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## CROSSED MODULES AS G-CATEGORIES

by Robert GORDON

**RESUME.** Une  $G$ -catégorie est une catégorie sur laquelle opère un groupe  $G$ . Cet article étudie, dans le contexte des modules croisés, les  $G$ -catégories (héréditairement) stablement fermées. En particulier, on montre que la  $G$ -catégorie associée à un module croisé est stablement fermée exactement si toute dérivation appartenant à un certain sous-groupe normal distingué du groupe de Whitehead des dérivations est principal.

### 1. Introduction.

$G$ -categories are categories upon which a group  $G$  acts. They have been studied, for their own sake, intensively in Gordon [7].

Now, as said to have been recognized by J.L. Verdier (see [4]), internal objects in the category of groups amount precisely to the crossed modules of Whitehead [15]. Indeed, the interpretation of a crossed module as a category makes it evident that category objects in  $Grp$  are  $G$ -categories for canonical  $G$  (namely, the object groups). These  $G$ -categories allow one to realize some of the basic ideas from [7] in a very concrete way, thereby enabling the construction with ease — in fact, our examples can all be taken to have only a few objects and arrows — of  $G$ -categories archetypical of certain of the phenomena addressed in [7].

This paper is primarily concerned, in the context of crossed modules, with the stably closed  $G$ -categories that constitute one of the central notions of [7]. The  $G$ -category associated with a crossed module is herein shown to be stably closed exactly when every derivation belonging to a certain distinguished normal subgroup of the Whitehead group of derivations [10,12,14] is principal.

Now actually, in [7], stably closed  $G$ -categories are studied chiefly in order to study hereditarily stably closed  $G$ -categories: that is,  $G$ -categories  $H$ -stably closed for all subgroups  $H$  of  $G$ . The  $G$ -category corresponding to a  $G$ -module  $M$  (seen as trivial crossed module), for example, is shown in this paper to be hereditarily stably closed iff the one-dimensional cohomology groups  $H^1(H,M)$  are trivial for every subgroup  $H$ . Notice

when  $G$  is finite any cohomologically trivial  $G$ -module  $M$  in the sense of Tate cohomology [2] has this property.

Concerning the importance of the hereditarily stably closed concept within the framework of  $G$ -categories we refer the reader to the introduction of [7] wherein this is detailed. We single out from [7] Theorem 13.13 which asserts any suitably nice (e.g. a Grothendieck category or complete topos) hereditarily stably closed  $G$ -category is cotripleable, in the sense of the 2-category of  $G$ -categories, over a functor- $G$ -category of the form  $B^G$ , where  $B$  is an ordinary category viewed as trivial  $G$ -category and  $G$  is viewed as discrete  $G^{OP}$ -category: and these  $B^G$  are, incidentally, hereditarily stably closed for any  $B$ . Additionally, in [8] categories of graded modules over  $G$ -graded algebras are characterised as being certain hereditarily stably closed  $G$ -categories.

We hasten to point out only minimal use of category theory is made in this article. All undefined categorical notions and terminology are to be found in MacLane [11]. Also, although the group theory used in this paper is elementary, we feel the point of view taken is of independent group-theoretic interest. We use standard group theoretical notation with no further comment.

Inasmuch as  $G$ -categories associated to crossed modules are automatically groupoids, they are  $G$ -groupoids in the sense of Taylor [13]. We suspect our results have applications to topology; for instance, to the orbit groupoids studied by Taylor in his paper just cited. We also refer to Brown's book [3] which suggests further topological applications.

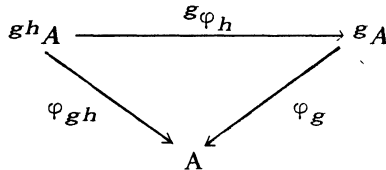
Finally, it may be of interest to study how the factor group  $CDer(G,M)/PDer(G,M)$  of Theorem 7.2 is related to the  $H^1$  of Datuashvili [5,6] for internal categories in the category of groups — cf. Theorem 6.1. Under a suitable hypothesis, this factor group is shown in Section 7 to be an invariant of the actor crossed module of Norrie [12]. We wish to thank the referee for making us aware of Norrie's article.

## 2. $G$ -categories.

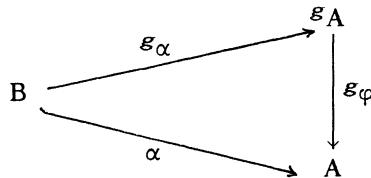
More explicitly than above, a  $G$ -category is a category  $\mathbf{A}$  together with actions of  $G$  on  $Ob\mathbf{A}$  and  $Ar\mathbf{A}$  such that the latter action respects composition and such that, for all  $g \in G$ ,  $\mathcal{E}\alpha \in \mathbf{A}(\mathcal{E}A, \mathcal{E}B)$  whenever  $\alpha \in \mathbf{A}(A, B)$ . Objects and arrows of  $\mathbf{A}$  are called *stable* if they are fixed under the action of  $G$ , and the subcategory,  $Stab\mathbf{A}$ , determined by the stable objects and arrows is called the *stable subcategory*. We introduce no notation for

