O}_{\mathcal{L}E}), \quad (\alpha \in I)$$

of $(E, \mathcal{O}_{\mathcal{L}E})$ in $\mathcal{H}$ such that for each $\alpha \in I, j_\alpha$ factors through F or G. Then, if we define $I_1 = \{\alpha \in I \mid j_\alpha$ factors through $F\}$ and $I_2 = \{\alpha \in I \mid j_\alpha$ factors through $G\}$, we have that $I_1 \cup I_2 = I$. Then if we define $Y = \bigcup_{\alpha \in I_1} E_\alpha$ and $Z = \bigcup_{\alpha \in I_2} E_\alpha$, we have that $I_1 \subset I_2 = I$. Then, if we define $Y = \bigcup_{\alpha \in I_1} E_\alpha$ and $Z = \bigcup_{\alpha \in I_2} E_\alpha$, we have that $Y \cup Z = \bigcup_{\alpha \in I} E_\alpha = E$. Since each E_α is open in E, then Y and Z are open in E, and since for each $\alpha \in I_1, j_\alpha$ factors through F, we have $(E_\alpha, \mathcal{O}_{\mathcal{L}E_\alpha}) \subset F$. Hence $(Y, \mathcal{O}_{\mathcal{L}Y}) \subset F$. Similarly, we have $(Z, \mathcal{O}_{\mathcal{L},Z}) \subset G$.

Converseley, suppose that $Y \subset E, Z \subset E$ are open subsets such that $Y \cup Z = E$, and that $(Y, \mathcal{O}_{\mathcal{L}Y}) \subset F, (Z, \mathcal{O}_{\mathcal{L}Z}) \subset G$. then, we have that:

$$k : (Y, \mathcal{O}_{\mathcal{L}Y}) \to (E, \mathcal{O}_{\mathcal{L}E}), \qquad r : (Z, \mathcal{O}_{\mathcal{L}Z}) \to (E, \mathcal{O}_{\mathcal{L}E})$$

is covering of $(E, \mathcal{O}_{\mathcal{L}E})$ in $\mathcal{H}$ such that k factors through F and r factors through G. This shows that $F \cup G = (E, \mathcal{O}_{\mathcal{L}E})$ in $\mathcal{T}$. □

2.4 Corollary. *Let $M \in \mathcal{M}$ and let F and G be subobjects of iM. Then, $F \cup G = iM$ in $\mathcal{T}$ iff there exist $V \subset M, W \subset M$ open such that $iV \subset F, iW \subset G$, and $V \cup W = M$.*

Proof. Suposse that $F \cup G = iM$. Let $\{U_\alpha\}_{(\alpha \in I)}$ be an open covering of M such that each U_α is biholomorphic to an open subset of $\mathbf{C}^n$. Then, $(U_\alpha, \mathcal{O}_{U_\alpha}) \in \mathcal{H}$ and we have $iU_\alpha = (U_\alpha, \mathcal{O}_{U_\alpha})$ in $\mathcal{T}$. Since i preserves open coverings, $\bigcup_{\alpha \in I}(iU_\alpha) = iM$ in $\mathcal{T}$. Let $F_\alpha = F \cap iU_\alpha$ and $G_\alpha = G \cap iU_\alpha$. Then $F_\alpha \cup G_\alpha = iU_\alpha = (U_\alpha, \mathcal{O}_{U_\alpha})$ in $\mathcal{T}$. Hence, by theorem 2.3 there exists $V_\alpha \subset U_\alpha$ and $W_\alpha \subset U_\alpha$ open such that $(V_\alpha, \mathcal{O}_{V_\alpha}) \subset F_\alpha$, $(W_\alpha, \mathcal{O}_{W_\alpha}) \subset G_\alpha$, and $V_\alpha \cup W_\alpha = U_\alpha$ (for each α).

Let $V = \bigcup_{\alpha \in I} V_\alpha$ and $W = \bigcup_{\alpha \in I} W_\alpha$; V and W are open subsets of M such that $V \cup W = M$. Moreover, as before, we have that $iV = \bigcup_{\alpha \in I} iV_\alpha$ and $iW = \bigcup_{\alpha \in I} iW_\alpha$. But, $iV_\alpha = (V_\alpha, \mathcal{O}_{V_\alpha}) \subset F_\alpha \subset F$ (for all $\alpha \in I$). Then, $iV \subset F$. Similarly, we have $iW \subset G$.

Converseley, suppose that there exist $V \subset M, W \subset M$ open such that $iV \subset F, iW \subset G$, and $V \cup W = M$. Then, $iV \cup iW = iM$ in $\mathcal{T}$. But, since $iV \subset F \subset iM$, and $iW \subset G \subset iM$, we have that $F \cup G = iM$. □

3. Penon's characterization of open sets in $\mathcal{T}$.

Recall that given an object X in a topos, a part $U \in \Omega^X$ is said to be *Penon open* iff the following condition holds (in the internal logic of the topos):

$$\forall\, x \in U \;\; \forall\, y \in X \;\; (y \neq x) \;\; \vee \;\; (y \in U). \quad \text{(see [10], [11], [2])}$$

We shall denote by Γ the global sections functor. Clearly, we have $\Gamma(iM) = M$ for all $M \in \mathcal{M}$.

3.1 Theorem. *Given an analytic complex manifold $M \in \mathcal{M}$, we have:*

a) If $N \subset M$ is any subset, then:

$$iN \subset iM \text{ is a Penon open subobject, and } N = \Gamma(iN).$$

b) If $F \subset iM$ is any Penon open subobject, then:

$$\Gamma(F) \subset M \text{ is an open subset and } F = i(\Gamma F).$$

(This theorem is similar to theorem 7 of [5]).

Proof. a) Let Δ be the diagonal and let "$\neg$" be the negation in $\mathcal{T}$. We have to show that $\neg\Delta_{iN}$ and $iN \times iN$ cover $iM \times iN$ in $\mathcal{T}$ (where we think Δ_{iN} and $iN \times iN$ as subobjects of $iM \times iN$ in $\mathcal{T}$). Let $(\Delta_N)^c$ be the complement of the diagonal Δ_N in $M \times N$. It is an open subset of $M \times N$, and we have $i(\Delta_N^c) \subset iM \times iN$ in $\mathcal{T}$. Moreover, since $i(\Delta_N) = \Delta_{iN}$ it follows that $i(\Delta_N^c) \subset \neg\Delta_{iN}$ (recall that i preserves products and $i\emptyset = 0$). Hence, it is sufficient to show that $i(\Delta_N^c) \cup (iN \times iN) = iM \times iN$. This follows from the fact that $(\Delta_N)^c$ together with $(N \times N)$ are an open cover of $M \times N$.

b) We have $\Gamma(F) \subset \Gamma(iM) = M$. Let $p \in \Gamma(F)$ be any point of $\Gamma(F)$, $p : 1 \to F$ in $\mathcal{T}$. Since F is a Penon open subobject of iM, we have that $\neg\{p\} \cup F = iM$ in $\mathcal{T}$. Then, by corollary 2.4 there exist open subsets $V \subset M$, $W \subset M$ such that $iV \subset \neg\{p\}, iW \subset F$ and $V \cup W = M$. Then, $W = \Gamma(iW) \subset \Gamma(F)$. We shall see now that $p \in W$. If $p \notin W$, since $p \in \Gamma(F) \subset M$ and $V \cup W = M$, it follows that $p \in V$, that is $p : 1 \to iV$ in $\mathcal{T}$. Since $iV \subset \neg\{p\}$, this implies $p : 1 \to \neg\{p\}$. Thus, the empty family covers 1. This shows that $p \in W$. We have proved that for each $p \in \Gamma(F)$, there exist an open subset $W \subset M$ such that $p \in W \subset \Gamma(F)$. This shows that $\Gamma(F)$ is open. It remains to see that $F = i(\Gamma F)$. We have stablished above that for each $p \in \Gamma(F)$, there exist W open such that $iW \subset F$ (and $p \in W \subset \Gamma(F)$). The the open sets W (one for each $p \in \Gamma(F)$) cover the open set $\Gamma(F)$. It follows that the subobjects iW cover $i(\Gamma(F))$. Since each $iW \subset F$, it follows that $i(\Gamma(F)) \subset F$. On the other hand, if $B = \Gamma(F)$, then B is an open subset of M such that $\Gamma(F) \subset B$. Hence by corollary 2.2, it follows that $F \subset iB$, that is $F \subset i(\Gamma(F))$. □

3.2 Observation. *It follows that i and Γ establish a bijection between open subsets of M and Penon open subobjects of iM. A subobject $F \subset iM$ is Penon open iff it is of the form $F = iN$ for some open subset $N \subset M$.* □

A similar fact holds for all the objects in $\mathcal{H}$ (and in fact it could be proved for all analytic schemes in the sense of 1.5 above):

3.3 Proposition. *Given any* $(E, \mathcal{O}_{\mathcal{L}E}) \in \mathcal{H}$ *and* $F \in \mathcal{T}$ *such that* $F \subset (E, \mathcal{O}_{\mathcal{L}E})$*, we have:*

F is a Penon open subobject of $(E, \mathcal{O}_{\mathcal{L}E})$ iff there exist an open subset $Y \subset E$ such that $F = (Y, \mathcal{O}_{\mathcal{L}Y})$.