the "stabilizer of  $\mathbf{A}$ " — the largest subgroup of  $G$  acting trivially on  $\mathbf{A}$  — because it is not used in this paper; but, see [7], §12. The subcategory  $Stab\mathbf{A}$  of  $\mathbf{A}$  is said to be *stably reflective* when it is reflective and when, given a reflector  $R:\mathbf{A}\rightarrow\mathbf{A}$  and unit of reflection  $\lambda:1_{\mathbf{A}}\rightarrow R$ , the family  $\{\varepsilon\lambda_{\mathbf{A}}:\varepsilon\mathbf{A}\rightarrow R\mathbf{A}\}_{g\in G}$  is a coproduct for each  $\mathbf{A}\in\mathbf{A}$ . Sections 7-13 of [7] are in good measure devoted to the study of stable reflections and show, in particular, that the distinction between stable reflections and ordinary ones is significant. Here, several examples are given confirming that the former notion is indeed considerably stronger than the latter.

In [7] the notion "stably closed" mentioned in the introduction is defined in terms of the existence of certain limits. The equivalent definition (cf. [7], Theorem 2.7) given now is designed to be, as well as self-contained, easy to apply. First, two auxiliary definitions: A *system of isomorphisms* at  $\mathbf{A}\in\mathbf{A}$  is a  $G$ -indexed family of isomorphisms  $\varphi_g:\varepsilon\mathbf{A}\rightarrow\mathbf{A}$  such that the triangle



commutes for all  $g, h \in G$ . Such a family  $\{\varphi_g\}$  is called *stabilizing* if, for some  $\mathbf{B}\in Stab\mathbf{A}$  and isomorphism  $\alpha\in\mathbf{A}(\mathbf{B},\mathbf{A})$ ,



commutes for all  $g$ . If every system of isomorphisms is stabilizing, then we say  $\mathbf{A}$  is *stably closed*.

The following easy basic result from [7] will be useful.

**LEMMA 2.1.** *An object  $\mathbf{A}$  of a  $G$ -category is isomorphic to a stable object iff there is a stabilizing system of isomorphisms at  $\mathbf{A}$ .*

Now, a major result of [7] is that a  $G$ -category is stably closed provided its stable subcategory is stably reflective and its insertion tripleable (see [1], p. 102). We will show that this is untrue if the stable subcategory is only assumed to be reflecti-

ve. Indeed, we will give several examples showing that a G-category being stably closed is neither implied by, nor implies, its stable subcategory being reflective; and in these examples insertions of reflective subcategories are always tripleable — see Proposition 3.1.

### 3. Crossed modules.

We recall that a crossed module consists of two groups G and M, an action of G on M — written  $(g, m) \mapsto g \cdot m$  and assumed to respect multiplication — and a homomorphism  $\mu: M \rightarrow G$  such that

$$\mu(g \cdot m) = g \mu(m) g^{-1}, \quad \mu(m) n = m n m^{-1}.$$

We shall deal most often with three common examples, namely, insertion of normal subgroups in G ( $\mu$  injective), central extension by G ( $\mu$  surjective) and G-modules ( $\mu$ -trivial).

The category **C** corresponding to  $\mu$  — see Loday [9] — has the elements of G as objects. An arrow  $g \rightarrow h$  of **C** is an element  $(m, g)$  of the semidirect product  $M \rtimes G$  of M by G satisfying  $\mu(m)g = h$ . Composition is given by

$$(m, g) \circ (n, h) = (m n, h), \quad \mu(n) h = g.$$

This **C** is a G-category under conjugation by elements of G (viewing G as subgroup of  $M \rtimes G$ ):

$$x g = x g x^{-1}, \quad x(m, g) = (x m, x g) \quad (x \in G).$$

Naturally **C** is also a  $G \times G^{\text{OP}}$ -category with G-action left multiplication and  $G^{\text{OP}}$ -action right multiplication. We find the action by conjugation of paramount interest and ignore the other actions.

Notice that  $\text{Ob}(\text{Stab } \mathbf{C}) = Z(G)$  whereas

$$(1) \quad \text{Ar}(\text{Stab } \mathbf{C}) = C_{M \rtimes G}(G) = M^G \times Z(G),$$

where  $M^G$  stands for the subgroup of fixed points of M. Plainly  $\text{Stab } \mathbf{C}$  is a full subcategory of **C** iff

$$(2) \quad \mu^{-1}(Z(G)) \subset M^G.$$

Thereby, since  $Z(G) \text{ im } \mu$  is manifestly the set of objects of **C** isomorphic to stable objects, all objects are isomorphic to stable objects iff

$$(3) \quad G = Z(G) \text{ im } \mu.$$

Thus inasmuch as  $\text{Stab } \mathbf{C}$ , as well as **C**, is a groupoid, we get

**PROPOSITION 3.1.** *The following statements are equivalent:*

- (i) *Stab C is reflective;*
- (ii) *inclusion Stab C → C is an equivalence (hence tripleable);*
- (iii) *the expressions in (2) and (3) are valid.*

Using the result, namely [7], Theorem 9.7, cited below Lemma 2.1, an easy consequence of the definition of stably closed G-categories as given in §1 of [7] is:

**COROLLARY 3.2.** *Stab C is stably reflective precisely when C is stably closed and has, for each object g, coproducts of the form  $\coprod_{x \in G} {}^xg$ .*