Proof. Assume F is a Penon open subobject of $(E, \mathcal{O}_{\mathcal{L}E})$. Then, for any $p \in \Gamma(F)$, that is $p : 1 \to F, \neg\{p\} \cup F = (E, \mathcal{O}_{\mathcal{L}E})$. Then by theorem 2.3 and arguing exactly as in the proof of theorem 3.1,b), it follows that there is an open set $W \subset E$ such that $p \in W$ and $(W, \mathcal{O}_{\mathcal{L}W}) \subset F$. Let Y be the union of all the sets W (one for each p). Then it follows that $(Y, \mathcal{O}_{\mathcal{L}Y}) \subset F$. Since Y is an open subset of E, there is an open subset V of $\mathbf{C}^n$ such that $Y = V \cap E$ (here we suppose that $E \subset \mathbf{C}^n$). By construction of Y, we have that $\Gamma(F) \subset Y$. Then, we have that F is a subobject of $(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) = i\mathbf{C}^n$ such that $\Gamma(F) \subset V$. Hence, by corollary 2.2, $F \subset iV = (V, \mathcal{O}_V)$. Since $F \subset (E, \mathcal{O}_{\mathcal{L}E})$, it follows that $F \subset (V \cap E, \mathcal{O}_{\mathcal{L}, V \cap E})$, that is $F \subset (Y, \mathcal{O}_{\mathcal{L}Y})$.

Converseley, suppose that $F = (Y, \mathcal{O}_{\mathcal{L}Y})$, where Y is an open subset of E. Then, there is an open subset V of $\mathbf{C}^n$ such that $Y = V \cap E$. By theorem 3.1,a), iV is a Penon open subobject of $i\mathbf{C}^n$. It follows immediately that $iV \cap (E, \mathcal{O}_{\mathcal{L}E})$ is a Penon open subobject of $(E, \mathcal{O}_{\mathcal{L}E})$. Moreover, $iV \cap (E, \mathcal{O}_{\mathcal{L}E}) = (V, \mathcal{O}_V) \cap (E, \mathcal{O}_{\mathcal{L}E}) = (Y, \mathcal{O}_{\mathcal{L}Y}) = F$. □

3.4 Proposition. *Let* $(E, \mathcal{O}_{\mathcal{L}E}), (Y, \mathcal{O}_{\mathcal{S}Y}) \in \mathcal{H}$ *and let* $F \subset (Y, \mathcal{O}_{\mathcal{S}Y})$ *be any Penon open subobject in* $\mathcal{T}$*. Let* f *be any map in* $\mathcal{H}$*,* $f : (E, \mathcal{O}_{\mathcal{L}E}) \to (Y, \mathcal{O}_{\mathcal{S}Y})$*.*

Consider a diagram:

$$
\begin{array}{ccc}
1 & \longrightarrow & (E, \mathcal{O}_{\mathcal{L}E}) \\
\big\downarrow & \swarrow & \big\downarrow f \\
F & \longrightarrow & (Y, \mathcal{O}_{\mathcal{S}Y})
\end{array}
$$

If for every point $p : 1 \to (E, \mathcal{O}_{\mathcal{L}E})$, there is a map $1 \to F$ such that the square commutes, then f factors through F as indicate by the diagonal arrow.

Proof. Immediate from proposition 2.1 and 3.3. □

3.5 Corollary. *Let $(Y, \mathcal{O}_{SY}) \in \mathcal{H}$ and let $F \subset (Y, \mathcal{O}_{SY})$ be any Penon open subobject in $\mathcal{T}$. Given any subobject $G \subset (Y, \mathcal{O}_{SY})$ in $\mathcal{T}$, if $\Gamma(G) \subset \Gamma(F)$, then $G \subset F$.*

Proof. Observe that objects in $\mathcal{H}$ generate $\mathcal{T}$, then utilize Proposition 3.4.

□

3.6 Proposition. *Given any $(E, \mathcal{O}_{\mathcal{L}E}) \in \mathcal{H}, (E, \mathcal{O}_{\mathcal{L}E})$ satisfies the covering principle with respect to the intrinsic Penon topology in $\mathcal{T}$ (see [2]). That is, the following conditions holds:*

$$\forall\ F, G \in \Omega^{(E, \mathcal{O}_{\mathcal{L}E})}\{F \cup G = (E, \mathcal{O}_{\mathcal{L}E}) \Rightarrow \ell F \cup \ell G = (E, \mathcal{O}_{\mathcal{L}E})\}$$

(where ℓ indicates the interior operator (largest Penon open inside)).

Proof. Let $F, G : (Y, \mathcal{O}_{SY}) \to \Omega^{(E, \mathcal{O}_{\mathcal{L}E})}$, the corresponding subobjects $F, G \subset (E, \mathcal{O}_{\mathcal{L}E}) \times (Y, \mathcal{O}_{SY})$ and are such that $F \cup G = (E, \mathcal{O}_{\mathcal{L}E}) \times (Y, \mathcal{O}_{SY})$. By 1.8 we have then $F \cup G = (E \times Y, \mathcal{O}_{(\mathcal{L},\mathcal{S})E\times Y})$. Hence, by 2.3, there are open subsets Z and W in $E \times Y$ such that $(Z, \mathcal{O}_{(\mathcal{L},\mathcal{S})Z}) \subset F, (W, \mathcal{O}_{(\mathcal{L},\mathcal{S})W}) \subset G$, and $Z \cup W = E \times Y$. By 3.3, $(Z, \mathcal{O}_{(\mathcal{L},\mathcal{S})Z})$ and $(W, \mathcal{O}_{(\mathcal{L},\mathcal{S})W})$ are Penon open subobjects of $(E \times Y, \mathcal{O}_{(\mathcal{L},\mathcal{S})E\times Y})$, and we have:

$$(Z, \mathcal{O}_{(\mathcal{L},\mathcal{S})Z}) \cup (W, \mathcal{O}_{(\mathcal{L},\mathcal{S})W}) = (E \times Y, \mathcal{O}_{(\mathcal{L},\mathcal{S})E\times Y}) \qquad \text{in } \mathcal{T}$$

The corresponding arrows:

$$(Z, \mathcal{O}_{(\mathcal{L},\mathcal{S})Z}) : (Y, \mathcal{O}_{SY}) \to \Omega^{(E, \mathcal{O}_{\mathcal{L}E})}, \text{ and } (W, \mathcal{O}_{(\mathcal{L},\mathcal{S})W}) : (Y, \mathcal{O}_{SY}) \to \Omega^{(E, \mathcal{O}_{\mathcal{L}E})}$$

factor through the object of Penon opens of $(E, \mathcal{O}_{\mathcal{L}E}), P(E, \mathcal{O}_{\mathcal{L}E}) \subset \Omega^{(E, \mathcal{O}_{\mathcal{L}E})}$

Thus we have:

$$x \in P(E, \mathcal{O}_{\mathcal{L}E}), y \in P(E, \mathcal{O}_{\mathcal{L}E}) \text{ such that } x \subset F, y \subset G, \text{ and } x \cup y = (E, \mathcal{O}_{\mathcal{L}E}).$$

This proves the statement. □

3.7 Proposition. *Given any $(E, \mathcal{O}_{\mathcal{L}E}) \in \mathcal{H}, (E, \mathcal{O}_{\mathcal{L}E})$ is separated for the intrinsic Penon topology in $\mathcal{T}$ (see [2]). That is, the following condition holds:*

$$\forall\ x \in (E, \mathcal{O}_{\mathcal{L}E}), \neg\{x\} \text{ is Penon open }.$$

Proof. Let $x \in (E, \mathcal{O}_{\mathcal{L}E})$. We have $\{x\} \in \Omega^{(E, \mathcal{O}_{\mathcal{L}E})}$. If $x : (Y, \mathcal{O}_{\mathcal{L}E}) \to (E, \mathcal{O}_{\mathcal{L}E})$, then $\{x\} : (Y, \mathcal{O}_{SY}) \to \Omega^{(E, \mathcal{O}_{\mathcal{L}E})}$ corresponds to the (subobject) graph of $x, \Gamma_x = (x, \mathrm{id})$. We have $\Gamma_x \subset (E, \mathcal{O}_{\mathcal{L}E}) \times (Y, \mathcal{O}_{\mathcal{S}Y})$. Thus, it is sufficient to prove that $\neg\Gamma_x$ is a Penon open subobject of $(E, \mathcal{O}_{\mathcal{L}E}) \times (Y, \mathcal{O}_{\mathcal{S}Y}) = (E \times Y, \mathcal{O}_{(\mathcal{L},\mathcal{S})E\times Y})$. [c.f. 1.8]. Let $x = (f, \phi)$. Since f is a continuous function, $f : Y \to E$, the complement of its graph, $Z = \Gamma_f^c$ is an open subset of $E \times Y$. It follows easily by 3.5 that $\neg\Gamma_x = (Z, \mathcal{O}_{(\mathcal{L},\mathcal{S})Z})$. Then, by 3.3, we have that $\neg\Gamma_x$ is a Penon open subobject of $(E \times Y,\ \mathcal{O}_{(\mathcal{L},\mathcal{S})E\times Y})$. This proves the statement. □

3.8 Proposition. *Let $(E, \mathcal{O}_{\mathcal{L}E})$ and $(Y, \mathcal{O}_{\mathcal{S}Y})$ in $\mathcal{H}$ and $p : 1 \to (E, \mathcal{O}_{\mathcal{L}E})$. Let G be any family of Penon opens indexed by $(Y, \mathcal{O}_{\mathcal{S}Y})$. That is, $G \subset (E, \mathcal{O}_{\mathcal{L}E}) \times (Y, \mathcal{O}_{\mathcal{S}Y})$ is Penon open in the slice topos $\mathcal{T}|_{(Y,\mathcal{O}_{\mathcal{S}Y})}$. Then, if $\{p\} \times (Y, \mathcal{O}_{\mathcal{S}Y}) \subset G$, there exist a subobject $F \subset (E, \mathcal{O}_{\mathcal{L}E}) \times (Y, \mathcal{O}_{\mathcal{S}Y})$, Penon open in $\mathcal{T}$, such that $\{p\} \times (Y, \mathcal{O}_{\mathcal{S}Y}) \subset F \subset G$.*

Proof. By [2 lemma II. 1.7] this will be true provided that the intrinsic topology satisfies the covering principle. But this is precisely the statement in proposition 3.6 □

4. The axiom of germ representability.

Recall that the object of infinitesimals, $\Delta(n) \subset (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$, is defined as follows:

$$\Delta(n) = \neg\neg\{0\} \subset (\mathbf{C}^n, \mathcal{O}^n_{\mathbf{C}}) \text{ in } \mathcal{T}.$$

It is the largest infinitesimal neighbourhood of $0 \in (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$ in $\mathcal{T}$.

4.1 Observation. *It is easy to see that $\Delta(n) = P_0(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$, where $P_0(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$ denotes the intersection of all Penon neighbourhoods of 0 in $(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$ [2.II 1.10].*

That is:

$$P_0(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) =$$
$$= [[x \in (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) | \forall\, U \in \Omega^{(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})} (U \in P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \wedge 0 \in U) \Rightarrow x \in U]]$$

4.2 Definition. *Consider the object of partial maps, and the 'domain' map ∂:*

$$\textit{Partial}\ ((\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), (\mathbf{C}, \mathcal{O}_{\mathbf{C}})) \xrightarrow{\ \partial\ } \Omega^{(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})}$$

A germ at 0 is an equivalence class of elements $f \in$ *Partial* $((\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), (\mathbf{C}, \mathcal{O}_{\mathbf{C}}))$ *such that* $0 \in \partial(f)$ *and* $\partial(f)$ *is Penon open, that is,* $\partial(f) \in P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$.

The equivalence relation is:

$$f \sim g \quad \Leftrightarrow \quad \exists U \in P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) | 0 \in U,\ U \subset \partial(f) \cap \partial(g),\ f|_U = g|_U.$$

The object of germs is denoted $\mathbf{C}_0^g((\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), (\mathbf{C}, \mathcal{O}_{\mathbf{C}}))$. Given any germ, since $0 \in \partial(f)$ and $\partial(f) \in P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$, it is clear that $\Delta(n) \subset \partial(f)$. Thus there is a map:

$$j : C_0^g((\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), (\mathbf{C}, \mathcal{O}_{\mathbf{C}})) \to (\mathbf{C}, \mathcal{O}_{\mathbf{C}})^{\Delta(n)},$$

which sends a germ into its restriction to $\Delta(n)$.

The axiom of germ representability, axiom III of S.D.G., says that this map is invertible, (see **[2]**). In the rest of this section we shall prove that it holds in $\mathcal{T}$.

4.3 Proposition.

$$\neg\{0\} = (\mathbf{C}^n - \{0\}, \mathcal{O}_{\mathbf{C}^n - \{0\}}) \qquad (in\ \mathcal{T})$$

that is,
$$\Delta(n) = \neg(\mathbf{C}^n - \{0\}, \mathcal{O}_{\mathbf{C}^n - \{0\}}) \qquad (in\ \mathcal{T})$$

Proof. Clearly $(\mathbf{C}^n - \{0\}, \mathcal{O}_{\mathbf{C}^n - \{0\}}) \cap \{0\} = \emptyset$. Thus we have to prove:

If $G \subset (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), G$ in $\mathcal{T}$, is such that $G \cap \{0\} = \emptyset$, then $G \subset (\mathbf{C}^n - \{0\}, \mathcal{O}_{\mathbf{C}^n - \{0\}})$.

In fact, by 3.3 we know that $(\mathbf{C}^n - \{0\}, \mathcal{O}_{\mathbf{C}^n - \{0\}})$, is a Penon open. On the other hand, $\Gamma(G) \subset \mathbf{C}^n - \{0\}$. The statement follows then by 3.5. □

4.4 Proposition. $\Delta(n)$ *is representable by* $(\{0\}, \mathcal{O}_{\mathbf{C}^n})$. *That is,* $\Delta(n) \in \mathcal{H}$ *and it is the analytic scheme* $\Delta(n) = (\{0\}, \mathcal{O}_{\mathbf{C}^n})$. *(see 1.2).*

Proof. By 4.3, it is sufficient to check:

1) $(\{0\}, \mathcal{O}_{\mathbf{C}^n}) \cap (\mathbf{C}^n - \{0\}, \mathcal{O}_{\mathbf{C}^n - \{0\}}) = \emptyset$

2) Given a subobject F of $(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$ in $\mathcal{T}$:

$$\text{if} \quad F \cap (\mathbf{C}^n - \{0\}, \mathcal{O}_{\mathbf{C}^n - \{0\}}) = \emptyset, \quad \text{then} \quad F \subset (\{0\}, \mathcal{O}_{\mathbf{C}^n}).$$

1) is clear by 1.13. Let now F be such as in 2), and let $i : F \to (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$ be the inclusion. We have to to prove that for all objects $(E, \mathcal{O}_{\mathcal{L}E}) \in \mathcal{H}$ and $\lambda : (E, \mathcal{O}_{\mathcal{L}E}) \to F$, the composite $i \circ \lambda$ factors through $(\{0\}, \mathcal{O}_{\mathbf{C}^n})$. Let $i \circ \lambda = (f, \phi) : (E, \mathcal{O}_{\mathcal{L}E}) \to (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$. By assumption it follows that $\Gamma(F) \subset \{0\}$. Thus f factors through $\{0\}$ and $\phi_p : \mathcal{O}_{n,0} \to \mathcal{O}_{\mathcal{L}E,p}$ (for all p in E) □

4.5 Theorem. *For each positive integer n, the restriction map j (defined in 4.2) is invertible. That is, j is injective and surjective. This mean the validity in $\mathcal{T}$ of the following two formulae, (Axiom of Germ Representability, [2 II. 3.1]):*

a) $\forall\ f, g \in \mathit{Partial}((\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}})),$

$$0 \in \partial(f) \cap \partial(g),\ \partial(f), \partial(g) \in P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}),$$

$$f|_{\Delta(n)} = g|_{\Delta(n)} \quad \Rightarrow \quad \exists U \in P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), 0 \in U \subset \partial(f) \cap \partial(g), f|_U = g|_U.$$

b) $\forall\ g \in (\mathbf{C}, \mathcal{O}_{\mathbf{C}})^{\Delta(n)}\ \exists f \in \mathit{Partial}((\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}})),$

$$0 \in \partial(f) \in P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), f|_{\Delta(n)} = g.$$

Proof of a). Let $(Y, \mathcal{O}_{SY}) \in \mathcal{H}$ and let

$$f, g : (Y, \mathcal{O}_{SY}) \to \text{Partial } ((\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}}))$$