**4.  $\Xi(G)$  and  $\mu$  injective.**

Consider, where  $[, ]$  denotes commutator, the sets

$$\begin{aligned} \Xi(G) &= \{g \in G \mid [g, G] \subset im\mu\}, \\ Set_g(G, M) &= \{f \in Set(G, M) \mid \mu \circ f = [g, -]\}. \end{aligned}$$

Concerning the latter, we point out that

$$Set_g(G, M) \cap Set_h(G, M) \neq \emptyset$$

exactly when  $h^{-1}g \in Z(G)$ , in which case  $Set_g(G, M) = Set_h(G, M)$ . Concerning the former, clearly

$$G \triangleright \Xi(G) \supset Z(G) im\mu.$$

In fact

$$(4) \quad \Xi(G) / im\mu = Z(G / im\mu).$$

If  $f \in Set(G, M)$  and  $x \in G$ , the element  $(f(x), {}^xg)$  of  $M \times G$  is an arrow  ${}^xg \rightarrow g$  of **C** exactly when  $\mu(f(x)) = {}^xg g^{-1} = [g, x]$ . It follows there is a bijection of  $Set_g(G, M)$  with the set of families of arrows of the form  $\{{}^xg \rightarrow g\}_{x \in G}$ . Thus, since evidently

$$\Xi(G) = \{g \in G \mid Set_g(G, M) \neq \emptyset\},$$

$\Xi(G)$  may be described as the set of objects  $g$  of **C** at which there is a G-indexed family of isomorphisms  ${}^xg \rightarrow g$ .

When  $\mu$  is injective, **C** is a preorder; and one can determine immediately necessary and sufficient conditions for **C** to be stably closed. For, if  $g \in G$ , all G-indexed families  ${}^xg \rightarrow g$  are systems of isomorphisms. Furthermore, a given system of isomorphisms at  $g$  is stabilizing iff  $g$  is isomorphic to a stable object. Hence, insofar as  $\mu^{-1}(Z(G)) = M^G$  for  $\mu$  injective, by Proposition 3.1 one has

**PROPOSITION 4.1.** *If  $\mu$  is injective, then  $\mathbf{C}$  is stably closed iff  $\Xi(G) = Z(G)im\mu$ , and  $Stab\mathbf{C}$  is reflective iff  $Z(G)im\mu = G$ .*

We add that, here,  $Stab\mathbf{C}$  is automatically stably reflective if it is reflective. Indeed, the same is true of any  $G$ -groupoid which is a preorder. Thus, when  $\mu$  is injective,  $Stab\mathbf{C}$  reflective implies  $\mathbf{C}$  stably closed, by Corollary 3.2 (or directly, as  $\Xi(G) \supset Z(G)im\mu$ ). However, by the preceding result and (4), if  $G$  is any nonabelian group,  $M$  is trivial and  $\mu$  is insertion, then  $\mathbf{C}$  is stably closed and  $Stab\mathbf{C}$  is not reflective.

### 5. Commutator derivations.

Given groups  $A$  and  $B$  and an action of  $A$  on  $B$ ,  $Der(A,B)$  will stand for the set of derivations  $d: A \rightarrow B$  (explicitly,  $d(a_1)^{a_2}d(a_2) = d(a_1a_2)$ ) and  $PDer(A,B)$  will stand for the set of principal derivations  $A \rightarrow B$  (explicitly,  $d_b(a) = b^a b^{-1}$ ). When we speak of derivations, say,  $A \rightarrow A$  the action, in the absence of a specified action, is understood to be conjugation.

We set

$$\begin{aligned} Der_g(G,M) &= Der(G,M) \cap Set_g(G,M), \\ CDer(G,M) &= \bigcup Der_g(G,M) \end{aligned}$$

and call elements of  $Der_g(G,M)$  *commutator derivations at  $g$* . Note that principal derivations  $G \rightarrow M$  are commutator derivations.

To see when  $d \in Set_g(G,M)$  corresponds to a system of isomorphisms in  $G$  at  $g$ , we set  $\varphi_x = (d(x), {}^xg) \in M \rtimes G$ . Since we know the  $\varphi_x$  constitute a family of arrows  ${}^xg \rightarrow g$  of  $\mathbf{C}$ ,

$$\begin{aligned} \varphi_x {}^x\varphi_y &= \varphi_{xy} \Leftrightarrow (d(x), {}^xg) \circ ({}^x d(y), {}^{xy}g) = (d(xy), {}^{xy}g) \\ &\Leftrightarrow (d(x) {}^x d(y), {}^{xy}g) = (d(xy), {}^{xy}g) \Leftrightarrow d \in Der_g(G,M). \end{aligned}$$

In addition, since  $d = d_m$  forces  $\mu(m)^{-1}g$  to be central (because  $d_m \in Der_{\mu(m)}(G,M)$ ), the triangle

$$\begin{array}{ccc} {}^xg & \xrightarrow{\varphi_x = (d(x), {}^xg)} & g \\ & \searrow ({}^x(m,c) = ({}^x m, c)) & \swarrow (m,c) \\ & & c \end{array}$$

commutes for some  $c \in Z(G)$  and  $m \in M$  iff  $d(x) = m {}^x m^{-1} = d_m(x)$  for some  $m \in M$ .

These considerations establish:

**LEMMA 5.1.** *Let  $g \in G$ . The assignment*

$$d \in \text{Der}_g(G, M) \mapsto \{(d(x), {}^xg) \mid {}^xg \rightarrow g\}_{x \in G}$$

*yields a 1-1 correspondence between the set of commutator derivations at  $g$  and the set of systems of isomorphisms at  $g$ . Under this correspondence, principal derivations correspond to stabilizing systems of isomorphisms.*

**THEOREM 5.2.** *The  $G$ -category associated to a crossed module  $M \rightarrow G$  is stably closed iff every commutator derivation  $G \rightarrow M$  is principal.*

**COROLLARY 5.3.** *If  $\mathbf{C}$  is stably closed,  $d\mu$  is a principal derivation for every commutator derivation  $d: G \rightarrow M$ .*

**PROOF.** We have  $d(x) = n^x n^{-1}$  for some  $n \in M$ . So

$$d(\mu(m)) = n^{\mu(m)} n^{-1} = n(mn^{-1}m^{-1}).$$

When  $\mu$  is surjective, the converse of this result is true — see the proof of Theorem 6.2.

In the light of the two last results, it would be of interest to know when every derivation is a commutator derivation. Before giving a criterion we study the set

$$\underline{\text{Set}}(G, M) = \{t \in \text{Set}(G, M) \mid t(x) {}^x t(y) t(xy)^{-1} \in \ker \mu\},$$

saying elements  $t$  and  $u$  of this set are *equivalent* if  $u = \delta t$  (pointwise product) for some  $\delta \in \text{Set}(G, \ker \mu)$ . Given  $t \in \underline{\text{Set}}(G, M)$  and considering the two ways of bracketing the product  $t(x) {}^x t(y) {}^{xy} t(z)$ , we find that  $f \in \text{Set}(G \times G, \ker \mu)$  defined by

$$t(x) {}^x t(y) = t(xy) f(x, y)$$

is a 2-cocycle (recall  $\ker \mu$  central). Moreover, the 2-cocycle defined by the product  $\delta t$  is just  $\delta(x) {}^x \delta(y) \delta(xy)^{-1} f(x, y)$ . Thus  $t$  is equivalent to an element of  $\text{Der}(G, M)$  precisely when  $f$  is a 2-coboundary. In particular we have

**LEMMA 5.4.** *If  $H^2(G, \ker \mu) = 1$ , then elements of  $\underline{\text{Set}}(G, M)$  are equivalent to elements of  $\text{Der}(G, M)$ .*

The converse of this result is, of course, invalid (consider a  $G$ -module  $M$  with  $H^2(G, M) \neq 1$ ).

**PROPOSITION 5.5.** *If every derivation  $G \rightarrow G$  that factors through  $\text{im} \mu$  is principal then  $\text{Der}(G, M) = \text{CDer}(G, M)$ . Furthermore, if  $H^2(G, \ker \mu) = 1$ , the converse is true.*

**PROOF.** Observe the principal derivations  $G \rightarrow G$  are precisely the



functions of the form  $[g, -]$ . In particular, if  $d: G \rightarrow M$  is a derivation and the derivation  $\mu d: G \rightarrow G$  is principal, then  $d \in \text{Der}(G, M)$ .

Conversely, suppose  $\text{Der}(G, M) = \text{CDer}(G, M)$ , and let  $f$  in  $\text{Der}(G, G)$  with  $\text{im } f \subset \text{im } \mu$ . Regarding  $\mu$  as epimorphism  $M \rightarrow \text{im } \mu$ , let  $s$  be one of its cross-sections. Then, since  $f$  is a derivation, the function  $d = s \circ f: G \rightarrow M$  is in fact in  $\underline{\text{Set}}(G, M)$ . So, provided  $H^2(G, \ker \mu) = 1$ , the preceding result provides a  $\delta: G \rightarrow \ker \mu$  such that  $\delta d: G \rightarrow M$  is a derivation. But

$$\mu \circ \delta d = \mu \circ d = \mu \circ s \circ f = f.$$

Thus, by supposition,  $f$  is principal.

The assumption  $H^2(G, \ker \mu) = 1$  is needed for the validity of the converse. An example can be gotten from the nonsplit extension of  $\mathbb{Z}_2$  by  $\mathbb{Z}_2$ .

We end the section with an easy consequence of Lemmas 5.1 and 5.4 and the fact  $\text{Set}_g(G, M) \subset \underline{\text{Set}}(G, M)$ .

**PROPOSITION 5.6.** *Given  $g \in \mathbf{C}$  and a  $G$ -indexed family of isomorphisms, if  $H^2(G, \ker \mu) = 1$ , there is a  $G$ -indexed family of automorphisms  $g \rightarrow g$  such that the composite family  ${}^x g \rightarrow g \rightarrow g$  is a system of isomorphisms.*

The assumption  $H^2(G, \ker \mu) = 1$  is not redundant here either, as shown by the example at the end of Section 7. However, we know of no such example where  $\mathbf{C}$  is stably closed.