Then, f and g are partial maps from $(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \times (Y, \mathcal{O}_{SY})$ into $(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$ in $\mathcal{T}$, and their domain (which we can assume to be the same) is a subobject $H \subset (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \times (Y, \mathcal{O}_{SY})$ such that the corresponding map $(Y, \mathcal{O}_{SY}) \to \Omega^{(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})}$ factors $(Y, \mathcal{O}_{SY}) \to P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \subset \Omega^{(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})}$, and the map $(0, \mathrm{id}) : (Y, \mathcal{O}_{SY}) \to (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \times (Y, \mathcal{O}_{SY})$ factors $(Y, \mathcal{O}_{SY}) \to H, H \subset (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \times (Y, \mathcal{O}_{SY})$. Since $0 \in H$ and $H \in P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$, by (4.1) we have that $\Delta(n) \times (Y, \mathcal{O}_{SY}) \subset H$. Then:

$$\begin{array}{ccc} \{0\} \times (Y, \mathcal{O}_{SY}) \subset \Delta(n) \times (Y, \mathcal{O}_{SY}) \subset & H & \subset (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \times (Y, \mathcal{O}_{SY}) \\ & f \downarrow g \downarrow & \\ & (\mathbf{C}, \mathcal{O}_{\mathbf{C}}), & \end{array}$$

($f = g$ on $\Delta(n) \times (Y, \mathcal{O}_{SY})$)

The object H is a "family" of Penon opens, that is, it is Penon open in the slice topos $\mathcal{T}/_{(Y, \mathcal{O}_{SY})}$. By proposition 3.8 there is a Penon open $F \subset (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \times (Y, \mathcal{O}_{SY})$ in $\mathcal{T}$ such that $\{0\} \times (Y, \mathcal{O}_{SY}) \subset F \subset H$. By 4.1, $\Delta(n) \times (Y, \mathcal{O}_{SY}) \subset F$. Thus, we have f, g, such that $f, g : F \to (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$ and $f = g$ on $\Delta(n) \times (Y, \mathcal{O}_{SY})$. Moreover, by 1.8 we can consider $F \subset (\mathbf{C}^n \times Y, \mathcal{O}_{(0,S)\mathbf{C}^n \times Y})$. Now, by 3.3, there is an open subset $Z \subset \mathbf{C}^n \times Y$ such that $F = (Z, \mathcal{O}_{(0,S)Z})$. Since $\Delta(n) \times (Y, \mathcal{O}_{SY}) = (\{0\}, \mathcal{O}_{\mathbf{C}^n}) \times (Y, \mathcal{O}_{SY}) = (\{0\} \times Y, \mathcal{O}_{(0,S)\{0\} \times Y})$ (this holds by 4.4 and 1.8), we have that $\{0\} \times Y \subset Z$. Let us denote $f = (f, \phi), g = (g, \varphi)$,

$f, g : (Z, \mathcal{O}_{(0,\mathcal{S})Z}) \to (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$. The fact that f and G are equal on $\Delta(n) \times (Y, \mathcal{O}_{\mathcal{S}Y})$ imply that $f(0, y) = g(0, y)$, and $\phi_{(0,y)} = \varphi_{(0,y)}$ (for all $y \in Y$).

Let the object $(Y, \mathcal{O}_{\mathcal{S}Y}) \in \mathcal{H}$ be defined by two coherent sheaves of ideals $\mathcal{J}, \mathcal{S}$, in $\mathcal{O}_V$, with V an open subset of $\mathbf{C}^k, \mathcal{S} \subset \mathcal{J}$. Since $Z \subset \mathbf{C}^n \times Y$ is open, there is an open subset W in $\mathbf{C}^{n+k}, W \subset \mathbf{C}^n \times V$, such that $Z = W \cap (\mathbf{C}^n \times Y)$. Let $y \in Y$. Then, $(0, y) \in Z$ and we have $(f, \phi), (g, \varphi), : (Z, \mathcal{O}_{(0,\mathcal{S})Z}) \to (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$. It follows by 1.6 that there are open neighbourhoods W_{1y} of 0 in $\mathbf{C}^n$ and W_{2y} of y in $\mathbf{C}^k, W_{1y} \times W_{2y} \subset W$, and holomorphic functions $f^{\sim}, g^{\sim} : W_{1y} \times W_{2y} \to \mathbf{C}$, such that:

$$(1) \qquad f^{\sim}(p,q) = f(p,q), \qquad g^{\sim}(p,q) = g(p,q) \;\; \forall\, (p,q) \in (W_{1y} \times W_{2y}) \cap Z$$

$$\phi_{(p,q)}(t_{f(p,q)}) = [(t \circ f^{\sim})_{(p,q)}]_{(0,\mathcal{S})} \qquad \forall\, (p,q) \in (W_{1y} \times W_{2y}) \cap Z, \forall\, t_{f(p,q)} \in \mathcal{O}_{1,f(p,q)}$$

$$\varphi_{(p,q)}(t_{g(p,q)}) = [(t \circ g^{\sim})_{(p,q)}]_{(0,\mathcal{S})} \qquad \forall\, (p,q) \in (W_{1y} \times W_{2y}) \cap Z, \forall\, t_{g(p,q)} \in \mathcal{O}_{1,g(p,q)}$$

(where $[\;]_{(0,\mathcal{S})}$ indicates class modulo $(0, \mathcal{S})$).
Since $f(0,y) = g(0,y)$ and $\phi_{(0,y)} = \varphi_{(0,y)}$, then $\phi_{(0,y)}(\mathrm{id}_{f(0,y)}) = \phi_{(0,y)}(\mathrm{id}_{g(0,y)})$. This shows that $[f^{\sim}{}_{(0,y)}]_{(0,\mathcal{S})} = [g^{\sim}{}_{(0,y)}]_{(0,\mathcal{S})}$, that is $f^{\sim}{}_{(0,y)} - g^{\sim}{}_{(0,y)} \in (0, \mathcal{S}y)$. Moreover, since $\mathcal{S}$ is a coherent sheaf of of ideals in $\mathcal{O}_V$, there is an open neighborhood T_y of y in $\mathbf{C}^k, T_y \subset V$, and holomorphic functions $h_1, \dots h_r : T_y \to \mathbf{C}$, such that, for each $q \in T_y, \mathcal{S}_q$ is generated by $\{h_{1q}, \dots h_{rq}\}$. It follows that there are open neighgorhoods T_{1y} of 0 in $\mathbf{C}^n, T_{1y} \subset W_{1y}$ and T_{2y} of y in $\mathbf{C}^k, T_{2y} \subset W_{2y} \cap T_y$, such that on $T_{1y} \times T_{2y}, f^{\sim} - g^{\sim} = \sum_i \alpha_i . h_i$, where α_i are holomorphic functions on $T_{1y} \times T_{2y}$. That is:

$$(2) \qquad f^{\sim}(p,q) - g^{\sim}(p,q) = \sum_i \alpha_i(p,q).h_i(q) \text{ for each } (p,q) \in T_{1y} \times T_{2y}.$$

Let $(p,q) \in (T_{1y} \times T_{2y}) \cap (\mathbf{C}^n \times Y)$. Since $Y = supp(\mathcal{O}_{V/\mathcal{J}})$, each $h_{iq} \in \mathcal{S}_q \subset \mathcal{J}_q$, and $q \in Y$, we have that $h_i(q) = 0 \;\; \forall\, i$. This implies that $f^{\sim}(p,q) = g^{\sim}(p,q)$, and since $T_{1y} \times T_{2y} \subset W_{1y} \times W_{2y} \subset W$, then $(p,q) \in W \cap (\mathbf{C}^n \times Y) = Z$. Thus, $f^{\sim}(p,q) = f(p,q)$ and $g^{\sim}(p,q) = g(p,q)$. Then, $f(p,q) = g(p,q)$. Moreover, by (2) and the fact that each $h_{iq} \in \mathcal{S}_q$ we have that $f^{\sim}_{(p,q)} - g^{\sim}_{(p,q)} \in (0, \mathcal{S}_q)$, that is $[f^{\sim}_{(p,q)}]_{(0,\mathcal{S})} = [g^{\sim}_{(p.q)}]_{(0,\mathcal{S})}$. Then by (1), if we denote $\beta = f(p,q) = g(p,q)$, we have that $\phi_{(p,q)}(\mathrm{id}_\beta) = \varphi_{(p,q)}(\mathrm{id}_\beta)$. Then, by the characterization of morphisms of analytic rings (with domain $\mathcal{O}_{1\beta}$), we have that $\phi_{(p,q)} = \varphi_{(p,q)}$. (see [6 prop.1.16]). It follows that if we define $M = \bigcup_{y \in Y}(T_{1y} \times T_{2y}), M$ is an open subset of $\mathbf{C}^{n+k}$, $M \cap (\mathbf{C}^n \times Y) \subset Z$, and $f(p,q) = g(p,q), \phi_{(p,q)} = \varphi_{(p,q)}$, for all $(p,q) \in M \cap (\mathbf{C}^n \times Y)$. Hence, if we denote