## 6. $\mu$ Trivial and $\mu$ surjective.

When  $\mu$  is trivial, by Theorem 5.2 and Proposition 5.5, the criterion for  $\mathbf{C}$  to be stably closed is given by

**THEOREM 6.1.** *The  $G$ -category associated with a  $G$ -module  $M$  is stably closed iff  $H^4(G, M) = 1$ .*

Further, when  $\mu$  is trivial, an easy consequence of Proposition 3.1 and (1) is that  $\text{Stab } \mathbf{C}$  is reflective when and only when  $G$  acts trivially on  $\mathbf{C}$ . So here examples that are reflective and not stably closed, and vice versa, are eminently constructible. Here, also, if  $\text{Stab } \mathbf{C}$  is reflective, one gets by the foregoing that  $\mathbf{C}$  is stably closed exactly when  $\text{Hom}(G, M) = 1$ . This can be seen as direct consequence of the given definition of stably closed (cf. [7], p.12) insofar as triviality of  $\mu$  implies  $\mathbf{C}(g, g) \approx M$ . Lastly, in the  $\mu$  trivial case, if  $\text{Stab } \mathbf{C}$  is reflective it is readily seen to be stably reflective iff every function  $G \rightarrow M$  is constant; that is, iff  $G = 1$  or  $M = 1$ . Thus stably closed examples  $\mathbf{C}$  with non-

stably reflective stable subcategory abound.

There is too a special criterion for  $\mathbf{C}$  to be stably closed when  $\mu$  is surjective. Beforehand we point out surjectivity of  $\mu$  implies triviality of the action of  $G$  on  $\ker \mu$ . Particularly,

$$\text{Der}(G, \ker \mu) = \text{Hom}(G, \ker \mu).$$

**THEOREM 6.2.** *If  $\mu$  is surjective, a necessary and sufficient condition for  $\mathbf{C}$  to be stably closed is that  $\sigma_\mu$  is a principal derivation for every homomorphism  $\sigma: G \rightarrow M$  that factors through  $\ker \mu$ .*

**PROOF.** The necessity is an immediate consequence of 5.3. For the sufficiency, let  $d \in \text{Der}_{\mathbf{g}}(G, M)$ , choose  $n \in M$  with  $\mu(n) = \mathbf{g}$  and let  $s$  be a cross-section of  $\mu$ . Then  $d(x) = [n, s(x)]\sigma(x)$  for some function  $\sigma: G \rightarrow \ker \mu$ . Thereto, because  $\ker \mu$  is central, the function  $[n, s-]$  is independent of the choice of cross-section  $s$ . Consequently one has

$$\begin{aligned} \sigma(xy) &= [s(xy), n]d(xy) = [s(x)s(y), n]d(x)^x d(y) \\ &= s(x)[s(y), n]s(x)^{-1}[s(x), n]d(x)^x d(y) \\ &= \mu(s(x))[s(y), n]\sigma(x)^x d(y) \\ &= \sigma(x)^x [s(y), n]^x d(y) = \sigma(x)^x \sigma(y) = \sigma(x)\sigma(y). \end{aligned}$$

So  $\sigma$  is a homomorphism.

By assumption,  $\sigma(x) = \sigma(\mu(s(x))) = [m, s(x)]$  for some  $m \in M$ .

Thus

$$\begin{aligned} d(x) &= [n, s(x)][m, s(x)] = n[m, s(x)]n^{-1}[n, s(x)] \\ &= [nm, s(x)] = nm^{\mu(s(x))}(nm)^{-1} = nm^x(nm)^{-1}. \end{aligned}$$

Therefore  $d \in \text{PDer}(G, M)$ , as required by 5.2 for  $\mathbf{C}$  to be stably closed.

For the next result we note that when  $\mu$  is surjective, by Proposition 3.1,  $\text{Stab } \mathbf{C}$  is reflective exactly when  $\mu^{-1}(Z(G)) \subset Z(M)$  ( $= M^G$ ).

**COROLLARY 6.3.** *If  $\mu$  is surjective, then  $\text{Hom}(G, \ker \mu) = 1$  iff  $\mathbf{C}$  is stably closed and  $\text{Stab } \mathbf{C}$  is reflective.*

**PROOF.**  $\Leftarrow$  Let  $\sigma \in \text{Hom}(G, \ker \mu)$ . Then  $\sigma_\mu = [m, -]$  for some  $m$ . But, since  $1 = \mu\sigma_\mu = [\mu(m), \mu-]$ ,  $\mu(m)$  is central. Thus  $m$  is central, implying  $\sigma_\mu = 1$ . So  $\sigma = 1$ .

$\Rightarrow$  Let  $\mu(m) \in Z(G)$  and consider  $d_m \in \text{PDer}(G, M)$ . Since  $\text{im } d_m \subset \ker \mu$ ,  $d_m$  may be regarded as element of  $\text{Hom}(G, \ker \mu)$ . Consequently  $m \in Z(M)$ .

This result, in one direction, can be stated in purely group

theoretical terms: If  $C$  is a central subgroup of a group  $E$  such that  $Hom(E/C, C) = 1$  then every element of  $E$  that is central modulo  $C$  is central in  $E$ .

We claim that for  $\mu$  surjective,  $Stab C$  is stably reflective when and only when  $\mu$  is an isomorphism. The sufficiency of  $\mu$  being an isomorphism is obvious. The necessity follows easily from the fact surjectivity of  $\mu$  forces  $id:1 \rightarrow 1$  to be the component of a unit of reflection at  $1 \in C$ . In particular, here too one may construct at will stably closed  $G$ -categories with reflective non-stably reflective stable subcategory.