$U_1 = M \cap (\mathbf{C}^n \times Y)$, we have that U_1 is an open subset of $\mathbf{C}^n \times Y$ such that $\{0\} \times Y \subset U_1 \subset Z \subset \mathbf{C}^n \times Y$, and $(f,\phi),(g,\varphi)$ are equal on $(U_1, \mathcal{O}_{(0,\mathcal{S})U_1})$. Then, by 3.3, we have that $U = (U_1, \mathcal{O}_{(0,\mathcal{S})U_1})$ is a Penon open of $(\mathbf{C}^n \times Y, \mathcal{O}_{(0,\mathcal{S})\mathbf{C}^n \times Y})$, thus a Penon open subobject of $(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \times (Y, \mathcal{O}_{\mathcal{S}Y})$ in the topos $\mathcal{T}$. It follows then that the corresponding map $(Y, \mathcal{O}_{\mathcal{S}Y}) \to \Omega^{(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})}$ factors $(Y, \mathcal{O}_{\mathcal{S}Y}) \to P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \subset \Omega^{(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})}$. That is. $U \in P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), 0 \in U$ and f, g are equal on U. □

Proof of b). Let $(Y, \mathcal{O}_{\mathcal{S}Y}) \in \mathcal{H}$ and let $g : (Y, \mathcal{O}_{\mathcal{S}Y}) \to (\mathbf{C}, \mathcal{O}_{\mathbf{C}})^{\Delta(n)}$. By 4.4 and 1.8, we have that $\Delta(n) \times (Y, \mathcal{O}_{\mathcal{S}Y}) = (\{0\}, \mathcal{O}_{\mathbf{C}^n}) \times (Y, \mathcal{O}_{\mathcal{S}Y}) = (\{0\} \times Y, \mathcal{O}_{(0,\mathcal{S})\{0\}\times Y})$. Let the object $(Y, \mathcal{O}_{\mathcal{S}Y}) \in \mathcal{H}$ be defined by two coherent sheaves of ideals $\mathcal{J}, \mathcal{S}$ in $\mathcal{O}_V$, where V is an open subset of $\mathbf{C}^k, \mathcal{S} \subset \mathcal{J}$, and let $g = (g, \phi) : (\{0\} \times Y, \mathcal{O}_{(0,\mathcal{S})\{0\}\times Y}) \to (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$. It follows by 1.6, that for each $y \in Y$, there are open neighborhoods W_{1y} of 0 in $\mathbf{C}^n$ and W_{2y} of y in $\mathbf{C}^k, W_{2y} \subset V$, and an holomorphic function $g^\sim : W_{1y} \times W_{2y} \to \mathbf{C}$ such that:

$$(3) \qquad g^\sim(0,q) = g(0,q) \qquad \forall\ (0,q) \in (W_{1y} \times W_{2y} \cap (\{0\} \times Y)$$

$$\phi_{(0,q)}(t_{g(0,q)}) = [(t \circ g^\sim)_{(0,q)}]_{(0,\mathcal{S})}$$
$$\forall\ (0,q) \in (W_{1y} \times W_{2y}) \cap (\{0\} \times Y), \quad \forall\ t_{g(0,q)} \in \mathcal{O}_{1,g(0,q)}$$

where $[\ \ ]_{(0,\mathcal{S})}$ indicates class modulo $(0, \mathcal{S})$.

We can assume that W_{1y} is a polydisk with center at 0 in $\mathbf{C}^n$. Let $y, z \in Y$, and let us denote $g^\sim$ and g° the holomorphic functions given by (3), corresponding to y and z respectively. We have, for each $q \in W_{2y} \cap W_{2z} \cap Y$:

$$\varphi_{(0,q)}(\mathrm{id}_{g(0,q)}) = [g^\sim{}_{(0,q)}]_{(0,\mathcal{S})} \text{ and } \varphi_{(0,q)}(\mathrm{id}_{g(0,q)}) = [g^\circ{}_{(0,q)}]_{(0,\mathcal{S})}.$$

It follows that $g^\sim{}_{(0,q)} - g^\circ{}_{(0,q)} \in (0, \mathcal{S}_q)$. Moreover, since $\mathcal{S}$ is a coherent sheaf of of ideals in $\mathcal{O}_V$, there is an open neighborhood W of q in $\mathbf{C}^k$, and $h_1, \dots h_r$: $W \to \mathbf{C}$ holomorphic functions such that, for each $w \in W, \mathcal{S}_W$ is generated by $\{h_{1w}, \dots h_{rw}\}$. Thus:

$$(4) \qquad g^\sim - g^\circ = \sum_i \alpha_i . h_i, \text{ on } V_1 \times V_2$$

where V_1, V_2, are open neighborhoods of 0 in $\mathbf{C}^n$ and q in $\mathbf{C}^k$ respectively, and α_i are holomorphic functions defined on $V_1 \times V_2$.

We can assume $V_1 \subset W_{1y} \cap W_{1z}$, $V_2 \subset W_{2y} \cap W_{2z} \cap W$. Since $Y = supp(\mathcal{O}_{V/\mathcal{J}})$, each $h_{iq} \in \mathcal{S}_q \subset \mathcal{J}_q$. For $q \in Y$ we have that $h_i(q) = 0\ \forall\ i$. Then, by (4), we have that $g^\sim(p,q) = g^\circ(p,q)\ \forall\ p \in V_1$. It follows that the holomorphic function $m : W_{1y} \cap W_{1z} \to \mathbf{C}$ given by $m(p) = g^\sim(p,q) - g^\circ(p,q)$ vanishes for $p \in V_1$. Since $W_{1y} \cap W_{1z}$ is a polydisk, it is connected, and it follows that $m(p) = 0 \ \forall p \in W_{1y} \cap W_{1z}$. Thus, $g^{\sim}(p,q) = g^{\circ}(p,q) \ \forall p \in W_{1y} \cap W_{1z}$. Then, we have that, $g^{\sim} = g^{\circ}$ on $(W_{1y} \cap W_{1z}) \times (W_{2y} \cap W_{2z} \cap Y)$. Hence, if we define $H = (\bigcup_{y \in Y}(W_{1y} \times W_{2y})) \cap (\mathbf{C}^n \times Y)$, we have that H is an open subset of $\mathbf{C}^n \times Y, \{0\} \times Y \subset H$, furnished with an arrow $(f, \phi) : (H, \mathcal{O}_{(0,\mathcal{S})H}) \to (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$ such that $(f, \phi)|_{(\{0\} \times Y, \mathcal{O}_{(0,\mathcal{S})\{0\} \times Y})} = (g, \varphi)$. That is, $(f, \phi)|_{\Delta(n) \times (Y, \mathcal{O}_{\mathcal{S}Y})} = (g, \varphi)$. Then, by 3.3, $U = (H, \mathcal{O}_{(0,\mathcal{S})H})$ is a Penon open subobject of $(\mathbf{C}^n \times Y, \mathcal{O}_{(0,\mathcal{S})\mathbf{C}^n \times Y})$. That is U is a Penon open subobject of $(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \times (Y, \mathcal{O}_{\mathcal{S}Y})$ in the topos $\mathcal{T}$. Thus the corresponding map $(Y, \mathcal{O}_{\mathcal{S}Y}) \to \Omega^{(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})}$ factors $(Y, \mathcal{O}_{\mathcal{S}Y}) \to P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$. This means $U \in P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$ (notice also $0 \in U$).

Thus, the arrow $(f, \phi) : (Y, \mathcal{O}_{\mathcal{S}Y}) \to$ Partial $((\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), (\mathbf{C}, \mathcal{O}_{\mathbf{C}}))$, shows that there is $f \in$ Partial $((\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}), (\mathbf{C}, \mathcal{O}_{\mathbf{C}}))$ such that $0 \in \partial(f) = U \in P(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$, and $f|_{\Delta(n)} = g$. □

5. The postulate of Δ-infinitesimal integration

In this section, the complex space $\mathbf{C}^m$ will be denoted by the letter M when it is consider as an analytic complex manifold. However, when it plays the role of the tangent space at a point (of any m-dimensional complex manifold) we shall write $\mathbf{C}^m$.

Given an Analytic model of S.D.G. $i : \mathcal{M} \to \mathcal{T}$ (cf definition 1.11), the basic ring object of line type (the one dimensional "line") is given by the object $i\mathbf{C}$ in $\mathcal{T}$, which by abuse of notation we shall also denote $\mathbf{C}$. Notice that in the model $\mathcal{T}$ introduced here in 1.12, $\mathbf{C}$ is representable and we have $\mathbf{C} = (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$.

The following considerations are meaningful in any model of S.D.G. (see [5]). We shall specify them here in the context of analytic models.

Given a function $g \in (\mathbf{C}^m)^M$ and a point $p \in M$, they determine a differential equation:

$$\frac{dy}{dz} = g(y), \qquad y(0) = p.$$

A solution of this equation is a map f on the variables (p, z), defined on some "neighbourhood" H of the axe, $M \times \{0\} \subset H \subset M \times \mathbf{C}$, such that:

$$\text{(1)} \qquad \frac{dy}{dz}(p, z) = g(f(p, z)), \qquad f(p, 0) = p$$

For example, if $D = [[z \in \mathbf{C} | z^2 = 0]]$, there is always (tautologically) a solution on $H = M \times D$, defined by $f(p, z) = p + z.g(p)$.

Recall that an integral flow of g is a map f on the variables $(p,z) \in M \times \mathbf{C}$ such that:

$$(2) \qquad \begin{aligned} f(p,d) &= p + d.g(p) \quad \forall d \in D \\ f(p, z+w) &= f(f(p,z), w) \end{aligned}$$

f will be defined on some part $H \subset M \times \mathbf{C}$, $M \times \{0\} \subset H \subset M \times \mathbf{C}$.

Any map $f(p,z)$ which satisfies (2) also satisfies (1), since

$$f(p, z+d) = f(f(p,z), d) = f(p,z) + d.g(f(p,z)).$$

Conversely, if $f(p,z)$ satisfies (1), for each p,z (fixed), the functions $y_1(w) = f(f(p,z),w)$ and $y_2(w) = f(p, z+w)$ both satisfy the differential equation $dy/dz = g(y)$ with initial condition $y(0) = p$. Thus, the uniqueness of solution to differential equations shows that $y_1 = y_2$.

We shall consider now the infinitesimal neighbourhood of the axis $H = M \times \Delta$, where $\Delta = \Delta(1) \subset \mathbf{C}$, $M \times \{0\} \subset M \times \Delta \subset M \times \mathbf{C}$. The previous considerations show the following:

5.1 Proposition. *The following two statements are equivalent in any analytic model of S.D.G.:*

1) $\forall\, g \in (\mathbf{C}^m)^M, \exists\, !\, f \in M^{M \times \Delta} \mid f(p,0) = p, \quad df/dz(p,z) = g(f(p,z))$.

2) $\forall\, g \in (\mathbf{C}^m)^M, \exists\, !\, f \in M^{M \times \Delta}$

$$\mid \forall d \in D \;\; f(p,d) = p + d.g(p), \;\; f(p, z+w) = f(f(p,z), w).$$

□

Any one of this two equivalent is postulate $WA2$ in [2,II; 3.1; 3.2]. Its validity (in the presence of the axiom of germ representability, c.f. 4.5 above) implies that we actually have local integration of vector fields in the topos.

We shall now prove that postulate $WA2$ is valid in the model $\mathcal{T}$ introduced in 1.12.

5.2 Observation. *Consider on the set* $A = \{$ *holomorphic functions defined on an open set* W *in* $\mathbf{C}^{m+1}$ *such that* $\mathbf{C}^m \times \{0\} \subset W\}$, *the relation* " $\sim$ " *given by:* $h \sim r$ *iff* $h = r$ *o n on an open set in* $\mathbf{C}^{m+1}$ *which contains* $\mathbf{C}^m \times \{0\}$. *Then we have a bijection:*

$$A/_{\sim} = [(\mathbf{C}^m \times \{0\}, \mathcal{O}_{\mathbf{C}^m \times \{0\}}), (\mathbf{C}, \mathcal{O}_{\mathbf{C}})], \quad \textit{thus} \;\; \Gamma(M^{M \times \Delta}) = (A|_{\sim})^m.$$

Proof. Recall that $M = (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m})$ in $\mathcal{T}$, and that Γ denotes global sections. We have:

$$\Gamma((\mathbf{C}^m)^M) = [(\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m}), (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m})] = (\mathcal{O}_m(\mathbf{C}^m))^m. \qquad \text{(see 1.6)}$$

Moreover, by (1.8) and (4.4) if follows that:

$$\Gamma(M^{M\times\Delta}) = [M\times\Delta,\ M] = [(\mathbf{C}^m,\mathcal{O}_{\mathbf{C}^m})\times(\{0\},\mathcal{O}_{\mathbf{C}}),(\mathbf{C}^m,\mathcal{O}_{\mathbf{C}^m})] =$$
$$= [(\mathbf{C}^m\times\{0\},\mathcal{O}_{\mathbf{C}^m\times\{0\}}),(\mathbf{C}^m,\mathcal{O}_{\mathbf{C}^m})] =$$
$$= [(\mathbf{C}^m\times\{0\},\mathcal{O}_{\mathbf{C}^m\times\{0\}}),(\mathbf{C}^m,\mathcal{O}_{\mathbf{C}^m}),(\mathbf{C},\mathcal{O}_{\mathbf{C}})]^m.$$

Let $(f,\phi)\ :\ (\mathbf{C}^m\times\{0\},\mathcal{O}^m_{\mathbf{C}}\times\{0\})\ \to\ (\mathbf{C},\mathcal{O}_{\mathbf{C}})$. By (1.6), for each $(p,0)\in\mathbf{C}^m\times\{0\}$, there is an open neighbourhood W_p of $(p,0)$ in $\mathbf{C}^{m+1}$ and an holomorphic function $f^p : W_p\to\mathbf{C}$ such that

$$\text{(3)}\quad \begin{aligned} &f^p(y,0)=f(y,0)\ \forall(y,0)\in W_p\cap(\mathbf{C}^m\times\{0\})\\ &\phi_{(y,0)}(t_{f(y,0)})=(t\circ f^p)_{(y,0)}\ \forall(y,0)\in W_p\cap(\mathbf{C}^m\times\{0\}),\ \forall t_{f(y,0)}\in\mathcal{O}_{1,f(y,0)}.\end{aligned}$$

We can assume that $W_p=U_p\times V_p$ where U_p is a polydisk in $\mathbf{C}^m$ centered at p and V_p is a disk in $\mathbf{C}$ centered at 0. Let $p,q\in\mathbf{C}^m,(y,z)\in W_p\cap W_q$. Then, $(y,0)\in W_p\cap W_q$. Thus, by (3), $\phi_{(y,0)}(\mathrm{id}_{f(y,0)})=f^p_{(y,0)}$ and $\phi_{(y,0)}(\mathrm{id}_{f(y,0)})=f^q_{(y,0)}$. That is, $f^p=f^q$ on an open neighbourhood of $(y,0)$ in $\mathbf{C}^{m+1}$. Since $W_p\cap W_q$ is connected, it follows that $f^p=f^q$ on $W_p\cap W_q$. Thus, there is an holomorphic function h on $W=\bigcup W_p$, such that $h=f^p$ on each W_p. (note that W is open in $\mathbf{C}^{m+1}$, and $\mathbf{C}^m\times\{0\}\subset W$). By (3) it follows that $h(p,0)=f(p,0)$, and $\phi_{(p,0)}(t_{f(p,0)})=(t\circ h)_{(p,0)}\forall(p,0)\in\mathbf{C}^m\times\{0\},\ \forall t_{f(p,0)}\in\mathcal{O}_{1,f(p,0)}$. Thus, if we consider the arrow $(h,h^*)\ :\ (W,\mathcal{O}_W)\ \to\ (\mathbf{C},\mathcal{O}_{\mathbf{C}})$, we have that $(h,h^*)|_{(\mathbf{C}^m\times\{0\},\mathcal{O}_{\mathbf{C}^m\times\{0\}})}=(f,\phi)$. Moreover, if r is any other holomorphic function defined on an open set W_1 in $\mathbf{C}^{m+1}$, and $\mathbf{C}^m\times\{0\}\subset W_1$, such that $(r,r^*)|_{(\mathbf{C}^m\times\{0\},\mathcal{O}_{\mathbf{C}^m\times\{0\}})}=(f,\phi)$, then $\phi_{(p,0)}(\mathrm{id}\ _{f(p,0)})=r_{(p,0)}$ $\forall(p,0)\in\mathbf{C}^m\times\{0\}$, and $\phi_{(p,0)}(\mathrm{id}\ _{f(p,0)})=h_{(p,0)}$. Thus, $h=r$ on an open neighbourhood of each $(p,0)$ in $\mathbf{C}^m\times\{0\}$. It follows that $h=r$ on an open set W_2 in $\mathbf{C}^{m+1},\mathbf{C}^m\times\{0\}\subset W_2$. □

Given an element g in $\Gamma((\mathbf{C}^m)^M), g:\mathbf{C}^m\to\mathbf{C}^m$, let f be the solution to the differential equation $df/dz(p,z)=g(f(p,z)), f(p,0)=p$, given by the classical theory of differential equations. f will be defined in an open set $W,\mathbf{C}^m\times\{0\}\subset W\subset\mathbf{C}^{m+1}, f:W\to\mathbf{C}^m$. Thus. each coordinate f_i of f, $f_i:W\to\mathbf{C}$, determines an element $f_i\in A$. The (local) uniqueness of f with respect to g implies that this determinates a map:

$$[M,(\mathbf{C}^m,\mathcal{O}_{\mathbf{C}^m})]\to(A/_\sim)^m,\ g\mapsto([f_1],\dots\ [f_m]).$$

Thus, we have an arrow:

$$[M,(\mathbf{C}^m,\mathcal{O}_{\mathbf{C}^m})]\to\Gamma(M^{M\times\Delta})$$

this arrow sends $g \in \Gamma((\mathbf{C}^m)^M)$ into $(f, f^*)|_{(\mathbf{C}^m \times \{0\}, \mathcal{O}_{\mathbf{C}^m \times \{0\}})}$.
(where $(f, f^*) : (W, \mathcal{O}_W) \to (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m})$).
We shall now show that this map lifts into a map in the topos.