We terminate the section with two examples both of which have  $\mu$  surjective.

**EXAMPLE.** *Stab C reflective and C not stably closed* : By 6.3, it is enough to construct a surjection  $\mu: M \rightarrow G$  with central kernel such that  $\mu^{-1}(Z(G)) \subset Z(M)$  and  $Hom(G, ker \mu) \neq 1$ . For this, take  $M = Z_2 \times S_3$ ,  $G = S_3$  and  $\mu$  the projection.

**EXAMPLE.** *C stably closed and Stab C not reflective* : Let  $M$  be the (quaternion) group with presentation  $a^4 = 1, ba = a^3b, a^2 = b^2$  and let  $G$  be the group with presentation  $\alpha^2 = 1 = \beta^2, \alpha\beta = \beta\alpha$ . Take  $\mu: M \rightarrow G$  to be defined by  $\mu(a) = \alpha$  and  $\mu(b) = \beta$ , so that  $\mu$  is a surjection with kernel  $\langle a^2 \rangle = Z(M)$ ; and let  $\sigma: G \rightarrow ker \mu$  be a non-trivial homomorphism. By symmetry, one may assume that either  $\sigma(\alpha) = a^2$  and  $\sigma(\beta) = 1$  or  $\sigma(\alpha) = a^2 = \sigma(\beta)$ . But, one checks that in the former case  $\sigma\mu = [b, -]$  while, in the latter,  $\sigma\mu = [a b, -]$ . Thus  $C$  is stably closed, by 6.2. By 3.1,  $Stab C$  is not reflective.

7. Whitehead derivations and  $\Xi'(G)$ .

For details of the following treatment of the Whitehead group of derivations with respect to  $\mu$  we refer to Whitehead ([14], §7). Recall that  $Der(G, M)$  is a monoid with multiplication, say  $\diamond$ , defined by

$$(d \diamond e)(x) = d(\tilde{e}(x))e(x), \tilde{e}(x) = \mu(e(x))x.$$

Note that the identity of this monoid is the trivial derivation and that  $\tilde{e}$  is an endomorphism of  $G$ . We call units *Whitehead derivations* and the group,  $WDer(G, M)$ , they form the *Whitehead group*. Notice the Whitehead derivations are precisely the  $d$  in  $Der(G, M)$  with  $d \in Aut G$ .

Now, according to Lue ([10], Corollary 2), the map

$$(5) \quad WDer(G, M) \rightarrow Aut G : w \mapsto \tilde{w}$$

is a homomorphism. Since, plainly, the kernel of the induced homomorphism  $WDer(G,M) \rightarrow OutG$  is  $CDer(G,M)$ ,  $CDer(G,M)$  is a normal subgroup of  $WDer(G,M)$ . Moreover, one has

**PROPOSITION 7.1.** *If  $\mu$  is surjective and  $H^2(G, ker \mu) = 1$ , then the map (5) induces an isomorphism  $WDer(G,M) / CDer(G,M) \approx OutG$ .*

**PROOF.** Let  $\alpha \in AutG$  and choose  $t \in Set(G,M)$  such that  $\mu(t(x)) = \alpha(x)x^{-1}$ . A routine computation divulges that  $t \in \underline{Set}(G,M)$ . Thus, by 5.4,  $t$  is equivalent to some  $d \in Der(G,M)$ ; and this  $d$  is a Whitehead derivation with  $\tilde{d} = \alpha$ .

Notice that, under the hypothesis of this result, every automorphism of  $G$  is inner iff every Whitehead derivation  $G \rightarrow M$  is a commutator derivation — cf. Proposition 5.5.

Next, inasmuch as  $PDer(G,M)$ , being a normal subgroup of the Whitehead group, is normal in  $CDer(G,M)$ , Theorem 5.2 may be rephrased as follows.

**THEOREM 7.2.** *The G-category corresponding to a crossed module  $M \rightarrow G$  is stably closed exactly when the factor group  $CDer(G,M) / PDer(G,M)$  is trivial.*

The paper of Norrie [12] affords an interesting interpretation of the factor group  $CDer(G,M) / PDer(G,M)$ : There is another homomorphism

$$WDer(G,M) \rightarrow Aut M: w \mapsto \hat{w},$$

$\hat{w}$  being defined by  $\hat{w}(m) = w(\mu(m))m$ . The homomorphisms ( $\sim$ ) and ( $\hat{\quad}$ ) together define a homomorphism  $\Delta$  from  $WDer(G,M)$  to the group of automorphisms,  $Aut \mu$ , of the crossed module  $\mu$  (see [12]). This  $\Delta$  is in fact a crossed module. Norrie calls it the actor,  $Actor \mu$ , of  $\mu$ . She further defines the outer actor,  $Out \mu$ , of  $\mu$  to be the factor crossed module of  $Actor \mu$  by its normal sub-crossed module  $im((\eta, \gamma): \mu \rightarrow \Delta)$ , where  $\eta: M \rightarrow WDer(G,M)$  and  $\gamma: G \rightarrow Aut \mu$  are the homomorphisms given by

$$\eta_m(g) = m^g m^{-1} \text{ and } \gamma(g) = (g(\quad), g(\quad)).$$

Now, if  $d \in Der_g(G,M)$ , plainly  $d(\mu(m)) = \alpha(m)(g m) m^{-1}$  for some  $\alpha \in Set(M, ker \mu)$ ; and, using centrality of  $ker \mu$ , it can be checked that  $\alpha$  is a homomorphism. Thus, if  $Hom(M, ker \mu) = 1$ , we see that  $CDer(G,M) / PDer(G,M)$  is the kernel of  $Out \mu$  regarded as a homomorphism

$$WDer(G,M) / PDer(G,M) \rightarrow Aut \mu / im \gamma.$$

In particular, by the preceding result, given that  $Hom(M, ker \mu) = 1$  the G-category  $\mathbf{C}$  corresponding to  $\mu$  is stably closed iff

$Out\ \mu$  has trivial kernel.