5.3 Theorem. *There is a map in $\mathcal{T}$, $(\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m})^M \to M^{M\times\Delta}$ which sends $g \in (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m})^M$ into a solution $f \in M^{M\times\Delta}$ off the differential equation (1) above. This actually means the following:*

Given any $(Y, \mathcal{O}_{SY})$ in $\mathcal{H}$, and $g : M \times (Y, \mathcal{O}_{SY}) \to (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m})$, There is $f : M \times (Y, \mathcal{O}_{SY}) \times \Delta \to M$, such that:

$$df/dz(p, y, z) = g(f(p, y, z), y)$$
$$f(p, y, 0) = p \qquad \forall y \in (Y, \mathcal{O}_{SY}),$$

and the correspondence $g \mapsto f$ is natural in $(Y, \mathcal{O}_{SY})$.

Proof. Let the object $(Y, \mathcal{O}_{SY}) \in \mathcal{H}$ be given by two coherent sheaves of ideals $\mathcal{J}, \mathcal{S}$ in $\mathcal{O}_U$, where U is an open subset of $\mathbf{C}^n, \mathcal{S} \subset \mathcal{J}$. By (1.8) we have:

$$M \times (Y, \mathcal{O}_{SY}) = (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m}) \times (Y, \mathcal{O}_{SY}) = (\mathbf{C}^m \times Y, \mathcal{O}_{(0,\mathcal{S})(\mathbf{C}^m \times Y)}).$$

Thus:

$g : (\mathbf{C}^m \times Y, \mathcal{O}_{(0,\mathcal{S})(\mathbf{C}^m\times Y)}) \to (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m})$ (we shall denote $g = (g, \phi)$).
It follows by (1.6) (working in each coordinate), that for each $(p_0, y_0) \in \mathbf{C}^m \times Y$, there is an open neighbourhood W of (p_0, y_0) in $\mathbf{C}^m \times \mathbf{C}^n$ and an holomorphic function $g^\sim : W \to \mathbf{C}^m$ such that:

$$g^\sim(p, y) = g(p, y)$$
$$(1) \qquad \phi_{(p,y)}(t_{g(p,y)}) = [(t \circ g^\sim)_{(p,y)}]_{(0,\mathcal{S})}$$
$$\forall\ (p, y) \in W \cap (\mathbf{C}^m \times Y),\ \forall\ t_{g(p,y)} \in \mathcal{O}_{m,g(p,y)}$$

(we can suppose that $W = W_1 \times W_2$, where W_1 is an open neighbourhood of p_0 in $\mathbf{C}^m$, and W_2 is an open neighbourhood of y_0 in $\mathbf{C}^n, W_2 \subset U$).

By the classical theory of differential equations, there is an open set $N \subset \mathbf{C}^m \times \mathbf{C}^n \times \mathbf{C}$ such that $W_1 \times W_2 \times \{0\} \subset N \subset W_1 \times W_2 \times \mathbf{C}$, and an holomorphic function $f : N \to \mathbf{C}^m$ such that $\forall (p, y, z) \in N, f(p, y, z) \in W_1$, and $df/dz(p, y, z) = g^\sim(f(p, y, z), y), f(p, y, 0) = p$. We have $(f, f^*) : (N, \mathcal{O}_N) \to (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m})$. Consider the restriction of this map:

$$(W_1, \mathcal{O}_{W_1}) \times (W_2 \cap Y, \mathcal{O}_{SW_2 \cap Y}) \times \Delta \to (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m}), \qquad \Delta = (\{0\}, \mathcal{O}_{\mathbf{C}})$$

We are going to prove now that all these arrows (one for each point of $\mathbf{C}^m \times Y$), define an arrow $(\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m}) \times (Y, \mathcal{O}_{\mathcal{S}Y}) \times \Delta \to (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m})$. It is sufficient to prove:

If $g^\circ : W' \to \mathbf{C}^m$ is an holomorphic function such that

$$g^\circ(p,y) = g(p,y)$$

$$(2) \quad \phi_{(p,y)}(t_{g(p,y)}) = [(t \circ g^\circ)_{(p,y)}]_{(0,\mathcal{S})}$$

$$\forall\ (p,y) \in W' \cap (\mathbf{C}^m \times Y),\ \forall\ t_{g(p,y)} \in \mathcal{O}_{m,g(p,y)}$$

$(W' = W_1' \times W_2', W_1'$ open in $\mathbf{C}^m, W_2'$ open in $\mathbf{C}^n, W_2' \subset U)$.

And $f' : N' \to \mathbf{C}^m$ is an holomorphic function defined on an open set $N' \subset \mathbf{C}^m \times \mathbf{C}^n \times \mathbf{C}, W_1' \times W_2' \times \{0\} \subset N' \subset W_1' \times W_2' \times \mathbf{C}$, such that

$$\forall (p,y,z) \in N', f'(p,y,z) \in W_1',$$
$$df'/dz(p,y,z) = g^\circ(f'(p,y,z),y), f'(p,y,0) = p, \quad \text{then,}$$

$(f', f'^*) = (f, f^*)$ on $(W_1 \cap W_1', \mathcal{O}_{W_1 \cap W_1'}) \times (W_2 \cap W_2' \cap Y, \mathcal{O}_{\mathcal{S}W_2 \cap W_2' \cap Y}) \times \Delta$.

By (1.8) and (4.4) we have that:

$$(W_1 \cap W_1', \mathcal{O}_{W_1 \cap W_1'}) \times (W_2 \cap W_2' \cap Y, \mathcal{O}_{\mathcal{S}W_2 \cap W_2' \cap Y}) \times \Delta =$$
$$= ((W_1 \cap W_1') \times (W_2 \cap W_2' \cap Y) \times \{0\}, \mathcal{O}_{(0,\mathcal{S},0)(W_1 \cap W_1') \times (W_2 \cap W_2' \cap Y) \times \{0\}}).$$

Moreover, $f(p,y,0) = f'(p,y,0) = p$. Thus, we have to prove that

$$[(t \circ f)_{(p,y,0)}]_{(0,\mathcal{S},0)} = [(t \circ f')_{(p,y,0)}]_{(0,\mathcal{S},0)}$$
$$\forall\ (p,y) \in (W_1 \cap W_1') \times (W_2 \cap W_2' \cap Y), \forall\ t_p \in \mathcal{O}_{m,p}.$$

Then, by the characterization of morphisms of analytic rings with domain $\mathcal{O}_{m,p}$ [6, 1.16], it suffices to prove that this holds for the coordinate functions $z_1, \ldots z_m$. Thus we have to prove that $[(f_i)_{(p,y,0)}]_{(0,\mathcal{S},0)} = [(f_i')_{(p,y,0)}]_{(0,\mathcal{S},0)} \quad i = 1, \ldots m$. That is, for each one of the m coordinates, (that by abuse of notation we shall write without subindexes), we should verify $f_{(p,y,0)} - f'_{(p,y,0)} \in (0, \mathcal{S}_y, 0)$. We do this now.

From (1) and (2) we have $(g^\sim)_{(p,y)} - (g^\circ)_{(p,y)} \in (0, \mathcal{S}_y)$. Let U' be an open neighbourhood of y in $\mathbf{C}^n, U' \subset U$, and $h_1, \ldots h_r$ holomorphic functions on U' such that for each $y' \in U', \mathcal{S}_{y'}$ is generated by $\{(h_1)_{y'}, \ldots (h_r)_{y'}\}$. Take an open neighbourhood H of (p,y) in $\mathbf{C}^m \times \mathbf{C}^n$, and holomorphic functions $\beta_1, \ldots \beta_r : H \to \mathbf{C}^m$, such that $g^\sim(q,x) - g^\circ(q,x) = \sum_i h_i(x).\beta_i(q,x)$ $\forall (q,x) \in H$. Consider now a parameter space $\mathbf{C}^r$ and the function λ defined by

$$\lambda(z_1, \ldots z_r, q, x) = g^\circ(q,x) + \textstyle\sum_i z_i.\beta_i(q,x), \qquad \lambda : \mathbf{C}^r \times H \to \mathbf{C}^m.$$

It is holomorphic and clearly $\lambda(0,q,x) = g^\circ(q,x)$, $\lambda(h_1(x),\dots,h_r(x),q,x) = g^\sim(q,x)$.

By the classical theory of differential equations, there is an open neighbourhood V of 0 in $\mathbf{C}^r$, $V = V_1 \times \dots \times V_r$ (where each V_i is an open neighbourhood of 0 in $\mathbf{C}$), an open neighbourhood G of (p,y) in $\mathbf{C}^m \times \mathbf{C}^n$ (G can be considered to be a product of open subsets of $\mathbf{C}$), an open disk D with center at 0 in $\mathbf{C}$, and an holomorphic function $\psi : V \times G \times D \to \mathbf{C}^m$ such that:

$(\psi(z_1,\dots z_r,q,x,z),x) \in H, \qquad \forall\,(z_1,\dots z_r,q,x,z) \in V \times G \times D,$

$d\psi/dz(z_1,\dots z_r,q,x,z) = \lambda(z_1,\dots z_r,\psi(z_1,\dots z_r,q,x,z),x)$,

$\psi(z_1,\dots z_r,q,x,0) = x$.

It follows that there are holomorphic functions $\xi_i : V \times G \times D \to \mathbf{C}^m$ such that:

$\psi(z_1,\dots z_r,q,x,z) - \psi(0,\dots 0,q,x,z) = \sum_i z_i.\xi_i(z_1,\dots z_r,q,x,z)$, [6, 0.10]. Since each $h_{i,y} \in \mathcal{S}_y \subset \mathcal{J}_y$, and $y \in Y = supp(\mathcal{O}_U/\mathcal{J})$, then $h_i(y) = 0\ \forall i$. Thus, there is an open neighbourhood T of y in $\mathbf{C}^n$ such that for all $x \in T$, $(h_1(x),\dots h_r(x)) \in V$. It follows that for all $(q,x.z) \in (G \cap (\mathbf{C}^m \times T)) \times D$ (which is an open neighbourhood of $(p,y,0)$ in $\mathbf{C}^m \times \mathbf{C}^n \times \mathbf{C}$) we have:

$$\psi(h_1(x),\dots h_r(x),q,x,z) - \psi(0,\dots 0,q,x,z) = \\ = \sum_i h_i(x).\xi_i(h_1(x),\dots h_r(x),q,x,z).$$

Moreover, by the uniqueness of solution to differential equations,

$f'(q,x.z) = \psi(0,\dots 0,q,x,z)$ and $f(q,x,z) = \psi(h_1(x),\dots h_r(x),q,x,z)$, (in a neighbourhood of $(p,y,0)$). This finishes the proof of $f_{(p,y,0)} - f'_{(p,y,0)} \in (0,\mathcal{S}_y,0)$.

Thus, we have an arrow (that by abuse of notation we shall denote f) such that:

$f : (\mathbf{C}^m,\mathcal{O}_{\mathbf{C}^m}) \times (Y,\mathcal{O}_{SY}) \times \Delta \to (\mathbf{C}^m,\mathcal{O}_{\mathbf{C}^m}), \qquad$ and

$df/dz(p,y,z) = g(f(p,y,z),y)\ f(p,y,0) = p \qquad \forall\, y \in (Y,\mathcal{O}_{SY})$

□

5.4 Lemma. *Let $(Y,\mathcal{O}_{SY}) \in \mathcal{H}$ be defined by two coherent sheaves of ideals $\mathcal{J},\mathcal{S}$ in $\mathcal{O}_U$, where U is an open subset of $\mathbf{C}^n$, $\mathcal{S} \subset \mathcal{J}$. Let $(p_0,y_0) \in \mathbf{C}^m \times Y$, W an open neighbourhood of (p_0,y_0) in $\mathbf{C}^m \times \mathbf{C}^n$, $W \subset \mathbf{C}^m \times U$, and $g^\sim W \to \mathbf{C}^m$ an holomorphic function. Let $f, f' : W' \to \mathbf{C}^m$ be holomorphic functions (where W' is an open subset of $\mathbf{C}^m \times \mathbf{C}^n \times \mathbf{C}$ such that $W \times \{0\} \subset W' \subset W \times \mathbf{C}$, and for all $(p,y,z) \in W'$, $f(p,y,z),y) \in W$, $f'(p,y,z) \in W$) such that for each one of the m coordinates, (that as before we denote without subindexes) we have:*

$$\begin{array}{lcl}
(df/dz(p,y,z) - g^{\sim}(f(p,y,z),y)_{(p_0,y_0,0)} & \in & (0,\mathcal{S}_{y_0},0). \\
(f(p,y,0) - p)_{(p_0,y_0,0)} & \in & (0,\mathcal{S}_{y_0},0). \\
(df'/dz(p,y,z) - g^{\sim}(f'(p,y,z),y)_{(p_0,y_0,0)} & \in & (0,\mathcal{S}_{y_0},0). \\
(f'(p,y,0) - p)_{(p_0,y_0,0)} & \in & (0,\mathcal{S}_{y_0},0).
\end{array}$$

$$\textit{Then,}\ \ (f(p,y,z) - f'(p,y,z))_{(p_0,y_0,0)} \quad \in \quad (0,\mathcal{S}_{y_0},0).$$

Proof. By hypothesis, there is an open neighbourhood H of $(p_0, y_0, 0)$, $H \subset W'$, and there are holomorphic functions a_1, a_2, b_1, b_2 on H such that their germs at $(p_0, y_0, 0)$ belong to $(0, \mathcal{S}_{y_0}, 0)$, and:

$$\begin{array}{lcllcl}
df/dz(p,y,z) & = & g^{\sim}(f(p,y,z),y) + a_1(p,y,z), & f(p,y,z) & = & p + b_1(p,y,z), \\
df'/dz(p,y,z) & = & g^{\sim}(f'(p,y,z),y) + a_2(p,y,z), & f'(p,y,z) & = & p + b_2(p,y,z).
\end{array}$$

(the functions b_1 and b_2 actually do not depend on z).

By the classical theory of differential equations, there are open neighbourhoods V of 0 in $\mathbf{C}^2, V = V_1 \times V_2, G$ of (p_0, y_0) in $\mathbf{C}^m \times \mathbf{C}^n$, an open disk D centered at 0 in $\mathbf{C}$, and an holomorphic function $\psi : V \times G \times D \to \mathbf{C}^M$ such that:

$$d\psi/dz(s,w,p,y,z) = g^{\sim}(\psi(s,w,p,y,z),y) + s, \qquad \psi(s,w,p,y,0) = p + w.$$

By the uniqueness of solution to differential equations, if follows that

$$\begin{aligned}
f(p,y,z) &= \psi(a_1(p,y,z), b_1(p,y,z), p, y, z) \\
f'(p,y,z) &= \psi(a_2(p,y,z), b_2(p,y,z), p, y, z).
\end{aligned}$$

Since $y_0 \in Y = supp(\mathcal{O}_U/\mathcal{J})$, and the germs at $(p_0, y_0, 0)$ of the functions a_1, a_2, b_1, b_2 are in $\mathcal{S}_{y_0} \subset \mathcal{J}_{y_0}$, we have that $a_1(p_0, y_0, 0) = 0$, $a_2(p_0, y_0, 0) = 0$, $b_1(p_0, y_0, 0) = 0$, and $b_2(p_0, y_0, 0) = 0$. Then, there is an open neighbourhood V' of $(p_0, y_0, 0), V' \subset H$, such that for all $(p, y, z) \in V'$, $(a_1(p,y,z), b_1(p,y,z)) \in V, (a_2(p,y,z), b_2(p,y,z)) \in V$ (and $(p, y) \in G,\ z \in D$).

As in 5.3, let r, u be holomorphic functions, $r, u : V \times V \times G \times D \to \mathbf{C}^m$ such that:

$$\begin{aligned}
&\psi(s',w',p,y,z) - \psi(s,w,p,y,z) = \\
&\qquad = (s' - s).r(s,w,s',w',p,y,z) + (w' - w)u(s,w,s',w',p,y,z).
\end{aligned}$$

It follows that for $(p, y, z) \in V'$,

$$\psi(a_1, b_1, p, y, z) - \psi(a_2, b_2, p, y, z) = \\ = (a_1 - a_2).r(a_2, b_2, a_1, b_1, p, y, z) + (b_1 - b_2).u(a_2, b_2, a_1, b_1, p, y, z).$$

That is,

$$f(p, y, z) - f'(p, y, z) = \\ = (a_1 - a_2).r(a_2, b_2, a_1, b_1, p, y, z) + (b_1 - b_2).u(a_2, b_2, a_1, b_1, p, y, z).$$

Hence, $(f - f')_{(p_0, y_0, 0)} \in (0, \mathcal{S}_{y_0}, 0)$. □

5.5 Theorem. *In the topos $\mathcal{T}$, the following holds:*

$$\forall\, g \in (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m})^M,\ \exists\,!\, f \in M^{M \times \Delta} \mid f(p, 0) = p,\ df/dz(p, z) = g(f(p, z)),$$

where $M = (\mathbf{C}^m, \mathcal{O}_{\mathbf{C}^m})$.

Proof. The existence is guaranteed by theorem (5.3) and the uniqueness follows immediately from lemma (5.4). □

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