We denote the set of objects of  $\mathbf{C}$  at which there is a system of isomorphisms by  $\Xi'(G)$ . Alternately, by Lemma 5.1,

$$\Xi'(G) = \{g \in G \mid Der_g(G, M) \neq \emptyset\}.$$

The nonobvious fact  $\Xi'(G)$  is a subgroup of  $G$  follows from the fact that  $CDer(G, M)$  is a subgroup of  $WDer(G, M)$ . Indeed, since  $\Xi'(G)$  contains  $im\ \mu$  and, being contained in  $\Xi(G)$ , is central modulo  $im\ \mu$ ,  $\Xi'(G)$  is a normal subgroup of  $G$ . One might define another set, say  $P\Xi'(G)$ , made up of the objects of  $\mathbf{C}$  at which there is a stabilizing system of isomorphisms. But, by Lemma 2.1,  $P\Xi'(G)$  would just be  $Z(G)im\ \mu$ .

We wish to consider when inclusions in the normal series

$$(6) \quad G \supset \Xi(G) \supset \Xi'(G) \supset Z(G)im\ \mu$$

are proper. Properness of the first inclusion is uninteresting — see (4). Properness of the second means there is a  $g \in \mathbf{C}$  admitting a  $G$ -indexed family of isomorphisms  ${}^xg \rightarrow g$ , but no system of isomorphisms. Notice this cannot happen when  $H^2(G, ker\ \mu) = 1$ , by Proposition 5.6. Properness of the third inclusion means there is an object, isomorphic to no stable object, at which there is a system of isomorphisms (but no stabilizing one). Of course, when  $\mathbf{C}$  is stably closed, this cannot be. Yet when, for instance,  $\mu$  is trivial, there can only be systems of isomorphisms at stable objects, so that equality of  $\Xi'(G)$  and  $Z(G)im\ \mu$ , is automatic; but, nevertheless — see Theorem 6.1 —  $\mathbf{C}$  need not be stably closed (ensuring, incidentally, the existence of a nonstabilizing system of isomorphisms at some stable object).

On trivial grounds, if  $\mu$  is surjective, trivial or injective, all but possibly the inclusion  $G \supset \Xi(G)$  in (6) are improper except that, for injective  $\mu$ ,  $\Xi'(G) = Z(G)im\ \mu$  precisely when  $\mathbf{C}$  is stably closed (it need not be). In particular, in all three of our usual cases plus the case  $H^2(G, ker\ \mu) = 1$ ,  $\Xi'(G) = \Xi(G)$ . The following example shows that this is not true in general.

**EXAMPLE.** A crossed module  $\mu: M \rightarrow G$  such that  $\Xi(G) = G$  and  $\Xi'(G) = Z(G) \neq G$ : Take  $M = \mathbb{Z}_4$ ,  $G$  nonabelian of order 8, let  $G$  act trivially on  $M$ , and let  $\mu$  be the composite  $M \twoheadrightarrow Z(G) \hookrightarrow G$ . Since  $im\ \mu = Z(G)$  and  $M$  is abelian,  $\mu$  is a crossed module. Since  $G/im\ \mu$  is abelian,  $\Xi(G) = G$ , by (4). Since  $Z(G)$  is the only normal subgroup of  $G$  of order 2 and  $G/Z(G)$  is noncyclic of order 4, the image of every homomorphism  $G \rightarrow M$  is contained in  $ker\ \mu$ . But, since  $M^G = M$ , all derivations  $G \rightarrow M$  are homomorphisms. Hence  $\Xi'(G) = Z(G)$ , by 5.1.

We remark that, in this example, by Theorem 7.2,  $\mathbf{C}$  is not stably closed (there are nontrivial homomorphisms  $G \rightarrow ker\ \mu$ ).

**8. Relativization.**

There are relative notions of commutator derivation and Whitehead group: Let  $H$  be a subgroup of  $G$ . We define

$$\Xi_G(H) = \{g \in G \mid [g, H] \subset im\mu\}.$$

Then

$$G \triangleright \Xi_G(H) \triangleright C_G(H)im\mu$$

and, where  $K^-$  denotes images of subgroups  $K$  of  $G$  in  $G/im\mu$ ,  $\Xi_G(H)^- = C_G(H^-)$ . The interpretation of  $C_G(H)im\mu$  and  $\Xi_G(H)$  is that they are the respective sets of objects of  $\mathbf{C}$  isomorphic to  $H$ -stable objects and objects  $g$  of  $\mathbf{C}$  at which there is a family of isomorphisms of the form  $\{^y g \rightarrow g\}_{y \in H}$ . In addition, we see that  $g \in \Xi_G(H)$  exactly when the set,  $Set_g(H, M)$ , of function  $f: H \rightarrow M$  for which  $\mu f$  is the principal derivation  $[g, -]$  is non-empty.

As before, we define

$$Der_g(H, M) = Der(H, M) \cap Set_g(H, M), \quad CDer(H, M) = \bigcup Der_g(H, M)$$

and call elements of  $Der_g(H, M)$  *H-commutator derivations at g*. We obtain, as in Lemma 5.1, a bijection of  $Der_g(H, M)$  with the set of  $H$ -systems of isomorphisms at  $g$  under which elements of  $PDer(H, M)$  correspond to  $H$ -stabilizing systems of isomorphisms. Thus, if we say  $\mathbf{C}$  is *hereditarily stably closed* if it is a stably closed canonical  $K$ -category for every subgroup  $K$  of  $G$ , we get:

**THEOREM 8.1.** *C is hereditarily stably closed iff, for every subgroup K of G, every K-commutator derivation is principal.*

The *relative Whitehead group*,  $WDer(H, M)$ , is the group (under Whitehead's multiplication ([14] — see §7)

$$\{d \in Der(H, M) \mid \mu(d(H)) \subset H, \bar{d} \in AutH\}.$$

Define

$$\Xi'_G(H) = \{g \in G \mid Der_g(H, M) \neq \emptyset\}$$

and consider the homomorphism  $\psi_H: N_G(H) \rightarrow AutH$  given by the action by conjugation of  $N_G(H)$  on  $H$ . Plainly

$$CDer(H, M \subset WDer(H, M) \text{ iff } \Xi'_G(H) \subset N_G,$$

in which case  $CDer(H, M)$  is the inverse image of  $im\psi_H$  under the evident homomorphism  $WDer(H, M) \rightarrow AutH$ ; and then, although  $CDer(H, M)$  is not a normal subgroup of  $WDer(H, M)$  unless images  $\tilde{w}$  of elements  $w$  of  $WDer(H, M)$  normalize  $im\psi_H$ ,  $\Xi'_G(H)$  is nevertheless a subgroup (normal if  $H$  is its own normalizer) of  $\Xi_G(H)$  containing  $C_G(H)im\mu$ .

Naturally, if  $\Xi'_G(H) \subset N_G(H)$ ,  $\mathbf{C}$  is  $H$ -stably closed iff  $CDer(H, M) / PDer(H, M) = 1$ . However, with respect to  $\mathbf{C}$  being

hereditarily stably closed, the assumption  $E'_G(H) \subset N_G(H)$  is not made without loss of generality. To see this, take  $G$  = any finite nonabelian simple group,  $M = A \times G$  for any nontrivial abelian subgroup  $A$  of  $G$ , and  $\mu$  = projection. Then  $\mathbf{C}$  is not hereditarily stably closed, since  $\mathbf{C}$  is clearly not  $A$ -stably closed (consider the composite monomorphism  $A \rightarrow A \times 1 \hookrightarrow M$ ). But  $\mathbf{C}$  is stably closed, by Corollary 6.3. Thus  $\mathbf{C}$  is  $K$ -stably closed for every subgroup  $K$  of  $G$  such that  $E'_G(K) \subset N_G(K)$ .

If  $\mu$  is trivial,  $E'_G(K) = C_G(K) \subset N_G(K)$  for every subgroup  $K$  of  $G$ ; and so we have:

**THEOREM 6.3.** *The  $G$ -category associated to a  $G$ -module  $M$  is hereditarily stably closed iff, for all subgroups  $K$  of  $G$ ,  $H^1(K, M) = 1$ .*

Thereto the evident relative versions of Proposition 4.1 and Theorem 6.2 yield criteria for  $\mathbf{C}$  to be hereditarily stably closed when  $\mu$  is, respectively, injective or surjective.

There is also a relativized notion of  $G$ -category associated to a crossed module; namely, the  $H$ -category  $\mathbf{C}_H$  (it is an  $H$ -subcategory of  $\mathbf{C}$ ) corresponding to the sub-crossed module  $\mu^{-1}(H) \rightarrow H$  induced by restriction of  $\mu$ . Concerning the relationship between  $\mathbf{C}_H$  being stably closed and  $\mathbf{C}$  being  $H$ -stably closed, using Theorem 5.2 and the comments preceding Theorem 8.1, it is fairly routine to prove:

**PROPOSITION 8.3.** *The  $H$ -category  $\mathbf{C}_H$  is stably closed and  $E'_G(H) \subset HC_G(H)$  iff  $\mathbf{C}$  is  $H$ -stably closed and  $M = \mu^{-1}(H)M^H$ .*

Note that, assuming  $E'_G(H) \subset N_G(H)$ , the exact condition needed to render the groups  $CDer(H, \mu^{-1}(H))$  and  $CDer(H, M)$  identifiable is  $E'_G(G) \subset HC_G(H)$ . Yet this identification may not identify  $PDer(H, \mu^{-1}(H))$  with  $PDer(H, M)$ , the condition therefore being exactly  $M = \mu^{-1}(H)M^H$ .

Lastly, concerning Proposition 8.3, the reader will find there are better results in the special cases  $\mu$  injective, surjective or trivial.

## GORDON — CROSSED MODULES AS G-CATEGORIES

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